Generalized Reflexive Structures Properties of Crossed Products Type

Eltiyeb Ali$^{1,2}$

$^1$ Department of Mathematics, College of Science and Arts, Najran University, KSA
$^2$ Department of Mathematics, Faculty of Education, University of Khartoum, Sudan

Abstract. Let $R$ be a ring and $M$ be a monoid with a twisting map $f : M \times M \rightarrow U(R)$ and an action map $\omega : M \rightarrow \text{Aut}(R)$. The objective of our work is to extend the reflexive properties of rings by focusing on the crossed product $R * M$ over $R$. In order to achieve this, we introduce and examine the concept of strongly CM-reflexive. Although a monoid $M$ and any ring $R$ with an idempotent are not strongly CM-reflexive in general, we prove that $R$ is strongly CM-reflexive under some additional conditions. Moreover, we prove that if $R$ is a left p.q.-Baer (semiprime, left APP-ring, respectively), then $R$ is strongly CM-reflexive. Additionally, for a right Ore ring $R$ with a classical right quotient ring $Q$, we prove $R$ is strongly CM-reflexive if and only if $Q$ is strongly CM-reflexive. Finally, we discuss some relevant results on crossed products.

2020 Mathematics Subject Classifications: 16S36, 16N60, 16U99.


1. Introduction

Unless otherwise stated, we assume that $R$ is an associative ring with identity and $M$ is a monoid. The concept of reflexive properties of rings was first studied by Mason [1]. In particular, a right ideal $I$ of $R$ is said to be reflexive if $xRy \subseteq I$ implies $yRx \subseteq I$ for any $x, y \in R$. This concept is also specialized to the zero ideal of a ring, where a ring $R$ is said to be reflexive if its zero ideal is reflexive. Moreover, a ring $R$ is called completely reflexive if $xy = 0$ implies $yx = 0$ for any $x, y \in R$. It is worth noting that reduced rings are completely reflexive, and every completely reflexive ring is semicommutative, as shown in the literature [1].

Several authors have discussed extensions of reflexive rings, including strongly reflexive rings, strongly $M$-reflexive rings, Armendariz rings, reversible rings, and reflexive on skew monoid rings, in numerous publications (see, for example, [2], [3], [4] and [5]). According to [6], a ring $R$ is said to be an $M$-Armendariz ring of crossed product type relative to the given

DOI: https://doi.org/10.29020/nybg.ejpam.v16i4.4918
Email addresses: eltiyeb76@gmail.com, emali@nu.edu.sa (E. Ali)
twisting \( f \) and action \( \omega \), or an \( M \)-quasi Armendariz ring (or simply a \( CM \)-Armendariz ring or \( CM \)-quasi Armendariz ring, respectively), if for any \( \phi = \sum_{i=1}^{n} a_i g_i, \psi = \sum_{j=1}^{m} b_j h_j \in R * M \) such that \( \phi \psi = 0 \) (resp., \( \phi(R * M)\psi = 0 \)), it follows that \( a_i \omega g_i(b_j) = 0 \) (resp., \( a_i R \omega g_i(b_j) = 0 \)) for all \( i, j \) and all \( g_i, b_j, l \in M \).

The focus of this paper is on investigating strongly \( CM \)-reflexive rings, which are a reflexive-like property defined for the monoid crossed product \( R * M \) with respect to the given twisting map \( f \) and action map \( \omega \). This concept is a generalization of several other reflexive properties, including reflexive rings, strongly reflexive rings, strongly \( M \)-reflexive rings, and skew monoid rings. Additionally, if \( R \) is a left \( p.q. \)-Baer (semiprime, left \( APP \)-ring, respectively), then \( R \) is strongly \( CM \)-reflexive for a strictly totally ordered monoid. Additionally, if \( R \) is an \( M \)-compatible ring and \( M \) is a monoid with twisting \( f \) and action \( \omega \) as above, then for any reduced ideal \( I \) of \( R \) such that \( R/I \) is strongly \( CM \)-reflexive, then \( R \) is strongly \( CM \)-reflexive. Moreover, for a right Ore ring \( R \) with classical right quotient ring \( Q \), we show that \( R \) is strongly \( CM \)-reflexive if and only if \( Q \) is strongly \( CM \)-reflexive. Finally, we discuss example and some results in the subject.

To begin with, we introduce some notions and notations relevant to this paper. Let \( \omega : M \to \text{Aut}(R) \) be a monoid homomorphism. For \( h \in M \), we denote by \( \omega h \) the automorphism \( \omega(h) \). The crossed product \( R * M \) over \( R \) is defined as the set of all finite sums \( R * M = \{ x_h h | x_h \in R, h \in M \} \), where addition is defined component-wise and multiplication is defined using the distributive law and two rules known as action and twisting. Specifically, for \( l, h, x \in R \), we have \( hx = \omega h(x)h \) and \( lh = f(l, h)lh \), where \( f : M \times M \to U(R) \) is a twisted function and \( U(R) \) denotes the set of units of \( R \). Here, the twisted function \( f \) and the action \( \omega \) of \( M \) on \( R \) satisfy the following conditions: \( \omega_l(\omega_h(x)) = f(l, h)\omega_l(\omega_h(x))f(l, h)^{-1}, \omega_l(f(h, k))f(l, hk) = f(l, h)f(l, h, k), f(1, l) = f(l, 1) = 1 \) for all \( l, h, k \in M \). It is worth noting that the monoid crossed product is a general ring construction.

Given a monoid crossed product \( R * M \) with twisting \( f \) and action \( \omega \), if the twisting \( f \) is trivial, (i.e., \( f(a, b) = 1 \)) for all \( a, b \in M \), then \( R * M \) is the skew monoid ring \( R * M \). If both the twisting \( f \) and the action \( \omega \) are trivial, then \( R * M \) is a monoid ring denoted by \( R[M] \) (see [7] and [8]). A monoid \( M \) is said to be a \( u.p.-\)monoid (unique product monoid) if, for any two nonempty finite subsets \( X \) and \( Y \) of \( M \), there exists a unique element \( h \in M \) that can be written in the form \( h = uv \) with \( u \in X \) and \( v \in Y \). An ordered monoid \( (M, \leq) \) is said to be strictly ordered if the following condition holds: whenever \( g, k, h \in M \) with \( g \prec k \), it follows that \( gh \prec kh \) and \( hg \prec hk \).

2. Generalized Reflexive rings of crossed product type

In this section, we will discuss the concept of strongly reflexive properties in the context of a monoid of crossed product \( R * M \), where \( R \) is a ring and \( M \) is a monoid with a twisting map \( f : M \times M \to U(R) \) and an action map \( \omega : M \to \text{Aut}(R) \).
Definition 1. A ring $R$ is said to be strongly $M$-reflexive of crossed product type with respect to the given twisting map $f$ and action map $\omega$ (or simply, strongly $CM$-reflexive) if for any $\phi = c_1 l_1 + c_2 l_2 + \cdots + c_n l_n$ and $\psi = a_1 h_1 + a_2 h_2 + \cdots + a_m h_m \in R * M$ satisfying that $\phi(R * M)\psi = 0$ implies that $c_i\omega_i(h_i(Ra_j)) = 0$, then $\psi(R * M)\phi = 0$ for each $i, j$ and for all $g, l_i, h_j \in M$.

Remark 1. (1) If a ring $R$ is strongly $CM$-reflexive with a trivial twisting map $f$, then we refer to the monoid $M$ as a skew strongly $M$-reflexive ring. If $R$ is strongly $CM$-reflexive with a trivial action map $\omega$, then we call $R$ a strongly $TM$-reflexive (i.e., twisted strongly $M$-reflexive) ring. Note that when both $f$ and $\omega$ are trivial, $R$ is simply strongly $M$-reflexive. In particular, if $M = (\mathbb{N} \cup \{0\}, +)$ and both $f$ and $\omega$ are trivial, then $R$ is strongly $CM$-reflexive if and only if $R$ is strongly reflexive.

(2) If $R$ is a strongly $CM$-reflexive ring with a trivial twisting map $f$, then any $M$-invariant subring $S$ (i.e., $\omega_g(S) \subseteq S$ for all $g \in M$) of $R$ is strongly $CM$-reflexive.

An ideal $I$ of a ring $R$ is considered to be right $s$-unital if there exists an element $e \in I$ for every $t \in I$ such that $te = t$. A ring is referred to as a left APP-ring if the left annihilator $l_t(Rt)$ is right $s$-unital as an ideal of $R$ for any element $t \in R$.

In their work [9], Nasr-Isfahani and Moussavi introduced a ring $R$ with an endomorphism $\omega$ and defined it as $\omega$-weakly rigid if the condition $cRt = 0$ holds if and only if $c\omega(Rt) = 0$ for any $c, t \in R$. It is worth noting that the category of $\omega$-rigid rings and $\omega$-compatible rings is a limited one, and it is evident that every $\omega$-compatible ring falls under the category of $\omega$-weakly rigid rings. However, there exist several classes of $\omega$-weakly rigid rings that do not belong to the category of $\omega$-compatible rings. By [10], $R$ is $\alpha$-rigid if and only if $R$ is $\alpha$-compatible and reduced. According to [9], any prime ring that has an automorphism $\omega$ is considered to be $\omega$-weakly rigid. If a monoid homomorphism $\omega : M \to \text{Aut}(R)$ is weakly-rigid (compatible), it means that the ring $R$ is also weakly rigid (compatible) with respect to each $g \in M$ under the automorphism $\omega_g$.

Lemma 1. [11, Lemma 1.1]. If $M$ is a u.p.-monoid, then $M$ is cancellative (i.e., for $\ell, h, \lambda \in M$, if $\ell \lambda = h \lambda$ or $\lambda \ell = \lambda h$, then $\ell = h$).

Lemma 2. Suppose $R$ and $M$ is a u.p.-monoid with a twisting map $f : M \times M \to U(R)$ and an action map $\omega : M \to \text{Aut}(R)$. If $R$ is an $M$-rigid ring, then the monoid ring $R * M$ is reduced.

Proof. Assume that $\phi = c_1 h_1 + \cdots + c_n h_n \in R * M$ satisfies $\phi^2 = 0$. According to Proposition 2.2 [6], $R$ is $CM$-Armendariz, this implies $c_i\omega_h(b_j)f(l_i, h_j)(l_i h_j) = 0$ for all $i$ and $j$, by Lemma 1, $M$ is a cancellative so $c_i\omega_h(b_j) = 0$. As $R$ is an $M$-rigid, then $R$ is a reduced, we can conclude that $c_i = 0$ for all $1 \leq i \leq n$. Consequently, $\phi = 0$, and hence $R * M$ is a reduced. 

Theorem 1. Let $R$ be a semiprime ring and $M$ be a u.p.-monoid with a twisting map $f : M \times M \to U(R)$ and an action map $\omega : M \to \text{Aut}(R)$. If $R$ is an $M$-compatible ring, then $R$ is strongly $CM$-reflexive.
Proof. The evidence has been modified from the Theorem 1.1 of [12]. Let \( \phi = cl_1 + cl_2 + \cdots + c_n l_n, \psi = a_1 h_1 + a_2 h_2 + \cdots + a_m h_m \in R \ast M \) satisfy \( \phi(R \ast M) \psi = 0 \). Then for any \( r \in R \) and \( g \in M \), we have

\[
(c_l l_1 + cl_2 + \cdots + c_n l_n)gr(a_1 h_1 + a_2 h_2 + \cdots + a_m h_m) = 0. \tag{2.1}
\]

We will employ mathematical induction on \( n \) to demonstrate that \( c_l R \omega_{l_i}(\omega_g(a_j)) = 0 \) for all \( 1 \leq i \leq n, 1 \leq j \leq m \), and for any \( g \in M \). This can be achieved by utilizing the fact that \( M \) is a compatible monoid. If we take \( l_1 = 1 \), then we have \( (c_l l_1)gr(a_1 h_1 + a_2 h_2 + \cdots + a_m h_m) = 0 \). Therefore, for each \( 1 \leq j \leq m \), we have \( c_l R \omega_{l_1}(\omega_g(a_j))(l_1 h_j) = 0 \). By Lemma 1, \( M \) is a cancellative, this means \( l_1 h_i \neq l_1 h_j \) for any \( i \) and \( j \) with \( 1 \leq i \neq j \leq m \). Thus, \( c_l R \omega_{l_1}(\omega_g(a_j)) = 0 \). For the case where \( n \geq 2 \), we can use the assumption that \( M \) is a uniquely presented monoid to find \( s \) and \( t \) with \( 1 \leq s \leq n \) and \( 1 \leq t \leq m \) such that \( l_s g h_t \) is uniquely represented by considering two subsets \( K = \{l_1 g, l_2 g, \ldots, l_n g\} \) and \( H = \{h_1, h_2, \ldots, h_m\} \) of the monoid \( M \). Without loss of generality, we may assume that \( s = 1 \) and \( t = 1 \). From Eq. (2.1), we can deduce that \( c_l \omega_{l_1}(\omega_g(Ra_1))(l_1 h_1)(l_1 h_1) = 0 \), which implies that \( c_l R \omega_{l_1}(\omega_g(a_1)) = 0 \). Since \( \omega_g \) and \( \omega_l \) are automorphisms of \( R \), we have \( c_l R \omega_{l_1}(\omega_g(a_1)) = 0 \). As a result, for every \( z \in R \), we have \( c_l R \omega_{l_1}(\omega_g(a_1 z a_1))(l_1 h_1) = 0 \), which implies that \( 0 = (c_l l_1 + cl_2 + \cdots + c_n l_n)gra_1 z (gra_1 z (a_1 h_1 + a_2 h_2 + \cdots + a_m h_m) = (c_2 l_2 + \cdots + c_n l_n)gra_1 z a_1 h_1 + a_1 z a_2 h_2 + \cdots + a_1 z a_m h_m). \]

By applying the induction hypothesis, it follows that \( c_l \omega_{l_1}(\omega_g(r a_1 z a_1)) = 0 \) for all \( 2 \leq i \leq n \) and \( 1 \leq j \leq m \). Thus, we have \( c_l R \omega_{l_1}(\omega_g(a_1)) R \omega_{l_1}(\omega_g(a_1)) = 0 \), which implies that \( c_l R \omega_{l_1}(\omega_g(a_1)) = 0 \) for all \( 1 \leq i \leq n \), as \( R \) is a semiprime ring. Therefore, we have \( c_l R \omega_{l_1}(\omega_g(a_1)) = 0 \) for all \( 1 \leq i \leq n \). As a result, the Eq. (2.1) becomes \( (c_l l_1 + cl_2 + \cdots + c_n l_n)gr(a_2 h_2 + \cdots + a_m h_m) = 0 \). We can repeat this process to show that \( c_l \omega_{l_1}(\omega_g(r a_1)) = 0 \) for all \( g \in M \) and all \( i, j \). This shows that \( c_l R \omega_{l_1}(\omega_g(a_1)) = 0 \). Consequently, we can see that \( a_j R \omega_{l_1}(\omega_g(c_i)) = 0 \) for all \( g \in M \), \( 1 \leq j \leq m \), and \( 1 \leq i \leq n \). Therefore, \( R \) is strongly \( CM \)-reflexive.

The following example demonstrates the existence of a ring \( R \) over a field \( F \) that is not strongly \( CM \)-reflexive.

**Example 1.** Let \( M \) be a monoid with at least two elements, and let \( S = M_2(F) \) be the matrix ring over a field \( F \) with a twisting map \( f : M \times M \rightarrow U(R) \), then \( S \) is not strongly \( CM \)-reflexive.

**Solution.** Take \( e \neq h \in M \), we define \( \omega : M \rightarrow Aut(S) \) by

\[
\omega_h \left( \begin{array}{cc} a & d \\ 0 & c \end{array} \right) = \left( \begin{array}{cc} a & -d \\ 0 & c \end{array} \right).
\]

If the twisting map \( f \) is trivial (i.e., \( f(x, y) = 1 \) for all \( x, y \in M \)), then the ring \( S \) is not strongly \( CM \)-reflexive. To see this, consider \( \phi = E_{12}e + E_{11}h \) and \( \psi = (E_{11} + E_{12})h \in S \ast M \). For \( \varphi = (E_{11} + E_{22})h \in S \ast M \), we can easily verify that \( \varphi \psi \neq 0 \). However, we have \( \psi \varphi \neq 0 \), which implies that \( S \) is not strongly \( CM \)-reflexive.

\( \Box \)
A ring $R$ is categorized as a right $PP$-ring or left $PP$-ring if the right or left annihilator of an element in $R$, respectively, is generated by an idempotent. A (quasi-) Baer ring is one where the right annihilator of every nonempty subset or every right ideal of $R$ is generated by an idempotent. Principally quasi-Baer rings, introduced by Birkenmeier et al. [13], extend the concept of quasi-Baer rings. A ring $R$ is referred to as left principally quasi-Baer or simply left $p.q.$-Baer if the left annihilator of a principal left ideal in $R$ is generated by an idempotent. It is important to note that biregular rings and quasi-Baer rings are examples of left $p.q.$-Baer rings. For more information and examples of left $p.q.$-Baer rings, see Birkenmeier et al. ([13], [14]) and Liu [15]. Since right $PP$-rings and left $p.q.$-Baer rings both fall under the category of left $APP$ [16], the following results can be deduced.

**Theorem 2.** Suppose $R$ is a reduced ring, $M$ is a strictly totally ordered monoid with a twisting map $f : M \times M \to U(R)$ and an action map $\omega : M \to \text{Aut}(R)$ that is compatible with the multiplication in $M$. If $R$ is a left $p.q.$-Baer ring, then $R$ is strongly $CM$-reflexive.

**Proof.** The proof is a variant of the proof given in Proposition 2.9 [17]. Let $\phi = c_1l_1 + c_2l_2 + \cdots + c_ml_m, \psi = a_1h_1 + a_2h_2 + \cdots + a_mh_m \in R \ast M$ satisfy $\phi(R \ast M)\psi = 0$. Since $M$ is a strictly totally ordered monoid, we can assume that $l_i \leq l_j$ and $h_i \leq h_j$ whenever $i < j$. Now, we claim $c_i\omega_l(\omega_g(Ra_j)) = 0$ for all $i, j$. Let $r$ be an element of $R$. Then, we have $\phi(re)\psi = 0$ since $\phi(R \ast M)\psi = 0$. Thus, we have

$$0 = \phi(re)\psi = c_1rf(l_1, e)a_1f(l_1, h_1)l_1h_1 + \cdots + [c_nrf(l_n, e)a_{m-2}f(l_n, h_{m-2})]n\ell h_m$$

It follows that $c_nrf(l_n, e)a_{m}f(l_n, h_m) = 0$ since $l_nh_m$ is of highest order in the $l_ih_j$s. Hence $c_nrf(l_n, e)a_{m} = 0$. This shows that $c_n \in \ell_R(Rf(l_n, e)a_{m}) = \ell_R(Ra_m)$. Hence, $\ell_R(Ra_m) = Re_m$ for some idempotent $e_m$ by hypothesis. Replacing $r$ by $re_m$ in Eq. (2.2) we obtain

$$0 = c_1re_mf(l_1, e)a_1f(l_1, h_1)l_1h_1 + \cdots + [c_nre_mf(l_n, e)a_{m-2}f(l_n, h_{m-2})]n\ell h_m$$

So $c_nre_mf(l_n, e)a_{m-1}f(l_n, h_{m-1}) = 0$, because $l_nh_{m-1}$ is of highest order in $\{l_ih_j | 1 \leq i, 1 \leq j \leq m \}$ \{l_nh_m, l_1h_m\}. Hence $c_nre_mf(l_n, e)a_{m-1} = 0$. Since $Re_m$ is an ideal of $R$ and $e_m \in Re_m$, we have $e_m r \in Re_m$ and thus $e_m r = e_m re_m$ for all $r \in R$. On the other hand, we also have $c_n = c_ne_m$ since $c_n \in \ell_R(Ra_m) = Re_m$. Hence $c_nrf(l_n, e)a_{m-1} = c_ne_mre_mf(l_n, e)a_{m-1} = c_nre_mf(l_n, e)a_{m-1} = 0$. This implies that $c_n \in \ell_R(Ra_m + Ra_{m-1})$, and hence $\ell_R(Ra_m + Ra_{m-1}) = Re_{m-1}$ for some idempotent $e_{m-1} \in R$ since $R$ is a left $p.q.$-Baer ring. Replacing $r$ by $re_{m-1}$ in equation (2.3) we obtain $c_nre_{m-1}f(l_n, e)a_{m-2}f(l_n, h_{m-2}) = 0$ in the same way as above. This shows that $c_n \in \ell_R(Ra_m + Ra_{m-1} + Ra_{m-2})$. Continuing this process we obtain $c_nRa_t = 0$ for all $t = 1, 2, \ldots, m$. So, we have $c_n = c_ne_m$ since $c_n \in \ell_R(Ra_{m}) = Re_{m}$. Hence $c_nrf(l_n, e)a_{m-1} = c_nre_mf(l_n, e)a_{m-1} = c_nre_mf(l_n, e)a_{m-1} = 0$. Using induction on $m + n$, we obtain $c_i\omega_l(\omega_g(Ra_j)) = 0$ for all $i, j$. So it is easy to see that $a_i\omega_l(\omega_g(Ra_j)) = 0$ by a reducedness. Therefore, $R$ is strongly $CM$-reflexive.

If $N$ is an ideal of the monoid $M$ with twisting $f : M \times M \to U(R)$ and action
\( \omega : M \rightarrow Aut(R) \), then the restrictions \( f|_{N \times N} : N \times N \rightarrow U(R) \) and \( \omega|_{N} : N \rightarrow Aut(R) \) are induced twisting and action.

**Proposition 1.** Let \( R \) be an \( M \)-compatible ring and \( M \) be a commutative, cancellative monoid and \( N \) an ideal of \( M \) with a center element \( \lambda \). If \( R \) is strongly \( CN \)-reflexive, then \( R \) is strongly \( CM \)-reflexive.

**Proof.** Let \( \phi = \sum_{i=1}^{n} c_{i}l_{i}, \psi = \sum_{j=1}^{m} a_{j}h_{j} \in R \ast M \) satisfying \( \phi \varphi \psi = 0 \) for any \( \varphi = \sum_{r=1}^{\nu} \ell_{r}g_{r} \in R \ast M \). Since \( \lambda \in N \) is a center element, this implies that
\[
\lambda l_{1}, \lambda l_{2}, \ldots, \lambda l_{n}, \lambda g_{1} \lambda, \lambda g_{2} \lambda, \ldots, \lambda g_{\nu} \lambda, h_{1} \lambda, h_{2} \lambda, \ldots, h_{\nu} \lambda \in N,
\]
such that \( \lambda l_{i} \neq \lambda l_{j}, \lambda g_{i} \lambda \neq \lambda g_{j} \lambda \) and \( h_{i} \lambda \neq h_{j} \lambda \) for all \( i \neq j \). Then, we have
\[
\phi_{1} \varphi_{1} \psi_{1} = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{r=1}^{\nu} (c_{i} \omega_{1}(\ell_{r} \omega_{1}(a_{j}))) f(l_{i} \lambda, h_{j})(\lambda^{2} l_{r} g_{r} h_{j} \lambda^{2}) = 0.
\]

Since \( \varphi, \phi \) and \( \psi \) are nonzero in \( R \ast M \), so \( \phi_{1} \) and \( \psi_{1} \) are nonzero elements in \( (R \ast M)[N] \). Moreover, from \( \phi \varphi \psi = 0 \) and \( \omega \) compatible automorphism, \( \lambda \) a center element of \( N \) one can easily obtain that \( \phi_{1} \varphi_{1} \psi_{1} = 0 \) for any \( \varphi_{1} \in (R \ast M)[N] \). Since \( R \) is strongly \( CN \)-reflexive. Then, \( c_{i} \omega_{1}(\omega_{1}(R a_{j})) f(l_{i} \lambda, h_{j}) (l_{i} \lambda h_{j}) = 0 \). So \( c_{i} \omega_{1}(\omega_{1}(R a_{j})) = 0 \). By a compatible automorphism, we have \( a_{j} \omega_{1}(\omega_{1}(R c_{i})) = 0 \). Therefore, \( R \) is strongly \( CM \)-reflexive. \( \square \)

**Corollary 1.** [4, Proposition 3.1] Let \( M \) be a cancellative monoid and \( N \) an ideal of \( M \). If \( R \) is strongly \( N \)-reflexive, then \( R \) is strongly \( M \)-reflexive.

Suppose \( I \) is an ideal of \( R \) and \( \omega : M \rightarrow Aut(R) \) is a monoid homomorphism. We define \( \bar{\omega} : M \rightarrow Aut(R/I) \) as \( \bar{\omega}(d + I) = \omega(d) + I \), where \( d \in R \) and \( g \in M \). It can be shown that \( \bar{\omega} \) is a monoid homomorphism. Additionally, the twisting map \( f : M \times M \rightarrow U(R) \) induces a twisting map \( \tilde{f} : M \times M \rightarrow U(R/I) \) given by \( \tilde{f}(x, y) = f(x, y) + I \). Furthermore, for every \( \phi = \sum_{i=1}^{n} c_{i}l_{i} \in R \ast M \), we denote \( \bar{\phi} = \sum_{i=1}^{n} \bar{c}_{i} l_{i} \in (R/I) \ast M \), where \( \bar{c}_{i} = c_{i} + I \) for \( 1 \leq i \leq n \). It can be easily verified that the mapping \( \bar{\theta} : R \times M \rightarrow (R/I) \times M \) defined as \( \bar{\theta}(\bar{\phi}) = \bar{\phi} \) is a ring homomorphism. In a proof presented [4], it was shown that when \( I \) is a reduced ideal of \( R \) and \( R/I \) is strongly \( M \)-reflexive, then \( R \) is strongly \( M \)-reflexive. Similarly, we can establish the following result.

**Theorem 3.** Let \( M \) be a u.p.-monoid and \( I \) an ideal of \( R \) with twisting \( f : M \times M \rightarrow U(R) \) and action \( \omega : M \rightarrow Aut(R) \). If \( I \) is a reduced and \( R/I \) is strongly \( CM \)-reflexive, then \( R \) is strongly \( CM \)-reflexive.

**Proof.** Let \( \phi = \sum_{i=1}^{n} c_{i}l_{i}, \psi = \sum_{j=1}^{m} a_{j}h_{j} \in R \ast M \) satisfying \( \phi(R \ast M) \psi = 0 \). We will show that \( c_{i} \omega_{1}(\omega_{1}(R a_{j})) = 0 \) for any \( i \) and \( j \).
Note that in \((R/I) \ast M\), \(\bar{\phi} = \sum_{i=1}^{n} \bar{c}_i l_i, \bar{\psi} = \sum_{j=1}^{m} \bar{a}_j h_j \in (R/I) \ast M\), we have

\[
\bar{\phi} = \sum_{i=1}^{n} \bar{c}_i l_i + \sum_{j=1}^{m} \bar{a}_j h_j = \sum_{i=1}^{n} \bar{c}_i l_i + \sum_{j=1}^{m} \bar{a}_j h_j
\]

Thus, we have \(c_i \omega_l (\omega_g(Ra_j)) f(l_i, h_j)(l_i h_j) \subseteq I\) for all \(i\) and \(j\) with \(1 \leq i \leq n\) and \(1 \leq j \leq m\) since \(R/I\) is strongly CM-reflexive.

By induction on both \(n\) and \(m\), considering every \(g\) in \(M\), and for \(1 \leq i \leq n\) and \(1 \leq j \leq m\). If we take \(n = 1\). Then \((c_1 l_1)(R \ast M)(a_1h_1 + a_2h_2 + \cdots + a_m h_m) = 0\). Thus, \((c_1 l_1)(r g)(a_1h_1) + (c_1 l_1)(r g)(a_2h_2) + \cdots + (c_1 l_1)(r g)(a_m h_m) = 0\). Similar to the previous case, we can deduce that \(c_1 \omega_l (\omega_g(Ra_1)) f(l_1, h_1)(l_1 h_1) = 0\), which implies that \(c_1 \omega_l (\omega_g(Ra_1)) = 0\).

Now suppose that \(n \geq 2\) and \(m \geq 2\). Since \(M\) is a up-monoid, there exist \(i, j\) with \(1 \leq i \leq n\) and \(1 \leq j \leq m\) such that \(l_i g h_j\) is uniquely presented by considering two subsets \(K = \{l_1 g, l_2 g, \ldots, l_n g\}\) and \(H = \{h_1 h_2, h_3, \ldots, h_m\}\) of the monoid \(M\).

Without loss of generality, we may assume that \(i = 1\) and \(j = 1\). We can deduce that \(c_1 \omega_l (\omega_g(Ra_1)) f(l_1, h_1)(l_1 h_1) = 0\), which implies that \(c_1 \omega_l (\omega_g(Ra_1)) = 0\). Since \(\omega_g\) and \(\omega_l\) are automorphisms of \(R\), we have \(a_1 \omega_l (Ra_1) = 0\). Let \(b = c_g r a_q\), where \(r \in R, 1 \leq k \leq n, 1 \leq q \leq m\). Then \(b \in I\).

By induction, we have \(a_1 b c_1 = 0\). Thus, \(a_1 b c_1 = 0\) and \(b \in I\), which implies that \(a_1 b c_1 (Ra_1) = 0\). Note that \(bc_1 \omega_l (Ra_1) \subseteq I\).

Thus, \(bc_1 \omega_l (Ra_1) = 0\). Now we have

\[
(bc_1 l_1 + bc_2 l_2 + \cdots + bc_n l_n)(R \ast M)(a_1h_1 + a_2h_2 + \cdots + a_m h_m) = 0
\]

By applying the induction hypothesis, it follows that \(bc_1 \omega_l (\omega_g(Ra_j)) = 0\) for all \(1 \leq i \leq n\) and \(2 \leq j \leq m\). Thus, we have \(c_1 \omega_l (\omega_g(Ra_j)) = 0\) for all \(i, j\) and all \(r \in R\). Particularly, we have \(bc_1 \omega_l (\omega_g(Ra_q)) = 0\) and so \(b^2 = 0\). Thus \(b = 0\). This shows that \(c_k \omega_l (\omega_g(Ra_q)) = 0\) for any \(1 \leq k \leq n\) and \(1 \leq q \leq m\). Consequently, we can see that \(a_1 \omega_l (\omega_g(Rc_1)) = 0\) for all \(g \in M, 1 \leq j \leq m, 1 \leq i \leq n\). Therefore, \(R\) is strongly CM-reflexive.

The notion of complete \(M\)-compatibility is important in the following result [18].

**Corollary 2.** Assuming \(R\) is a ring that is completely \(M\)-compatible, where \(M\) is a monoid with twisted \(f : M \times M \to U(R)\) and action \(\omega : M \to \text{Aut}(R)\), and \(I\) is an ideal of \(R\) such that \(I\) is reduced and \(R/I\) is \(CM\)-quasi-Armendariz, then \(R\) is strongly \(CM\)-reflexive.

**Proof.** As \(CM\)-quasi-Armendariz rings are strongly \(CM\)-reflexive, the result can be
obtained from Theorem 3.

**Corollary 3.** Suppose that \( R \) is a completely \( M \)-compatible ring, where \( M \) is a monoid with twisting \( f : M \times M \rightarrow U(R) \) and action \( \omega : M \rightarrow \text{Aut}(R) \). Let \( I \) be an ideal of \( R \) such that \( I \) is reduced and \( R/I \) is CM-Armendariz. Then, \( R \) is strongly CM-reflexive.

**Proof.** Since CM-Armendariz is a CM-quasi-Armendariz, the result can be derived from Corollary 2.

**Proposition 2.** Assuming \( R \) is a ring that is both \( M \)-compatible and CM-quasi-Armendariz, where \( M \) is a monoid with twisting \( f : M \times M \rightarrow U(R) \) and action \( \omega : M \rightarrow \text{Aut}(R) \), then \( R \) is strongly CM-reflexive if and only if \( R \ast M \) is strongly CM-reflexive.

**Proof.** To prove a necessary condition is sufficient. Let \( \phi = \sum_{i=1}^{n} c_i l_i, \psi = \sum_{j=1}^{m} a_j h_j \in R \ast M \) satisfying \( \phi(R \ast M)\psi = 0 \). Since \( R \) is CM-quasi-Armendariz, we have \( c_i \omega_l(\omega_g(R a_j))f(l_i,h_j)(h_j l_i) = 0 \) for all \( i, j \). This implies that \( c_i \omega_l(\omega_g(R a_j)) = 0 \) for all \( i, j \) since \( R \) is \( M \)-compatible. Because \( R \) is a reflexive ring, \( a_j R c_i = 0 \). Then, \( a_j \omega_h(\omega_g(R c_i)) \neq 0 \) for all \( i, j \), and hence for any \( r \in R, g \in M \), we have

\[
\psi(R \ast M)\phi = \sum_{j=1}^{m} \sum_{i=1}^{n} a_j \omega_h(\omega_g(r c_i))f(h_j, l_i)(h_j l_i) = 0.
\]

Thus, \( a_j \omega_h(\omega_g(r c_i)) = 0 \) since \( R \) is \( M \)-compatible and CM-quasi-Armendariz. Therefore, \( R \) is strongly CM-reflexive.

Every left APP-ring is quasi-Armendariz, but not conversely [19, 20].

**Proposition 3.** Let \( M \) be a strictly totally ordered monoid with twisting \( f : M \times M \rightarrow U(R) \) and action \( \omega : M \rightarrow \text{Aut}(R) \). Let \( R \) be an \( M \)-compatible left APP-ring. Then \( R \) is strongly CM-reflexive if and only if \( R \ast M \) is strongly CM-reflexive.

**Proof.** If \( R \) is a left APP-ring, then it is \( M \)-quasi-Armendariz [21]. Therefore, the result follows from Proposition 2.

**Corollary 4.** Let \( R \) be a ring, \( M \) be a monoid with twisting \( f : M \times M \rightarrow U(R) \) and action \( \omega : M \rightarrow \text{Aut}(R) \). If \( R \) is reduced, then \( R \) is strongly CM-reflexive.

**Proof.** Since \( R \) is reduced, it is quasi-Armendariz. Therefore, the result can be derived from Proposition 2.

3. Some results on ring extensions of Crossed product type

Let \( \Delta \) be a multiplicative monoid consisting of central regular elements of \( R \). Then, the set \( \Delta^{-1} R := \{ u^{-1}c | u \in \Delta, c \in R \} \) forms a ring. Suppose \( \omega : M \rightarrow \text{Aut}(R) \) is a monoid homomorphism such that \( \omega_h(\Delta) \subseteq \Delta \) for every \( h \in M \). Then, \( \omega \) can be extended to \( \bar{\omega} : M \rightarrow \text{Aut}(\Delta^{-1} R) \) defined by \( \bar{\omega}_h(u^{-1}c) = \omega_h(u)^{-1}\omega_h(c) \). If \( f : M \times M \rightarrow U(R) \) is a twisted function, then it can be viewed as a twisted function from \( M \times M \) to \( U(\Delta^{-1} R) \) by noting that \( U(R) \subseteq U(\Delta^{-1} R) \).
Theorem 4. Assuming $R$ is an $M$-compatible ring, where $M$ is a cancellative monoid with twisting $f : M \times M \to U(R)$ and action $\omega : M \to \text{Aut}(R)$, then $R$ is strongly $CM$-reflexive if and only if $\Delta^{-1}R$ is strongly $CM$-reflexive, where $\Delta$ is the multiplicative subset of $R$ consisting of all elements that are not zero divisors modulo $M$.

Proof. It is enough showing necessary. Assume that $R$ is strongly $CM$-reflexive. Let $\phi = \sum_{i=1}^nu_i^{-1}c_i l_i, \psi = \sum_{j=1}^mv_j^{-1}a_j h_j$ be elements in $\Delta^{-1}R*M$ satisfying $\phi \varphi \psi = 0$, where $\varphi = \sum_{k=1}^q \lambda_k^{-1}b_k \ell_k$ is any nonzero element in $\Delta^{-1}R*M$. Then, we have $\alpha = (u_n u_{n-1} \ldots u_1) \phi, \theta = (\lambda_q \lambda_{q-1} \ldots \lambda_1) \varphi, \beta = (v_m v_{m-1} \ldots v_1) \psi$ are in $R*M$. Since $R$ is strongly $CM$-reflexive and $\alpha \beta = 0$ we have

$$(u_n u_{n-1} \ldots u_3 u_2^{-1} c_1) \omega_l (\omega_g (b (v_m v_{m-1} \ldots v_1 v_2^{-1} a_1))) f(l, h_j)_j (l, h_j) (v_j u_i)^{-1} = 0$$

for all $i, j$ and $b \in R$. It follows that $c_i \omega_l (\omega_g (R a_j)) f(l, h_j) (l, h_j) = 0$ for any $g \in M$, because $\Delta$ is a multiplicative monoid consisting of central regular elements of $R$ and all $u_i, v_j, \lambda_k \in \Delta$. Hence, $(u_i^{-1} c_i \omega_l (\omega_g (R v_i^{-1} a_1))) = c_i \omega_l (\omega_g (R a_j)) (\omega_l (v_j) u_i)^{-1} = 0$ for all $i, j$ and $\omega$ is automorphism. Therefore, $\Delta^{-1}R$ is strongly $CM$-reflexive. \qed

The following statement describes how the strongly $CM$-reflexive property of a ring $R$ is related to the property of its subrings, which are created by a central idempotent.

Proposition 4. The following conditions are equivalent for a ring $R$, a monoid $M$ with twisting $f : M \times M \to U(R)$, an action $\omega : M \to \text{Aut}(R)$, and a central idempotent $e$ of $R$ such that $\omega_g(e) = e$:

1. $R$ is strongly $CM$-reflexive.
2. $e R$ and $(1 - e) R$ are strongly $CM$-reflexive.

Proof. $(1) \Rightarrow (2)$. It is easy.

$(2) \Rightarrow (1)$. Assume that both $e R$ and $(1 - e) R$ are strongly $CM$-reflexive. Let $\phi = \sum_{i=1}^nc_i l_i, \psi = \sum_{j=1}^m a_j h_j \in R*M$ satisfying $\phi (R*M) \psi = 0$. Let

$\phi_1 = \sum_{i=1}^n e c_i l_i, \psi_1 = \sum_{j=1}^m e a_j h_j, \phi_2 = \sum_{i=1}^m (1 - e) c_i l_i, \psi_2 = \sum_{j=1}^m (1 - e) a_j h_j$.

clear that $\phi_1, \psi_1 \in (e R) * M$ and $\phi_2, \psi_2 \in ((1 - e) R) * M$. Since $e$ is a central idempotent of $R$ such that $\omega_g(e) = e$ for each $g \in M$ and for any $r \in R$ we have

$\phi_1 ((e R) * M) \psi_1 = e c_1 (e r) \omega_l (\omega_g (c a_1)) f(l, h_1)_1 l_1 h_1 + \cdots + c_n (e r) \omega_l (\omega_g (c a_m)) f(l, h_m)_1 l_1 h_m$

$= e c_1 (e r) \omega_l (\omega_g (e c_1 a_1)) f(l, h_1)_1 l_1 h_1 + \cdots + c_n (e r) \omega_l (\omega_g (e c_n a_m)) f(l, h_m)_1 l_1 h_m$

$= e c_1 (e r) \omega_l (\omega_g (e c_1 a_1)) f(l, h_1)_1 l_1 h_1 + \cdots + c_n (e r) \omega_l (\omega_g (e c_n a_m)) f(l, h_m)_1 l_1 h_m$

$= e c_1 (e r) \omega_l (\omega_g (e c_1 a_1)) f(l, h_1)_1 l_1 h_1 + \cdots + c_n (e r) \omega_l (\omega_g (e c_n a_m)) f(l, h_m)_1 l_1 h_m$.

$= e c_1 (e r) \omega_l (\omega_g (e c_1 a_1)) f(l, h_1)_1 l_1 h_1 + \cdots + c_n (e r) \omega_l (\omega_g (e c_n a_m)) f(l, h_m)_1 l_1 h_m$.

$= e c_1 (e r) \omega_l (\omega_g (e c_1 a_1)) f(l, h_1)_1 l_1 h_1 + \cdots + c_n (e r) \omega_l (\omega_g (e c_n a_m)) f(l, h_m)_1 l_1 h_m$.
Proposition 5. Let $R$ be a ring and $M$ is a strictly ordered monoid with a twisting $f : M \times M \rightarrow U(R)$ and an action $\omega : M \rightarrow \text{Aut}(R)$. Assume that $R$ is $CM$-quasi-Armendariz. Let $e$ be a nonzero idempotent in $R$ such that $\omega_g(e) = e$ for all $g \in M$. Then, the subring $eRe$ is strongly $CM$-reflexive.

Proof. The proof is a variant of the proof given in Proposition 2.9 [17]. Let $\phi = c_1l_1 + c_2l_2 + \cdots + c_nl_n$ and $\psi = a_1h_1 + a_2h_2 + \cdots + a_nh_m \in (eRe) * M$ satisfy $\phi((eRe) * M)\psi = 0$. Since $M$ is a strictly totally ordered monoid, we can assume that $l_i \leq l_j$ and $h_i \leq h_j$ whenever $i < j$. Since $R$ is $CM$-quasi-Armendariz, then so is $eRe$. Thus, we have $c_i\omega_i((eRe)a_j)f(l_i,h_j)(l_ii) = 0$ for all $i,j$. This implies that $c_i\omega_i((eRe)a_j) = 0$ for all $i,j$ since $R$ is $M$-compatible and $\omega$ is an automorphism. Therefore, by Proposition 2, $eRe$ is strongly $CM$-reflexive.

Corollary 5. [20, Proposition 3.7] Let $e \in R$ be an idempotent. If $R$ is a left APP, then $eRe$ is a left APP-ring.

Corollary 6. [22, Corollary 3.19] Let $M$ be a strictly totally ordered monoid and $\omega : M \rightarrow \text{End}(R)$ a monoid homomorphism. Assume that $e$ be an idempotent. If $R$ is left APP, then $eRe$ is $(M, \omega)$-quasi-Armendariz.

Proposition 6. Let $M$ be a strictly totally ordered monoid with twisting $f : M \times M \rightarrow U(R)$ and action $\omega : M \rightarrow \text{Aut}(R)$. Assume that $e$ be an idempotent. If $R$ is a left APP, then $eRe$ is strongly $CM$-reflexive.

Proof. By Corollary 5, $eRe$ is a left APP. So, $eRe$ is $(M, \omega)$-quasi-Armendariz by Corollary 6. Thus, the result follows from Proposition 5.

Let $I$ be an index set and $R_i$ be a ring for each $i \in I$. Let $M$ be a strictly ordered monoid and $\omega^i : M \rightarrow \text{End}(R_i)$ a monoid homomorphism. Then the mapping $\omega : M \rightarrow \text{End}(\prod_{i \in I} R_i)$ is a monoid homomorphism given by $\omega_g(\{r_i\}_{i \in I}) = \{(\omega^i)_g(r_i)\}_{i \in I}$ for all $g \in M$. 
Proposition 7. Let $R_i$ be a ring for each $i$ in a finite index set $I$, and let $M$ be a monoid with a twisting $f : M \times M \to \bigcup_{i \in I} U(R_i)$ and an action $\omega^i : M \to \text{Aut}(R_i)$ on each $R_i$. Suppose that each $R_i$ is strongly $CM$-reflexive. Then, the direct product $R = \prod_{i \in I} R_i$, equipped with the product action $\omega = \prod_{i \in I} \omega^i$, is strongly $CM$-reflexive.

Proof. Let $R = \prod_{i \in I} R_i$ be the direct product of rings $(R_i)_{i \in I}$ and $R_i$ is strongly $CM$-reflexive for each $i \in I$. Denote the projection $R \to R_i$ as $P_i$. Suppose that $\phi, \psi \in R * M$ are such that $\phi(R * M) \psi = 0$. Set $\phi_i = \prod_{i \in I} \phi, \psi_i = \prod_{i \in I} \psi$ and $\varphi_i = \prod_{i \in I} \phi_i$. Then $\phi_i, \psi_i \in R_i * M$. For any $u, v \in M$, assume $\phi(u) = (c_i^u)_{i \in I}, \psi(v) = (a_i^v)_{i \in I}$. Now, for any $r \in R$ and any $g \in M$,

$$\phi(r * M) \psi = \sum_{(u,v) \in X_\phi(\phi, \psi)} \phi(u) \omega_u(\omega_g(r \psi(v))) f(u_m, v_n) u_m^i v_n^j$$

for each $i \in I$, where

$$\phi = \sum_{k=1}^m \alpha_k j_i, \psi = \sum_{k=1}^n \gamma_k h_k$$

be elements in $Q * M$ satisfying $\phi \varphi \psi = 0$, where $\varphi = \sum_{j=1}^n \beta_j g_j$ is any nonzero element in $Q * M$. By Proposition 2.1.6 [23], we may assume that $\alpha_k = a_i u^{-1}, \beta_j = b_j v^{-1}$ and $\gamma_k = c_k w^{-1}$ with regular $u, v, w \in R$. Also, Proposition 2.1.6 [23], for each $j$ and $k$, there exist $d_j, e_k \in R$ and regular $s, t \in R$ such that $u^{-1}b_j = d_j s^{-1}$ and $e_k w^{-1} = c_k r_t$. Therefore, $\psi(R * M) \phi = 0.$

Since $\phi(R * M) \psi = 0$, we have $\phi_i(R_i * M) \psi_i = 0.$

Now it follows $\phi_i(u) \omega_u^i(\omega_g(r \psi_i(v))) = 0$ for any $r \in R$, any $u, v, g \in M$ and any $i \in I$, since $R_i$ is strongly $CM$-reflexive. Hence, for any $u, v \in M$,

$$\psi(v) \omega_v(\omega_g(r \phi(u))) = (\psi_i(v) \omega_v^i(\omega_g(r \phi_i(u))))_{i \in I} = 0$$

since $I$ is finite. Thus, $\psi(v) \omega_v(\omega_g(r \phi(u))) = 0$ by the compatibility of $\omega$. Therefore, $\psi(R * M) \phi = 0.$ This means that $R$ is strongly $CM$-reflexive.

\[ \square \]

Theorem 5. Assuming that $R$ is an $M$-compatible ring and $M$ is a cancellative monoid with a twisting map $f : M \times M \to U(R)$ and an action map $\omega : M \to \text{Aut}(R)$, and considering $R$ as a right Ore ring with the classical right quotient ring $Q$, the $R$ is strongly $CM$-reflexive if and only if $Q$ is strongly $CM$-reflexive.

Proof. It is enough showing necessary. Assume that $R$ is strongly $CM$-reflexive. Let $\phi = \sum_{i=1}^n \alpha_i j_i, \psi = \sum_{k=1}^n \gamma_k h_k$ be elements in $Q * M$ satisfying $\phi \varphi \psi = 0$, where $\varphi = \sum_{j=1}^n \beta_j g_j$ is any nonzero element in $Q * M$. By Proposition 2.1.6 [23], we may assume that $\alpha_i = a_i u^{-1}, \beta_j = b_j v^{-1}$ and $\gamma_k = c_k w^{-1}$ with regular $u, v, w \in R$. Also, Proposition 2.1.6 [23], for each $j$ and $k$, there exist $d_j, e_k \in R$ and regular $s, t \in R$ such that $u^{-1}b_j = d_j s^{-1}$ and $e_k w^{-1} = c_k r_t$. Therefore, $\psi(R * M) \phi = 0.$

Since $\phi(R * M) \psi = 0$, we have $\phi_i(R_i * M) \psi_i = 0.$

Now it follows $\phi_i(u) \omega_u^i(\omega_g(r \psi_i(v))) = 0$ for any $r \in R$, any $u, v, g \in M$ and any $i \in I$, since $R_i$ is strongly $CM$-reflexive. Hence, for any $u, v \in M$,

$$\psi(v) \omega_v(\omega_g(r \phi(u))) = (\psi_i(v) \omega_v^i(\omega_g(r \phi_i(u))))_{i \in I} = 0$$

since $I$ is finite. Thus, $\psi(v) \omega_v(\omega_g(r \phi(u))) = 0$ by the compatibility of $\omega$. Therefore, $\psi(R * M) \phi = 0.$ This means that $R$ is strongly $CM$-reflexive.
and $(vs)^{-1}c_k = e_k t^{-1}$. Suppose $\phi_1 = \sum_{i=1}^m a_i l_i, \varphi_1 = \sum_{j=1}^n b_j g_j, \varphi_2 = \sum_{j=1}^n d_j g_j, \psi_1 = \sum_{k=1}^p c_k h_k, \psi_2 = \sum_{k=1}^p c_k h_k \in R \ast M$. Since $M$ is a cancellative monoid by Lemma 1, thus, $g_i h_1 \neq g_i h_1$ for $g_i \neq g_j$. Then, we have $0 = \varphi_1 \psi = \sum_{i=1}^m \sum_{k=1}^p (a_i u^{-1}) \omega_i (\omega (R c_k u^{-1})) f(l_i, h_k)(l_i h_k) = \omega_i (\omega (R c_k u^{-1})) f(l_i, h_k)(l_i h_k)(\omega_i(t) w^{-1}) = 0 = \phi_1 \varphi_2 \psi_2 (w t)^{-1}$. Therefore, $\phi_1 \varphi_2 \psi_2 = 0$. Since $R$ is strongly CM-reflexive, then $\psi_2 \varphi_1 \phi_1 = 0$. This implies that $\psi_1 \varphi_1 \phi_1 = 0$ since $u^{-1} b_j = d_j s^{-1}$, then $s \psi_2 \varphi_1 \phi_1 = 0$ and $(vs) \psi_2 \varphi_1 \phi_1 = 0$, so $\psi_1 \varphi_1 \phi_1 = 0$ since $(vs)^{-1}c_k = e_k t^{-1}$. Using Proposition 2.1.16 [23] again, for each $i, j$ there exist $\varphi_i, \psi_j \in R \ast M$ and regular element $q, p \in R$ such that $w^{-1} b_j = \phi_j q^{-1}$ and $(v q)^{-1} a_i = \varphi_i p^{-1}$. Let $\psi_2 = \sum_{i=1}^m \sum_{j=1}^n a_i l_i, \varphi_3 = \sum_{j=1}^n \phi_j g_j$. Then, 

$$q \psi_1 \varphi_1 \phi_1 = \sum_{i=1}^m \sum_{k=1}^p q(c_k u^{-1}) \omega_{h_k} (\omega (Ra_i u^{-1})) f(h_k, l_i)(h_k l_i) = \sum_{i=1}^m \sum_{k=1}^p q(c_k) \omega_{h_k} (\omega (Ra_i)) \times \omega_{h_k}(u) w^{-1} = 0 \text{ since } \psi_1 \varphi_1 \phi_1 = 0. \text{ Thus, for all } k, i \text{ we have } c_k \omega_{h_k} (\omega (Ra_i)) = 0, \text{ and it follows that } \psi_1 \varphi_2 q \psi_2 = \sum_{i=1}^m \sum_{k=1}^p w a_i \omega_{l_i} (\omega (R e_k)) = 0 \text{ since } w^{-1} b_j = \phi_j q^{-1}. \text{ Therefore, } \psi_1 \varphi_3 \phi_1 w = 0 \text{ since } R \text{ is strongly CM-reflexive, and so } \psi_1 \varphi_3 \phi_1 = 0. \text{ Therefore, } \psi_1 \varphi_3 \phi_1 p = \sum_{k=1}^p \sum_{i=1}^m c_k \omega_{h_k} (\omega (Ra_i)) f(h_k, l_i)(h_k l_i) p = \psi_1 \varphi_3 \phi_1 (v q) = \sum_{k=1}^p \sum_{i=1}^m c_k \omega_{h_k} (\omega (R d_j))(pv) = 0, \text{ and thus } \psi_1 \varphi_3 \phi_2 = 0. \text{ Therefore, } 

$$

$$\psi \varphi = \sum_{k=1}^p \sum_{i=1}^m (c_k u^{-1}) \omega_{h_k} (\omega (Ra_i u^{-1})) = \sum_{k=1}^p \sum_{i=1}^m c_k \omega_{h_k} (\omega (Ra_i)) (\omega_{h_k} (u) w^{-1}) = 0. \text{ Therefore, } Q \text{ is strongly CM-reflexive. } \square$$

References


[22] E. Ali and A. Elshokry. A note on \((s, \omega)\)-quasi-armendariz rings. Accepted.