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New Variants of Newton's Method for Solving Nonlinear Equations

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Abstract. Two Newton-type iterative techniques have been created in this work to locate the true root of univariate nonlinear equations. One of these can be acquired by modifying the double Newton's method in a straightforward manner, while the other can be gotten by modifying the midpoint Newton's method. The iterative approach developed by McDougall and Wortherspoon is employed for the change. The study demonstrates that the modified double Newton's approach outperforms the current one in terms of both convergence order and efficiency index, even though both methods assess the same amount of functions and derivatives every iteration. In comparison to the midpoint Newton's technique, which has a convergence order of 3, the modified midpoint Newton's method has a convergence order of 5.25 and requires two extra functions to be evaluated per iteration. In order to evaluate the effectiveness of recently introduced approaches with current methods, some numerical examples are shown in the final section.

2020 Mathematics Subject Classifications: 65H05

Key Words and Phrases: Newton's method, Numerical method, Nonlinear equation, Convergence, Efficiency index

1. Introduction

A collection of various nonlinear equations with objective functions and/or nonlinear constraint functions is referred to as a nonlinear equation system [1]. Finding the roots of some nonlinear equation systems requires numerical approaches. The numerical approach is a strategy for figuring out mathematical issues so that they may be resolved using standard operations like add, subtract, multiply, and divide. One of the numerical techniques for resolving a system of nonlinear equations is the Newton method. Some mathematicians are modifying Newton methods to arrive at a novel technique. The Midpoint Newton approach, which can find the equation's root more quickly [13] is the result

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of combining the Newton method and the Midpoint rule. Singh MK [14] developed Newton's method based on contra harmonic mean with order of convergence six. He JH extended the frequency-amplitude formulation to solve nonlinear conservative oscillators with general initial conditions which is similar to the Hamiltonian approach. Finding the solutions to nonlinear equations is one of the most important and challenging problems in almost all areas of science and engineering, but solving such equations analytically is almost impossible. In those situations, the construction of efficient numerical methods based on an iterative procedure for finding the roots of such equations becomes a fascinating and attractive task. In literature, there are numerous methods available for finding the roots of such equations in different situations. Newton's method is one of the best-known and most frequently used basic numerical methods to solve nonlinear equations. It's an iterative formula for obtaining the simple root of the nonlinear equation In similar circumstances, developing effective numerical algorithms based on an iterative process to locate the roots of such equations turns out to be an intriguing and alluring task. There are many techniques described in the literature for locating the roots of these equations in various contexts. One of the most popular and widely used fundamental numerical techniques for solving nonlinear equations is Newton's method. It is an iterative formula for finding the nonlinear equation's simple root.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

This method is quadratically convergent [2] and its efficiency index is 1.414 [9]. During the last two decades, several higher-order variants of Newton's methods have appeared. Some of them can be found in [3]-[17]. Here, we will briefly describe those methods that encouraged us to carry out our own work. Now, consider double Newton's methods

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'(y_n)}. \end{aligned}\right\} \tag{2}$$

This method is fourth-order convergent [11, 15] and its efficiency index is the same as Newton's method.

Many researchers use the numerical integration technique to improve the local order of convergence of Newton's method. Some of them may be found in [3–8, 10, 16, 17] and references therein. To get the improved convergence order of Newton's method, Özban in [13] considered Newton's theorem

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)\, dt \tag{3}$$

and used the mid-point rule to approximate the integral, that is,

$$\int_{x_n}^{x} f'(t)\, dt = (x - x_n) f' \frac{(x + x_n)}{2}. \tag{4}$$

Then, they explored the variant of Newton's method given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(\frac{x_n+x_n^*}{2})}, \tag{5}$$

where $x_n^* = x_n - \frac{f(x_n)}{f'(x_n)}$. This is the midpoint Newton's method and it has been proven that its convergence order is 3.
Recently, McDougall and Wortherspoon [12] obtained a variant of Newton's method having the order of convergence $1+\sqrt{2}$ with a minor change in Newton's method. The method is given below:

If x_0 is the initial approximate solution, we can write

$$\left.\begin{aligned} x_0^* &= x_0 - \frac{f(x_0)}{f'(x_0)}, \\ x_1 &= x_0 - \frac{f(x_0)}{f'[\frac{1}{2}(x_0+x_0^*)]}. \end{aligned}\right\} \tag{6}$$

For $n \geq 1$, we have

$$\left.\begin{aligned} x_n^* &= x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_{n-1}+x_{n-1}^*)]}, \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_n+x_n^*)]}. \end{aligned}\right\} \tag{7}$$

This is a kind of predictor-corrector method. In Newton's method, the derivatives are calculated at the previously iterated point, but in the methods (6)-(7), the derivatives are calculated at some convenient point. As the supreme goal of this paper, two iterative methods are constructed as a variant of methods (2) and (5) using methods (6)-(7). Also, it has to be shown that both newly introduced methods can easily compete with similar existing methods.

2. Methods with Convergence Analysis

At first, we present the iterative method as a variant of method (2) using the view of method (6)-(7) as given below:
If x_0 is taken as the initial approximate solution,

$$\left.\begin{aligned} x_0^* &= x_0 \\ x_0^{**} &= x_0 - \frac{f(x_0)}{f'\left(\dfrac{x_0+x_0^*}{2}\right)} = x_0 - \frac{f(x_0)}{f'(x_0)} \\ x_1 &= x_0^{**} - \frac{f(x_0^{**})}{f'(x_0^{**})} \end{aligned}\right\} \tag{8}$$

followed by (for $n \geq 1$)

$$\left.\begin{aligned} x_n^* &= x_n - \frac{f(x_n)}{f'\left(\dfrac{x_{n-1}+x_{n-1}^*}{2}\right)} \\ x_n^{**} &= x_n - \frac{f(x_n)}{f'\left(\dfrac{x_n+x_n^*}{2}\right)} \\ x_{n+1} &= x_n^{**} - \frac{f(x_n^{**})}{f'(x_n^{**})} \end{aligned}\right\} \tag{9}$$

To analyze it's convergence, we prove:

Theorem 1. *Let a sufficiently differentiable function $f : D \subset R \to R$ has a simple root α in the interval D. If x_0 is sufficiently close to α, then to solve the nonlinear equation $f(x) = 0$, (8)-(9) has a convergence of order* 4.45*.*

Proof. Let $e_n = x_n - \alpha$ be the error in the nth*iteration*x_n and denote
$c_j = \frac{1}{j!} \cdot \frac{f^j(\alpha)}{f'(\alpha)}, j = 2, 3, 4....$
It is standard to work out that the error equation in the Newton method 1 :

$$e_{n+1} = c_2 e_n^2, \tag{10}$$

where the terms with higher powers of e_n are omitted.
Let us go on to discuss the convergence of the methods (8)-(9). Let e_n^*, e_n^{**} and e_n denote errors at iterates x_n^*, x_n^{**} and x_n respectively. Then clearly $e_0^* = e_0$ and in the view of (10), from (8), we get the error equation

$$e_0^{**} = c_2 e_0^2, \tag{11}$$

and

$$e_1 = c_2 (e_0^{**})^2 = c_2^3 e_0^4 \tag{12}$$

Here,

$$x_1^* = x_1 - \frac{f(x_1)}{f'[\frac{1}{2}(x_0 + x_0^*)]} = x_1 - \frac{f(x_1)}{f'(x_0).}$$

Taylor series and binomial expansion gives the error equation for x_1^*, we have

$$\begin{aligned} \alpha + e_1^* &= \alpha + e_1 - \frac{f(\alpha + e_1)}{f(\alpha + e_0)} \\ e_1^* &= e_1 - \frac{e_1 + c_2 e_1^2 + c_3 e_1^3 + O(e_1^4)}{1 + 2c_2 e_0 + 3c_3 e_0^2 + O(e_0^3)} \\ &= e_1 - \left(e_1 + c_2 e_1^2 + c_3 e_1^3 + O(e_1^4)\right)\left(1 - 2c_2 e_0 - 3c_3 e_0^2 + 4c_2^2 e_0^2 + O(e_0^3)\right) \\ &= 2c_2 e_0 e_1 \\ &= 2c_2^4 e_0^5, \end{aligned} \tag{13}$$

neglecting the higher powers of e_0.

Next, the error in x_1^{**} can be found as follows:

$$\begin{aligned} e_1^{**} &= e_1 - \frac{f(\alpha+e_1)}{f'\left(\alpha+\frac{e_1+e_1^*}{2}\right)} \\ &= e_1 - \left(e + c_2e_1^2 + c_3e_1^3\right)\left(1 + 2c_2(\frac{e_1+e_1^*}{2}) + 3c_3(\frac{e_1+e_1^*}{2})^2\right)^{-1} \\ &= c_2e_1e_1^* \\ &= 2c_2^5e_0^9 \end{aligned} \tag{14}$$

by using (12) and (13). Also, from the view of the error equation (10), the error equation for x_2 :

$$e_2 = c_2(e_1^{**})^2 = 2^2c_2^{11}e_0^{18} \tag{15}$$

In fact, for $n \geq 2$, the errors at x_n^*, x_n^{**} and x_n respectively, can be found by the relations

$$e_n^* = c_2e_ne_{n-1}, \tag{16}$$

$$e_n^{**} = c_2e_ne_n^* = c_2^2e_n^2e_{n-1}, \tag{17}$$

$$\text{and} \qquad e_{n+1} = c_2e_n^{**2} = c_2^5e_n^4e_{n-1}^2. \tag{18}$$

To calculate the order of convergence of (8)-(9), the following relation is needed:

$$e_{n+1} = Ae_n^p \tag{19}$$

Thus,

$$e_n = Ae_{n-1}^p \qquad \text{or} \qquad e_{n-1} = A^{-\frac{1}{p}}e_n^{\frac{1}{p}}. \tag{20}$$

From (16), (17) and (18), we have

$$Ae_n^p = c_2^5e_n^4A^{-2/p}e_n^{2/p}.$$

Equating the power of e_n, we obtain

$$\begin{aligned} p &= 4 + \frac{2}{p} \\ \text{or,} \qquad p^2 - 4p - 2 &= 0 \\ \text{or,} \qquad p &= \frac{4 \pm \sqrt{24}}{2}. \end{aligned}$$

For positive sign, $p = 2 + \sqrt{6} \approx 4.45$. Consequently, the convergence order of methods (8)-(9) is 4.45.

Alternative Proof: Accumulating the relations from the proof given above, we get the errors at x_n, x_n^*, and x_n^{**} stages as given in the following Table-1: We construct the table

Table 1: Successive errors.

n	e_n	e_n^*	e_n^{**}
0	e_0	e_0	$c_2e_0^2$
1	$c_2^3e_0^4$	$2c_2^4e_0^5$	$2c_2^5e_0^9$
2	$2^2c_2^{11}e_0^{18}$	$2^2c_2^{14}e_0^{22}$	$2^4c_2^{26}e_0^{40}$
3	$2^8c_2^{53}e_0^{80}$	$2^{10}c_2^{65}e_0^{98}$	$2^{18}c_2^{119}e_0^{178}$
4	$2^{38}c_2^{239}e_0^{356}$	$2^{46}c_2^{293}e_0^{436}$	$2^{84}c_2^{533}e_0^{792}$
5	$2^{168}c_2^{1067}e_0^{1584}$	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

as done in [12]. The sequence formed by the power of e_0 in the errors at each iteration is

$$4,\ 18,\ 80,\ 356,\ 1584,\ \cdots \tag{21}$$

The sequence of their successive ratios is

$$\frac{18}{4},\ \frac{80}{18},\ \frac{356}{80},\ \frac{1584}{356},\ \cdots$$

or

$$4.5,\ 4.44,\ 4.45,\ 4.45,\ \cdots .$$

If $\{R_i\}$ denotes the i^{th} term of the sequence (21), then it holds relation

$$R_i = 4R_{i-1} + 2R_{i-2},\ i = 2,\ 3,\ 4,\ \cdots .$$

If we take the limit

$$\frac{R_i}{R_{i-1}} = \frac{R_{i-1}}{R_{i-2}} = p$$

Then, we get

$$p^2 - 4p - 2 = 0, \tag{22}$$

which gives its positive root $p = 2 + \sqrt{6} \approx 4.45$. Consequently, convergence order of the method (8)-(9) is at least 4.45. □

Next, we modify method (5) by using methods (6)-7) as below:

If x_0 is the initial approximation, then

$$\left.\begin{aligned} &x_0^* = x_0, \\ &x_1 = x_0^* - \frac{f(x_0^*)}{f'[\frac{1}{2}(x_0^* + z_1)]}, \quad \text{with } z_1 = x_0 - \frac{f(x_0)}{f'\left(\frac{x_0+x_0^*}{2}\right)}. \end{aligned}\right\} \tag{23}$$

$$\left. x_1^* = x_1 - \frac{f(x_1)}{f'[\frac{1}{2}(x_1 + z_1^*)]}, \quad \text{with } z_1^* = x_1 - \frac{f(x_1)}{f'\left(\frac{x_0+x_0^*}{2}\right)}. \quad \right\} \tag{24}$$

For $n \geq 1$, we get the following iterations:

$$\left.\begin{aligned} x_n^* &= x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_n + z_n^*)]}, \text{where } z_n^* = x_n - \frac{f(x_n)}{f'\left(\frac{x_{n-1}+x_{n-1}^*}{2}\right)} \\ x_{n+1} &= x_n^* - \frac{f(x_n^*)}{f'\left(\frac{x_n^*+z_{n+1}}{2}\right)}, \text{where } z_{n+1} = x_n - \frac{f(x_n)}{f'\left(\frac{x_n+x_n^*}{2}\right)}. \end{aligned}\right\} \tag{25}$$

The proof of the convergence results of the above theorem is given below.

Theorem 2. *Let a sufficiently differentiable function $f : D \subset R \to R$ has a simple root α in the interval D. If x_0 is sufficiently close to α, then the methods (23)-(25) to solve the nonlinear equation $f(x) = 0$, have convergence order* 5.1925.

Proof. Let e_n and e_n^* be the errors in x_n and x_n^*,respectively. Again, suppose $c_k = \frac{f^k(\alpha)}{k!f'(\alpha)}$, $k = 2, 3, 4....$
The error equation of method (5) as obtained by Özban [13] :

$$e_{n+1} = ae_n^3, \tag{26}$$

where $a = c_2^2 - \frac{1}{4}c_3$ and the higher powers of e_n being neglected.
In the same line of proof of above result, e_0^* and e_1, in x_0^* and x_1 in (23)-(25) are given by

$$\begin{aligned} e_0^* &= e_0 \\ e_1 &= ae_0^3, \end{aligned} \tag{27}$$

where higher powers of small quantity are neglected. Next, we find e_1^* in x_1^*. Taylor and binomial series expansion gives:

$$\begin{aligned} \frac{f(x_1)}{f'(x_0)} &= \frac{f(\alpha + e_1)}{f'(\alpha + e_0)} \\ &= \left(e_1 + c_2e_1^2 + c_3e_1^3 + O(e_1^4)\right)\left(1 + 2c_2e_0 + 3c_3e_0^2 + O(e_0^3)\right)^{-1} \\ &= e_1 - 2c_2e_0e_1 + O(e_0^5), \end{aligned}$$

and hence

$$z_1^* = x_1 - \frac{f(x_1)}{f'(x_0)} = \alpha + 2c_2e_0e_1 + O(e_0^5).$$

Also,

$$\frac{x_1 + z_1^*}{2} = \alpha + \frac{e_1 + 2c_2e_0e_1 + ...}{2},$$

$$\text{and} \qquad f'\left(\frac{x_1 + z_1^*}{2}\right) = f'(\alpha)\left(1 + c_2e_1 + 2c_2^2e_0e_1 + O(e_0^5)\right). \tag{28}$$

Consequently, using (27) and (28), e_1^* in x_1^* at the equation (24) is given by

$$\begin{aligned} e_1^* &= e_1 - \big(e_1 + c_2 e_1^2 + O(e_1^3)\big)\big(1 + c_2 e_1 + 2c_2^2 e_0 e_1 + O(e_0^5)\big)^{-1} \\ &= 2c_2^2 e_0 e_1^2 \\ &= b e_0^7, \end{aligned}$$

where $b = 2c_2^2 a^2$. From (25), using e_1^*, we now find e_2 in

$$x_2 = x_1^* - \frac{f(x_1^*)}{f'\big(\frac{x_1^*+z_2}{2}\big)}, \qquad \text{where} \qquad z_2 = x_1 - \frac{f(x_1)}{f'\big(\frac{x_1+x_1^*}{2}\big)}.$$

Now,

$$\begin{aligned} f'\big(\frac{x_1+x_1^*}{2}\big) &= f'\big(\alpha + \frac{e_1+e_1^*}{2}\big) \\ &= f'(\alpha)\big(1 + c_2 e_1 + c_2 e_1^* + \frac{3}{4}c_3 e_1^2 + O(e_0^9)\big) \end{aligned}$$

in turn,

$$\begin{aligned} \frac{f(x_1)}{f'\big(\frac{x_1+x_1^*}{2}\big)} &= \big(e_1 + c_2 e_1^2 + O(e_1^3)\big)\big(1 + c_2 e_1 + c_2 e_1^* + \frac{3}{4}c_3 e_1^2 + O(e_0^9)\big)^{-1} \\ &= e_1 + \frac{1}{4}c_3 e_1^3 - c_2 e_1 e_1^* + ... \end{aligned}$$

and therefore

$$z_2 = \alpha - \frac{1}{4}c_3 e_1^3 + c_2 e_1 e_1^* + ...$$

Here,

$$\begin{aligned} \frac{x_1^*+z_2}{2} &= \alpha + \frac{1}{2}\big(e_1^* - \frac{1}{4}c_3 e_1^3 + c_2 e_1 e_1^* + ...\big), \\ \text{and} \qquad f'\big(\frac{x_1^*+z_2}{2}\big) &= f'(\alpha)\big(1 + c_2 e_1^* - \frac{1}{4}c_2 c_3 {e_1}^3 + c_2^2 e_1 e_1^* + ...\big) \end{aligned}$$

Thus, the error e_2 in x_2 is :

$$e_2 = c e_1^3 e_1^*,$$

where $c = -\frac{1}{4}c_2 c_3$ and higher powers of small quantities are neglected. Consequently, for $n \geq 1$, we get:

$$e_{n+1} = c e_n^3 e_n^*. \tag{29}$$

To compute e_{n+1} explicitly, the values of e_n^* are needed. The value of e_1^* is already known. Like above, e_2^* in x_2^* can be found, and its value is given by

$$e_2^* = d e_1 e_2^2,$$

where $d = c_2^2$ and, again, for $n \geq 2$, the following formula can be obtained:

$$e_n^* = de_{n-1}e_n^2. \tag{30}$$

To investigate the convergence order of methods (23)-(25), the following form should be obtained:

$$e_{n+1} = Ae_n^p. \tag{31}$$

Also,

$$e_n = Ae_{n-1}^p \qquad \text{or} \qquad e_{n-1} = A^{-\frac{1}{p}}e_n^{\frac{1}{p}}. \tag{32}$$

From relations (29), (30) and (31), we have

$$Ae_n^p = e_{n+1} = ce_n^3e_n^* = ce_n^3de_{n-1}e_n^2 = cde_n^5A^{-\frac{1}{p}}e^{\frac{1}{p}}$$

Equating the power of e_n, we get

$$\begin{aligned} p &= 5 + \frac{1}{p} \\ or, p^2 - 5p + 1 &= 0 \\ \therefore p &= \frac{5 \pm \sqrt{29}}{2} \end{aligned}$$

Hence, the convergence order of method (23)-(25)is $\frac{5+\sqrt{29}}{2} \approx 5.1925$

Note: The above theorem can also be proved in the same line of alternative proof of Theorem 1.

3. Numerical Examples

Here, we will be discussing some numerical examples to exhibit the performance of the newly developed methods. We will compare these methods with Newton's method (1), McDougall and Wotherspoon (8-(9), midpoint Newton's method (5), and double Newton's method (2). For the numerical computations, MATLAB software is used under the stopping criteria $|x_{n+1} - x_n| < 10^{-15}$ or $|f(x_n)| < 10^{-15}$. To determine the initial approximation, Intermediate Value Theorem or graphical method can be used. For the comparison, the following function has been used:

(i) $f_1 = x^3 + 4x^2 - 10$ (ii) $f_2 = x^6 - x - 1$

(iii) $f_3 = \sin^2 x - x^2 + 1$ (iv) $f_4 = xe^{x^2} - \sin^2 x + 3\cos x + 5$

(v) $f_5 = \cos x - xe^x + x^2$ (vi) $f_6 = e^{x^2+7x-30} - 1$

(vii) $f_7 = e^x \sin x + ln(x^2 + 1)$ $(viii)$ $f_8 = (x-2)^{23} - 1$

where, NM,MWM, MNM, DNM denote Newton's method, McDougall and Wotherspoon method, mid-point Newton's method and double Newton's method respectively.

Table 2: Comparison of various methods with distinct initial approximation x_0 .

Function	x_0	Number of Iterations						Root
		NM	MWM	MNM	DNM	Method (8)-(9)	Method (23)-(25)	
f_1	3	6	5	4	3	3	3	1.365230013414097
	1	5	4	3	3	3	3	-
	-1	24	32	9	12	5	7	-
f_2	3	10	9	7	5	5	5	1.34724138401519
	1	6	5	4	3	3	3	-
	0	7	7	4	4	4	3	-0.778089598678601
f_3	10	8	7	5	4	4	4	1.404491648215341
	4	6	5	4	3	3	3	-
	-3	6	5	4	3	3	3	-1.404491648215341
f_4	1	8	7	Div	5	11	80	-1.207647827130919
	-2	9	8	6	5	5	5	-
	-1.5	7	6	5	4	4	4	-
f_5	-1.5	7	5	5	4	4	4	0.639154096332008
	5	11	9	7	6	6	5	-
	10	17	14	10	9	8	7	-
f_6	6	53	46	34	27	26	22	3
	5	35	29	22	18	17	15	-
	4	19	16	12	10	10	8	-
f_7	0.5	6	5	4	3	3	3	0
	1	7	6	5	4	4	3	-
	2	6	6	5	4	3	3	-
f_8	3.5	14	12	9	7	7	6	3
	4	21	17	13	11	10	9	-
	4.5	26	21	16	13	13	11	-

4. Conclusion

Here, we explored two modified double Newton's type iterative methods (8)-(9) and (23)-(25) of convergence of order 4 and modified midpoint Newton's methods of convergence order 3 as iterative Newton's type methods, respectively. In Theorem-1, we established that the technique (8)-(9) has a convergence order of 4.45 when evaluating the same number of functions per iteration. As a result, this newly discovered method's convergence order and efficiency index are higher than those of the double Newton's approach. Additionally, the modified method's order of convergence (23)-(25) is greater than Newton's method's mid-point; however, after the initial iteration, we have to evaluate two more functions. The computational result shown in table-2 compares the practical performance of the recently investigated approaches.

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