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Some Properties of Zero Forcing Hop Dominating Sets in a Graph

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Abstract. In this paper, we initiate the study of a zero forcing hop domination in a graph. We establish some properties of this parameter and we determine its connections with other known parameters in graph theory. Moreover, we obtain some exact values or bounds of the parameter on the generalized graph, some families of graphs, and graphs under some operations via characterizations.

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Key Words and Phrases: Zero forcing hop dominating set, zero forcing hop domination number, zero forcing pointwise non-domination

1. Introduction

Hop domination was introduced by Natarajan et al. in [12]. This parameter is incomparable with the standard domination and just like domination, hop domination has many applications in different fields and in networks. A subset S of a vertex set $V(G)$ is called a *hop dominating* in G if $N_G^2[S] = V(G)$, where $N_G^2[S]$ is the closed hop neighborhood of S in G. The minimum cardinality among all hop dominating sets in G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. This concept had been studied on different types of graphs and graph theorists found some interesting results (see [1, 2, 10]). Since then, several researchers had studied this concepts and they had extended this parameter by

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introducing variants, that is, imposing additional properties or conditions on the standard hop domination (see [3–9, 11]).

In this paper, we introduce and investigate zero forcing hop domination in a graph. Let G be a graph. A subset Z of a vertex-set $V(G)$ of G is said to be a *zero forcing hop dominating* if Z is both zero forcing and hop dominating in G. The minimum cardinality among all zero forcing hop dominating sets in G, denoted by $\gamma_{zh}(G)$, is called the *zero forcing hop domination number* of G. We study this parameter on some classes of graphs and graphs under some operations. We determine its connections with other known parameters in graph theory such as zero forcing and hop domination. We believe that this study and its results would contribute a lot to the rapidly increasing number of studies in domination theory.

2. Terminology and Notation

Let G be a graph. The *distance* $d_G(u,v)$ of two vertices u, v in G is the length of a shortest *u-v path* in G. The greatest distance between any two vertices in G, denoted by $diam(G)$, is called the *diameter* of G.

Two distinct vertices v, w of G are said to be *neighbors*, if $d_G(v,w) = 1$. The *open neighborhood* (resp. *closed neighborhood*) of v in G is the set defined by $N_G(v) = \{w \in V(G) : d_G(v,w) = 1\}$ (resp. $N_G[v] = N_G(v) \cup \{v\}$). If $X \subseteq V(G)$, then the *open neighborhood* (resp. closed neighborhood) of X in G is the set defined by $N_G(X) = \bigcup_{x\in X} N_G(x)$ (resp. $N_G[X] = N_G(X) \cup X$).

The *color change rule* is: If u is a blue vertex and exactly one neighbor w of u is white, then change the color of w to blue. We say u forces w and denote this by $u \to w$.

A *zero forcing set* for G is a subset of vertices B such that when the vertices in Z are colored blue and the remaining vertices are colored white initially, repeated application of the color change rule can color all vertices of G blue. The *zero forcing number* of G, denoted by $Z(G)$, is the minimum cardinality among all zero forcing sets in G.

A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u,v) = 2$. The set $N^2_G(u) = \{v \in V(G) : d_G(v,u) = 2\}$ (resp. $N^2_G[u] = N^2_G(u) \cup \{u\}$) is called the *open hop neighborhood* (resp. *closed hop neighborhood*) of u. Let A be a subset of $V(G)$. Then the *open hop neighborhood* (resp. *closed hop neighborhood*) of A is the set defined by $N^2_G(A) = \bigcup_{u\in A} N^2_G(u)$ (resp. $N^2_G[A] = N^2_G(A) \cup A$).

A subset S of $V(G)$ is called a *hop dominating* of G if for every $v \in V(G)\backslash S$, there exists $u \in S$ such that $d_G(u,v) = 2$. The minimum cardinality among all hop dominating sets of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-set of G.

A subset C of $V(G)$ is called a *pointwise non-dominating* (PND) if for every $v \in V(G) \setminus C$, there exists $u \in C$ such that $v \notin N_G(u)$. The minimum cardinality of a pointwise non-dominating (PND) set of G, denoted by $pnd(G)$, is called the *pointwise non-domination number* of G. Any PND set of G with cardinality $pnd(G)$ is called a minimum PND set or a *pnd*-set of G.

Let G and H be two graphs. The *join* $G+H$ of G and H is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set

$$E(G+H) = E(G) \cup E(H) \cup \{ab : a \in V(G), b \in V(H)\}$$

.

The *corona* $G \circ H$ of G and H is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. We denote by H^a the copy of H in $G \circ H$ corresponding to the vertex $a \in V(G)$.

3. Results

We begin this section by defining the concept of zero forcing hop domination in a graph as follows:

Definition 1. Let G be a graph. A subset Z of $V(G)$ is said to be a *zero forcing hop dominating* if Z is both a zero forcing and a hop dominating in G. The minimum cardinality among all zero forcing hop dominating sets in G, denoted by $\gamma_{zh}(G)$, is called the *zero forcing hop domination number* of G. A zero forcing hop dominating set Z with $|Z| = \gamma_{zh}(G)$, is called the minimum zero forcing hop dominating set of G or a γ_{zh}-set of G.

Example 1. Consider the graph G below.

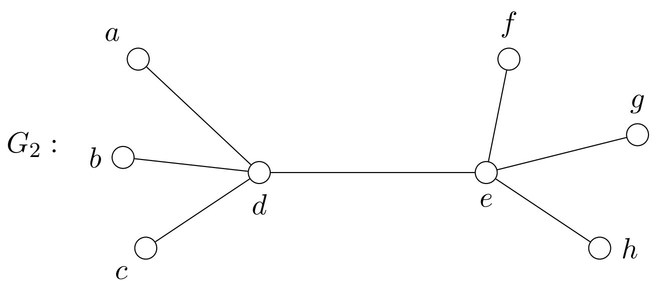


Figure 1: Graph G with $\gamma_{Zh}(G) = 5$

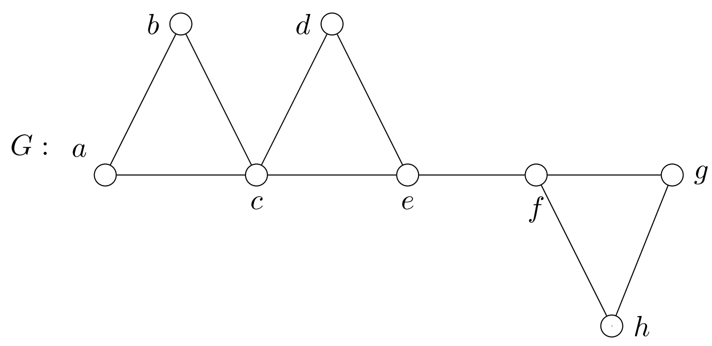
Let $Z = \{a, b, e, f, g\}$. Then Z is a zero forcing set in G. Observe that $N_G^2[a] = \{a, b, c, e\} = N_G^2[b] = N_G^2[e]$ and $N_G^2[f] = \{d, f, g, h\} = N_G^2[g]$. Thus, $N_G^2[Z] = \{a, b, c, d, e, f, g, h\} = V(G)$, showing that Z is a hop dominating set in G. Hence, Z is a zero forcing hop dominating set of G. Moreover, since Z is a minimum zero forcing set of G, it follows that Z is a minimum zero forcing hop dominating set of G, and so $\gamma_{zh}(G) = 5$.

Proposition 1. *Let G be a graph. Then*

(*i*) *a zero forcing set may not be a hop dominating; and*

(*ii*) *a hop dominating set may not be a zero forcing.*

Proof. (i) Consider the graph G below.



Let $Z = \{a, b, e, h\}$. Then, Z is a zero forcing set in G. However, $c, f \notin N_G^2[Z]$. Thus, $N_G^2[Z] \neq V(G)$, showing that Z is not a hop dominating set of G. Hence, the result follows.

(ii) Consider again the graph G in (i) and let $S = \{c, d, e\}$. Then, $N_g^2[S] = V(G)$, and so S is a hop dominating set in G. However, S is not a zero forcing set in G since it cannot forces vertices a, b, g and h in G. Thus, the assertion follows. □

Remark 1. *The Proposition 1 says that a zero forcing (resp. hop dominating) set may not be a zero forcing hop dominating set.*

Theorem 1. *Let G be any graph. Then*

(*i*) $Z(G) \leq \gamma_{zh}(G)$;

(*ii*) $\gamma_h(G) \leq \gamma_{zh}(G)$;

(*iii*) $1 \leq \gamma_{zh}(G) \leq |V(G)|$; *and*

(*iv*) $\gamma_{zh}(G) = |V(G)|$ *if and only if* $\gamma_h(G) = |V(G)|$.

Proof. (i) Let G be a graph and let Z be a γ_{zh}-set of G. Then Z is a zero forcing in G and $|Z| = \gamma_{zh}(G)$. Since $Z(G)$ is the minimum cardinality among all zero forcing sets in G, we have

$$Z(G) \leq |Z| = \gamma_{zh}(G).$$

(ii) Let S be a γ_{zh}-set of G. Then S is a hop dominating set in G and $|S| = \gamma_{zh}(G)$. Since $\gamma_h(G)$ is the minimum cardinality among all hop dominating sets in G, hence

$$\gamma_h(G) \leq | S | = \gamma_{zh}(G).$$

(iii) Since $\gamma_h(G) \geq 1$ for any graph G, it follows that $\gamma_{zh}(G) \geq 1$ by (ii). Since any zero forcing hop dominating set S' is always a subset of $V(G)$, we have $\gamma_{zh}(G) \leq | V(G) |$. Consequently,

$$1 \leq \gamma_{zh}(G) \leq |V(G)|.$$

(iv) Suppose that $\gamma_{zh}(G) = |V(G)|$. Then $V(G)$ is the minimum zero forcing hop dominating set in G. Assume that G is connected. Suppose further that G is non-complete. Then $d_G(v,w) = 2$ for some $v, w \in V(G)$. Hence, $Z' = V(G) \setminus \{w\}$ is a zero forcing hop dominating set of G, showing that $\gamma_{zh}(G) \leq |V(G)| - 1$, which is a contradiction. Therefore, G is complete, and so $\gamma_h(G) = |V(G)|$. Now, let $G_1, \ldots, G_k$, $k \geq 2$ be components of G. Suppose that G_i is non-complete for some $i \in \{1, \ldots, k\}$. Then $d_{G_i}(s,t) = 2 = d_G(s,t)$ for some $s, t \in V(G_i)$. Thus, $Z'' = V(G) \setminus \{t\}$ is a zero forcing hop dominating set of G, and so $\gamma_{zh}(G) \leq |V(G)| - 1$, a contradiction. Hence, every component of G is complete. Therefore, $\gamma_h(G_i) = |V(G_i)|$ for each $i \in \{1 \ldots, k\}$. Consequently,

$$\gamma_h(G) = \gamma_h(G_1) + \cdots + \gamma_h(G_k) = |V(G_1)| + \cdots + |V(G_k)| = |V(G)|.$$

Conversely, suppose that $\gamma_h(G) = |V(G)|$. Then by (ii) and (iii), $\gamma_{zh}(G) = |V(G)|$. □

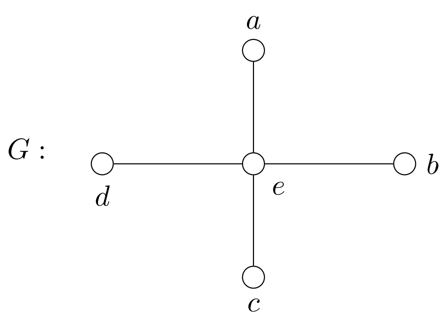
The following result follows immediately from Theorem 1(iv).

Corollary 1. *$\gamma_{zh}(K_r) = r = \gamma_{zh}(\overline{K}_r)$ for all positive integer $r \geq 1$.*

Proposition 2. *Let G be any graph with $|V(G)| \geq 2$. If $\gamma_{zh}(G) = 2$, then $\gamma_h(G) = 2$. However, the converse is not true.*

Proof. Let G be a graph with $|V(G)| \geq 2$. Then $\gamma_h(G) \geq 2$. Since $\gamma_{zh}(G) = 2$, $\gamma_h(G) \leq 2$ by Theorem 1(ii). Hence, $\gamma_h(G) = 2$.

To see that the converse is not true, consider the graph G below.



Let $Z_1 = \{a, e\}$. Then $N_G^2[Z_1] = V(G)$, showing that Z_1 is a hop dominating set of G. Thus, $\gamma_h(G) \leq 2$. Since, $|V(G)| \geq 2$, it follows that $\gamma_h(G) \geq 2$. Hence, $\gamma_h(G) = 2$. Now, let $Z_2 = \{a, b, c, e\}$. Then, Z_2 is minimum zero forcing hop dominating set in G. Thus, $fz_G(G) = 4$

□

Theorem 2. *Let $r, q \in \mathbb{N}$ with $2 \leq r \leq q$. Then there exists a connected graph K such that $\gamma_h(K) = r$ and $\gamma_{zh}(K) = q$.*

Proof. Suppose that $r < q$. Let $l = q - r$ and consider the graph K below.

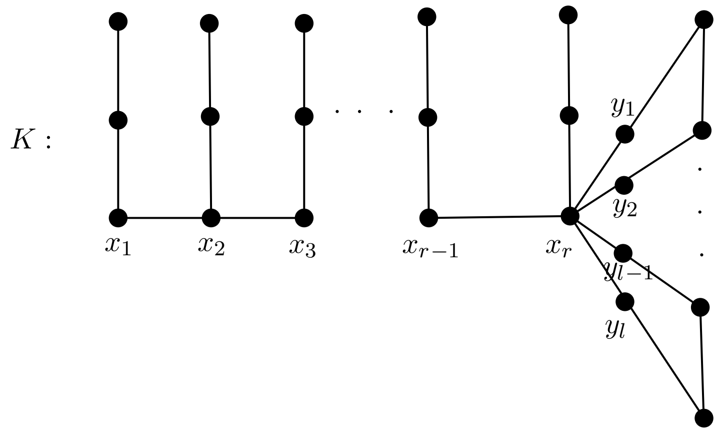


Figure 2: Graph K with $\gamma_h(K) < \gamma_{zh}(K)$

Let $A = \{x_1, x_2, \ldots, x_r\}$ and $B = \{x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_l\}$. Then A is a minimum hop dominating set of K. Thus, $\gamma_h(K) = r$. Observe that B is a minimum zero forcing set of K. Since $A \subseteq B$, it follows that B is also a hop dominating set of K. Hence, B is a minimum zero forcing hop dominating set of K. Consequently,

$$\gamma_h(K) = r < q = l + r = \gamma_{zh}(K).$$

For $r = q$, consider a complete graph G with order r. Then the sharpness of $\gamma_{zh}(G)$ and $\gamma_h(G)$ follows. □

The next definition will be used to calculate the exact value of parameter of the join of two graphs.

Definition 2. Let J be any graph. Then $F \subseteq V(J)$ is called a *zero forcing pointwise non-dominating* (ZFPND) in J if F is both a zero forcing and a pointwise non-dominating (PND) in J. The minimum cardinality among all zero forcing pointwise non-dominating (ZFPND) sets in J, denoted by $zfpnd(J)$, is called the *zero forcing pointwise non-domination number* of J. Any ZFPND set F with $|F| = zfpnd(J)$, is called the minimum ZFPND set or a $zfpnd$-set of J.

Example 2. Consider the graph G below.

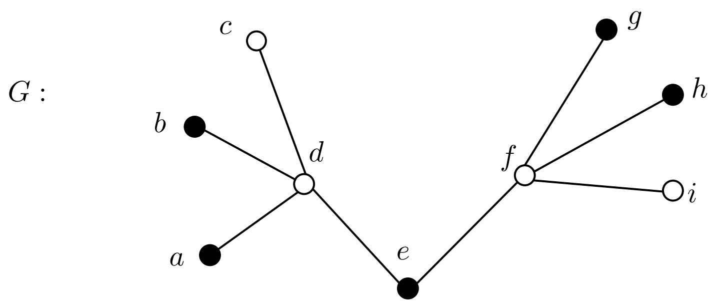
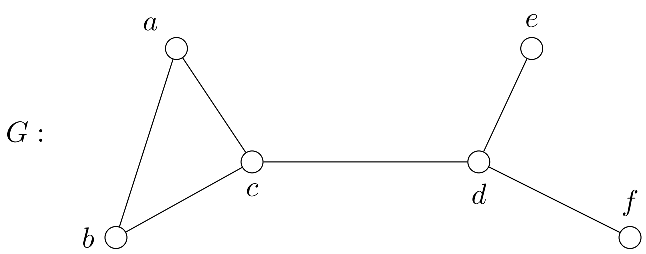


Figure 3: A graph G with $zfpnd(G) = 5$

Let $F = \{a, b, e, g, h\}$. Notice that $c, f, i \notin N_G(a)$ and $d \notin N_G(g)$. It follows that F is a PND set of G. Since F is a minimum zero forcing set of G, F is a minimum ZFPND set of G. Thus, $zfpnd(G) = 5$. Consequently, F is a $zfpnd$-set of G.

Proposition 3. *Let G be any graph. Then every ZFPND set $F \subseteq V(G)$ is a PND. But the converse is not true.*

Proof. Let F be a ZFPND. Then F is a PND set (by definition). To see that the converse is not true, consider the graph G below.



Let $N = \{a, b, c\}$. Observe that $d, e, f \notin N_G(a)$. It follows that N is a PND set in G. However, N is not a zero forcing set in G since it cannot forces vertices e and f. Hence, N is not a ZFPND set of G, and so the assertion follows. □

Theorem 3. *Let G be any graph. Then*

(i) $pnd(G) \leq zfpnd(G)$;

(ii) $1 \leq zfpnd(G) \leq |V(G)|$; *and*

(iii) $zfpnd(G) = 1$ *if and only if* $G = K_1$.

Proof. (i) Let G be any graph and let F be a minimum ZFPND set of G. Then $zfpnd(G) = |F|$ and F is a PND set of G. Since $pnd(G)$ is the minimum cardinality among all PND sets in G, it follows that

$$pnd(G) \leq |F| = zfpnd(G).$$

(ii) Since $pnd(G) \geq 1$ for any graph G, we have $zfpnd(G) \geq 1$ by (i). Moreover, since any ZFPND set F is always a subset of $V(G)$, it follows that $zfpnd(G) \leq |V(G)|$. Therefore,

$$1 \leq zfpnd(G) \leq |V(G)|.$$

(iii) Suppose that $zfpnd(G) = 1$. Assume that $G \neq K_1$. If G is connected, then $pnd(G) \geq 2$, a contradiction. Assume that G is disconnected. Let $G_1, \ldots, G_k, k \geq 2$ be components of G. Then $Z(G) \geq 2$. Since every ZFPND set is a zero forcing, we have $zfpnd(G) \geq Z(G)$. Thus, $zfpnd(G) \geq 2$, a contradiction. Therefore, $G = K_1$.

The converse is clear. □

Theorem 4. *Let G be non-trivial graph. Then $zfpnd(G) = |V(G)|$ if and only if every component of G is complete.*

Proof. Suppose that $zfpnd(G) = |V(G)|$. Then $V(G)$ is the minimum ZFPND set in G. Assume that G is connected. Suppose further that G is non-complete. Then $d_G(v, w) = 2$ for some $v, w \in V(G)$. Hence, $Z' = V(G) \setminus \{w\}$ is a ZFPND set of G, showing that $zfpnd(G) \leq |V(G)| - 1$, a contradiction. Therefore, G is complete. Now, let $Q_1, \ldots, Q_k, k \geq 2$ be components of G. Suppose that Q_i is non-complete for some $i \in \{1, \ldots, k\}$. Then $d_{Q_i}(s, t) = 2 = d_G(s, t)$ for some $s, t \in V(Q_i)$. Thus, $V(G) \setminus \{t\}$ is a ZFPND set of G, and so $zfpnd(G) \leq |V(G)| - 1$, a contradiction. Hence, every component of G is complete.

Conversely, let $G_1, \ldots, G_k, k \geq 2$ be complete components of G. If G_i is non-trivial for each $i \in \{1, \ldots, k\}$, then $pnd(G) = |V(G)| = k$. Thus, $zfpnd(G) = k$ by Theorem 3(i). Assume that G_i is trivial for some $i \in \{1, \ldots, k\}$. Since every ZFPND set F is a zero forcing, $V(G_i) \subseteq F$. Since vertices of every non-trivial complete component of G are also in any ZFPND set of G, it follows that $V(G)$ is the minimum ZFPND set of G. Thus, $zfpnd(G) = |V(G)|$. □

The following result follows from Theorem 3(iii) and Theorem 4.

Corollary 2. *$zfpnd(K_q) = q = zfpnd(\overline{K}_q)$ for all positive integer $q \geq 1$.*

Proposition 4. *Let n be any positive integer. Then each of the following holds.*

(i) $zfpnd(P_n) = \begin{cases} n & if \quad n = 1, 2 \\ 2 & if \quad n \geq 3. \end{cases}$

(ii) $zfpnd(C_n) = \begin{cases} 3 & if \quad n = 3 \\ 2 & if \quad n \geq 4. \end{cases}$

Proof. (i) Clearly, $zfpnd(P_n) = n$ for $n = 1, 2$. Suppose that $n \geq 3$. Let $V(P_n) = \{a_1, a_2, \ldots, a_n\}$ and consider $F = \{a_1, a_2\}$. Clearly, F is a zero forcing set of P_n. Observe that for every $w \in V(P_n) \setminus F$, $w \notin N_{P_n}(a_1)$. Thus, F is a PND set of P_n, showing that F is a ZFPND set of P_n. Since $\{x\}$ is not a ZFPND set in P_n $\forall$ $x \in V(P_n)$, it follows that F is a minimum ZFPND set of P_n. Hence, $zfpnd(P_n) = 2$ for all $n \geq 3$.

(ii) Since $pnd(C_3) = 3$, it follows that $zfpnd(C_3) = 3$ by Theorem 3(i)(ii). Suppose that $n \geq 4$. Let $V(C_n) = \{x_1, x_2, \ldots, x_n\}$ and consider $F' = \{x_1, x_2\}$. Notice that for all $j \in \{3, 4, ...n\}$ $x_j \notin N_{C_n}(x_1)$ and $x_n \notin N_{C_n}(x_2)$. Thus, F' is a PND set of C_n, and so F' is a ZFPND set of C_n. Since $\{x_i\}$ is not a ZFPND set of C_n for each $i \in \{1, 2, \ldots, n\}$, it follows that F' is a minimum ZFPND set of C_n. Consequently, $zfpnd(C_n) = 2$ for all $n \geq 4$. □

Theorem 5. *[10] Let G and H be two graphs. A set $S \subseteq V(G + H)$ is hop dominating set of $G + H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are PND sets of G and H, respectively.*

Theorem 6. *Let S and T be two non-complete graphs. A subset Z of $V(S+T)$ is a zero forcing hop dominating set in $S + T$ if and only if $Z = Z_S \cup Z_T$ and satisfies one of the following conditions:*

(i) *$Z_S = V(S)$ and Z_T is a ZFPND set in T.*

(ii) *$Z_T = V(T)$ and Z_S is a ZFPND set in S.*

(iii) *$Z_S = V(S) \setminus \{a\}$ and $Z_T = V(T) \setminus \{b\}$ are ZFPND sets in S and T, respectively, for some $a \in V(S), b \in V(T)$.*

Proof. Let $Z = Z_S \cup Z_T$ be a zero forcing hop dominating set in $S+T$. Then Z is a zero forcing in $S+T$. Suppose that $Z_S = V(S)$. If $Z_T = V(T)$, then we are done. Assume that $Z_T \neq V(T)$. Suppose Z_T is not a zero forcing set in T. Then there exists $w \in Z_T$ such that w cannot be forced by any element in Z_T. Thus, w cannot be forced by any element of Z, which is a contradiction. Hence, Z_T is a zero forcing set in T. Since Z is a hop dominating, Z_T is a PND set in T by Theorem 5. Consequently, Z_T is a ZFPND set in T, and so (i) holds. The (ii) can be proved in similar manner. Next, suppose that $Z_S \neq V(S)$ and $Z_T \neq V(T)$ then there exists $u \in V(S) \setminus Z_S$ and $v \in V(T) \setminus Z_T$. If $|Z_S| \leq |V(G)|-2$, then there exist at least two vertices $s, t \in V(S) \setminus Z_S$. However, any element of Z_S and Z_T cannot forces vertices s and t, a contradiction. Thus, $|Z_S| = |V(S)|-1$. Similarly, $|Z_T| = |V(T)|-1$. Let $V(S) = \{v_1, v_2, \ldots, v_m\}$ and $V(T) = \{u_1, u_2, \ldots, u_n\}$ and let $Z_S = V(S) \setminus \{v_i\}$ and $Z_T = V(T) \setminus \{u_j\}$ for some $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$. Clearly, Z_S and Z_T are zero forcing sets in S and T, respectively. Since $Z = Z_S \cup Z_T$ is a hop dominating set in $S+T$, it follows that Z_S and Z_T are PND sets in S and T, respectively, by Theorem 5. Therefore, Z_S and Z_T are ZFPND sets in S and T, respectively, showing that (iii) holds.

Conversely, suppose that (i) holds. Since Z_T is a PND set in T, it follows that $Z = V(S) \cup Z_T$ is a hop dominating set $S+T$ by Theorem 5. Since Z_T is also a zero forcing set in T, $Z = V(S) \cup Z_T$ is a zero forcing set in $S+T$. Hence, $Z = V(S) \cup Z_T$ is a zero forcing hop dominating set in $S+T$. Similarly, if (ii) holds, then the assertion follows. Now, suppose that (iii) holds. Then Z is a hop dominating set in $S+T$ by Theorem 5. Since S is non-complete, there exist $x, y \in V(S)$ such that $d_S(x,y) = 2$. Since $Z_S = V(S) \setminus \{a\}$ for some $a \in V(S)$, we let $y = a$ and so $x \in Z_S$. Then x forces all the vertices in $V(S+T) \setminus Z$, that is, Z is a zero forcing set in $S+T$. Therefore, Z is a zero forcing hop dominating set in $S+T$. □

The following result follows from Theorem 6.

Corollary 3. *Let S and T be two non-complete graphs. Then*

$$\gamma_{zh}(S+T) = min\{|V(S)| + |V(T)| - 2, |V(S)| + zfpnd(T), |V(T)| + zfpnd(S)\}.$$

Theorem 7. *Let J and K be complete and non-complete graphs, respectively. A subset Z of $V(J+K)$ is a zero forcing hop dominating set in $J+K$ if and only if $Z = V(J) \cup Z_K$, where Z_K is a ZFPND set in K.*

Proof. Let Z be a zero forcing hop dominating set in $J+K$. Since J is complete, $Z = V(J) \cup Z_K, Z_K \neq \varnothing$. Thus, by Theorem 6(i), Z_K is a ZFPND set in K.

Conversely, suppose that $Z = V(J) \cup Z_K$, where Z_K is a ZFPND set in K. Since Z_K is a zero forcing in K, $Z = V(J) \cup Z_K$ is a zero forcing in $J+K$. Moreover, since Z_K is PND set in K, it follows that $Z = V(J) \cup Z_K$ is a hop dominating set in $J+K$ by Theorem 5. Therefore, Z is a zero forcing hop dominating set of $J+K$. □

Corollary 4. *Let J and K be complete and non-complete graphs, respectively. Then*

$$\gamma_{zh}(J+K) = |V(J)| + zfpnd(K).$$

In particular, for any positive integers $m, n \geq 1$, we have

$$(i)\ \gamma_{zh}(K_m + P_n) = \begin{cases} m+n & if \quad n = 1,2 \\ m+2 & if \quad n \geq 3, \ and \end{cases}$$

$$(ii)\ \gamma_{zh}(K_m + C_n) = \begin{cases} m+3 & if \quad n = 3 \\ m+2 & if \quad n \geq 4 \end{cases}$$

Proof. Let Z be a minimum zero forcing hop dominating set in $J+K$. Then by Theorem 7, $Z = V(J) \cup Z_K$, where Z_K is a ZFPND set in K. Hence,

$$\gamma_{zh}(J+K) = |Z| = |V(J)| + |Z_K| \geq |V(J)| + zfpnd(K).$$

Conversely, suppose that $Z = V(J) \cup Z_K$, where Z_K is a minimum ZFPND set in K. Then Z is a zero forcing hop dominating set of $J+K$ by Theorem 7. Thus,

$$|V(J)| + zfpnd(K) = |Z| \geq \gamma_{zh}(J+K).$$

Consequently,

$$\gamma_{zh}(J+K) = |V(J)| + zfpnd(K).$$

The particular case, follows from Proposition 4. □

Theorem 8. *Let J and K be any non-trivial connected and any graph, respectively. Then, $M = V(J) \cup (\bigcup_{v \in V(K)} M_v)$ is a zero forcing hop dominating set in $J \circ K$ if M_v is a ZFPND set in K^v for each $v \in V(J)$. Moreover,*

$$\gamma_{zh}(J \circ K) \leq |V(J)| \cdot zfpnd(K) + |V(J)|.$$

Proof. Let $M = V(J) \cup (\bigcup_{v \in V(K)} M_v)$, where M_v is a ZFPND set in K^v for each $v \in V(J)$. Let $u \in V(J \circ K) \backslash M$. Then $u \in K^w$ for some $w \in V(G)$. Since M_w is PND set in K_w, there exists $y \in M_w$ such that $d_{J \circ K}(u, y) = 2$. Hence, M is a hop dominating set of $J \circ K$. Now, since M_v is a zero forcing set in K^v for each $v \in V(J)$, it follows that $M = V(J) \cup (\bigcup_{v \in V(G)} M_v)$ is a zero forcing set in $J \circ K$. Therefore, M is a zero forcing hop dominating set in $J \circ K$. Since $\gamma_{zh}(J \circ K)$ is the minimum cardinality among all zero forcing hop dominating sets in $J \circ K$, we have $\gamma_{zh}(J \circ K) \leq M = |V(J)| \cdot zfpnd(K) + |V(J)|$.

□

Remark 2. *The sharpness and strict inequality given in Theorem 8 are attainable.*

For the sharpness, consider the graph $P_3 \circ C_4$ below.

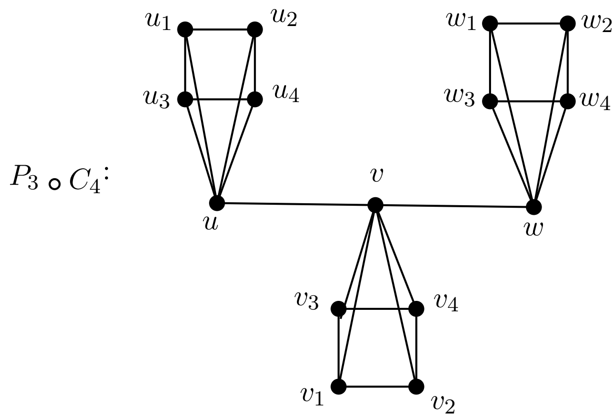


Figure 4: Graph $P_3 \circ C_4$ with $\gamma_{zh}(P_3 \circ C_4) = |V(P_3)| \cdot zfpnd(C_4) + |V(P_3)|$.

Let $M = \{u_1, u_2, u, v_1, v_2, v, w_1, w_2, w\}$. Then, $N^2_{P_3 \circ C_3}[M] = V(P_3 \circ C_4)$. Thus, M is a hop dominating set of $P_3 \circ C_4$. Observe that $\{u_1, u_2\}$, $\{v_1, v_2\}$ and $\{w_1, w_2\}$ are zero forcing sets in C_4^u, C_4^v and C_4^w, respectively. Hence, M is a zero forcing set in $P_3 \circ C_4$, and so M is a zero forcing hop dominating set in $P_3 \circ C_4$. Moreover, it can be verified that

$$\gamma_{zh}(P_3 \circ C_4) = |V(P_3)| \cdot zfpnd(C_4) + |V(P_3)| = 3 \cdot 2 + 3 = 9.$$

For strict inequality, consider the graph $C_3 \circ K_4$ below.

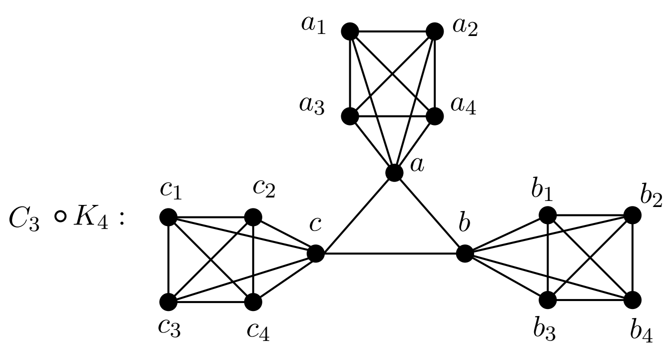


Figure 5: Graph $C_3 \circ K_4$ with $\gamma_{zh}(C_3 \circ K_4) < |V(C_3)| \cdot zfpnd(K_4) + |V(C_3)|$.

Let $S = \{a_1, a_2, a_3, a_4, b, c, c_1, c_2, c_3, b_1, b_2, b_3\}$. Clearly, S is a zero forcing set in $C_3 \circ K_4$. Notice that $N^2_{C_3 \circ K_4}[S] = V(C_3 \circ K_4)$. Thus, S is a zero forcing hop dominating set in $C_3 \circ K_4$, showing that $\gamma_{zh}(C_3 \circ K_4) \leq |S| = 12$. Now, since, $zfpnd(K_4) = 4$, it follows that $|V(C_3)| \cdot zfpnd(K_4) + |V(C_3)| = 3 \cdot 4 + 3 = 15$. Consequently,

$$\gamma_{zh}(C_3 \circ K_4) \leq 12 < 15 = |V(C_3)| \cdot zfpnd(K_3) + |V(K_4)|.$$

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