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m-polar Q-hesitant anti-fuzzy set in BCK/BCI-algebras

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Abstract. The main objective of this paper is to effectively define a new concept of the fabulous fuzzy set theory that is called m-polar Q-hesitant anti-fuzzy set and apply it to the BCK/BCI-algebras. The m-polar Q-hesitant anti-fuzzy set is an astonishing development of the combination between the m-polar fuzzy set and the Q-hesitant fuzzy set. However, we introduce knowledge of the m-polar Q-hesitant anti-fuzzy subalgebra, m-polar Q-hesitant anti-fuzzy ideal, closed m-polar Q-hesitant anti-fuzzy ideal, m-polar Q-hesitant anti-fuzzy commutative ideal, m-polar Q-hesitant anti-fuzzy implicative ideal, and m-polar Q-hesitant anti-fuzzy positive implicative of BCK/BCI-algebras. In addition, we investigate several theorems, examples, and properties of these notions.

2020 Mathematics Subject Classifications: 93C42, 68T05, 90C29, 68T37

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1. Introduction

In 1965, Zadeh [20] introduced the concept of fuzzy sets, which are sets whose elements have degrees of membership. These sets are an extension of the classical notion of a set. A fuzzy set is considered a class of objects with a continuum of grades of membership. Such a set is characterized by a membership (characteristic) function which assigns to each object a grade of membership ranging between zero and one. The notions of inclusion, union, intersection, complement, relation, convexity, etc., are extended to such sets, and various properties of these notions in the context of fuzzy sets are established. In particular, a separation theorem for convex fuzzy sets is proved without requiring that the fuzzy sets be disjoint.

Next, in 1966, Imai and Iseki [5] introduced BCI and BCK algebras, which are algebraic structures in universal algebra that describe fragments of the propositional calculus involving implication known as BCI and BCK logics. These structures are defined as follows: a triple $(X, *, 0)$ is called a BCI algebra if it satisfies five conditions, as we show

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them in the next section. The BCI algebra $(X, *, 0)$ is called a BCK algebra if it satisfies one condition.

In 2010, Torra [16] introduced the notion of the hesitant fuzzy set, which is used in decision-making problems. It is a very useful tool to deal with uncertainty and can be accurately and perfectly described in terms of the opinions of decision makers. It serves as a link between classical soft sets and hesitant fuzzy sets. Also, in recent years have witnessed a growing interest in the application of (hesitant) fuzzy sets to various algebraic structures. Notable contributions to this field have been made by researchers [11–13, 18]. Their studies have shed light on the potential of (hesitant) fuzzy sets in the context of algebraic structures and opened up new avenues for research and application. In this context see also, [14, 17]. In other hand Talee and all. In 2020 study explores the application of hesitant fuzzy sets to ideal theory in ordered Γ-semigroups [15]. This work offers a unique perspective on ideal theory, enhancing our understanding of ordered Γ-semigroups. Muhiuddin and all. In 2021 apply fuzzy soft set theory to investigate commutative ideals in BCK-algebras [8, 9]. Their research, published in the International Journal of Advanced and Applied Sciences, provides insights into this specific algebraic context. Muhiuddin and Aldhafeeri. In 2018 delve into subalgebras and ideals within BCK/BCI-algebras using uni-hesitant fuzzy set theory [10]. This paper explores the potential of uni-hesitant fuzzy set theory in understanding these algebraic structures. In this context, see, for instance, the study by Alleheb and Alsager [2].

In 2014, Chen et al. [1, 4, 21] introduced the notion of m-polar fuzzy sets as a generalization of bipolar fuzzy sets and showed that bipolar fuzzy sets and 2-polar fuzzy sets are cryptomorphic mathematical notions. The notions of fuzzy ideals and fuzzy subalgebras were considered by Yehia [19]. Since then, the concepts and results of Lie algebras have been broadened to the fuzzy setting.

In this paper, we introduce the notion of an m-polar fuzzy Lie ideal of a Lie algebra and investigate some properties of the nilpotency of m-polar fuzzy Lie ideals. We also introduce the concept of an m-polar fuzzy adjoint representation of Lie algebras and discuss the relationship between this representation and nilpotent m-polar fuzzy Lie ideals.

There are three theories: the theory of probability, the theory of fuzzy sets, and interval mathematics, which we can consider as mathematical tools for dealing with uncertainties. However, all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, the theory of intuitionistic fuzzy sets, the theory of vague sets, the theory of interval mathematics, and the theory of rough sets.

The results of this paper are organized as follows: In Section 2, we present some relevant notions that helped us with our work. In Section 3, we discuss the concept of m-polar Q-hesitant anti-fuzzy subalgebra. In Section 4, we define the characterization of m-polar Q-hesitant anti-fuzzy ideal. In Section 5, we describe the notion of a closed m-polar Q-hesitant anti-fuzzy set. In Section 6, we introduce the concept of a m-polar Q-hesitant anti-fuzzy commutative ideal. In Section 7, we investigate the notion of a m-polar Q-hesitant anti-fuzzy implicative ideal. In Section 8, we define the concept of a m-polar Q-hesitant anti-fuzzy positive implicative ideal. Finally, the study is concluded in Section

9.

2. Preliminaries

In this section, we recall some basic definitions and axioms that will be used in our work.

Definition 1. *An algebra* $(B, *, 0)$ *of type* $(2, 0)$ *is called a BCK-algebra if it satisfies the following conditions:*

B1:
$$\forall \chi, \omega, \tau \in B,\ (((\chi * \omega) * (\chi * \tau)) * (\tau * \omega) = 0) \tag{1}$$

B2:
$$\forall \chi, \omega \in B,\ ((\chi * (\chi * \omega)) * \omega = 0) \tag{2}$$

B3:
$$\forall \chi \in B,\ (\chi * \chi = 0) \tag{3}$$

B4:
$$\forall \chi, \omega \in B,\ (\chi * \omega = 0, \omega * \chi = 0 \Rightarrow \chi = \omega) \tag{4}$$

B5:
$$\forall \chi \in B,\ (0 * \chi = 0) \tag{5}$$

Then, B is called a BCK-algebra.
Any BCK-algebra B satisfies the following axioms:

B6:
$$\forall \chi \in B,\ (\chi * 0 = \chi) \tag{6}$$

B7:
$$\forall (\chi, \omega, \tau \in B)(\chi \leq \omega \Rightarrow \chi * \tau \leq \omega * \tau, \tau * \omega \leq \tau * \chi) \tag{7}$$

B8:
$$\forall \chi, \omega, \tau \in B,\ (((\chi * \omega) * \tau = (\chi * \tau) * \omega)) \tag{8}$$

B9:
$$\forall \chi, \omega, \tau \in B,\ (((\chi * \tau) * (\omega * \tau) \leq \chi * \omega) \tag{9}$$

Where, $\chi \leq \omega$ *means that* $\chi * \omega = 0$.
Any BCI-algebra B satisfies the following axiom:

BI1:
$$\forall \chi, \omega, \tau \in B,\ (0 * (0 * ((\chi * \tau) * (\omega * \tau))) = (0 * \omega) * (0 * \chi)) \tag{10}$$

Definition 2. *[6] A non-empty subset s of a BCK/BCI-algebra is called a Subalgebra of B if $\chi * \omega \in s$ for all $\chi, \omega \in s$.*

Definition 3. *[6] A nonempty subset D of a BCK/BCI-algebra B is called an ideal if it satisfies the following:*

ID1:

$$0 \in D. \tag{11}$$

ID2:

$$\text{For all}\, \chi, \omega \in B,\, (\chi * \omega \in D) \wedge (\chi \in D \Rightarrow \omega \in D). \tag{12}$$

Definition 4. *[7] Let B be a BCK/BCI-algebra. A hesitant fuzzy set,*

$$\Phi = \{(\chi, \mu_\Phi(\chi)) \,|\, \chi \in B\}, \tag{13}$$

on B is called a hesitant fuzzy subalgebra of B if it satisfies:

$$\text{For all}\, \chi, \omega \in B,\, (\mu_\Phi(\chi * \omega) \supseteq \mu_\Phi(\chi) \setminus \mu_\Phi(\omega)). \tag{14}$$

Definition 5. *[7] Let B be a BCK/BCI-algebra. A hesitant fuzzy set,*

$$\Phi = \{(\chi, \mu_\Phi(\chi)) \,|\, \chi \in B\}, \tag{15}$$

on B is called a hesitant fuzzy ideal of B if it satisfies:

$$\text{For all}\, \chi, \omega \in B,\, (\mu_\Phi(\chi * \omega) \setminus \mu_\Phi(\omega) \subseteq \mu_\Phi(\chi) \subseteq \mu_\Phi(0)). \tag{16}$$

Definition 6. *[3] Let B be a non-empty finite universe and Q be a non-empty set. A Q-hesitant fuzzy set Φ_Q is a set given by:*

$$\Phi_Q = \{((\chi, q), \mu_Q(\chi, q)) \,|\, \chi \in B, q \in Q\}, \tag{17}$$

where $\Phi_Q : B \times Q \to [0, 1]$.

Definition 7. *An m-polar Q-hesitant fuzzy set on a non-empty set B is the mapping $\Phi_Q : B \times Q \to [0, 1]^m$. The membership value of every element $\chi \in B$ is denoted by:*

$$\Phi_Q = \{((\chi_i, q), \mu_{\Phi_Q}(\chi_i, q)) \,|\, \chi_i \in B, q \in Q\}, \tag{18}$$

which can be written as:

$$\Phi_i = \{(\chi, q), \mu_i(\chi, q) \,|\, \chi \in B, q \in Q\}, \tag{19}$$

where $\chi \in B$ and $q \in Q$ for all $i = 1, 2, \ldots, m$.

3. m-polar Q-hesitant anti-fuzzy subalgebra

In this section, we'll explore a concept: m-polar Q-hesitant anti-fuzzy subalgebras. These help us understand complex algebraic structures in the real world, where things aren't always clear-cut.

Definition 8. *Let B be a BCK/BCI-algebra. The m-polar Q-hesitant anti-fuzzy set*

$$\Phi_i = \{(\nu, q), \mu^i_\Phi(\nu, q) \; j \; \nu \in B, q \in Q\} \tag{20}$$

on B is called an m-polar Q-hesitant anti-fuzzy subalgebra if it satisfies the following:

$$\forall \chi, \omega \in B, q \in Q : \mu_{\Phi_i}(\chi * \omega, q) \subseteq \mu_{\Phi_i}(\chi, q) \vee \mu_{\Phi_i}(\omega, q) \tag{21}$$

For all $i = 1, 2, \ldots, m$.

Example 1. *Let $B = \{a, b, c\}$ be a BCK-algebra with a binary operation "$*$," which is given in the following Cayley table.*

$*$	a	b	c
a	a	a	a
b	b	a	b
c	c	c	a

Define the set $Q = \{a, b, c, d\}$ and a 2-polar fuzzy set on B as follows:

(a, a)	$\{0.9, 0.8, 0.8\}, \{0.9\}$
(a, b)	$\{0.7, 0.8, 0.7\}, \{0.8, 0.8\}$
(a, c)	$\{0.8, 0.9, 0.7\}, \{0.9, 0.8\}$
(a, d)	$\{0.7, 0.8, 0.9\}, \{0.8, 0.8\}$
(b, a)	$\{0.6, 0.9, 0.6\}, \{0.9, 0.3, 0.2\}$
(b, b)	$\{0.1, 0.8, 0.4\}, \{0.5, 0.3, 0.8\}$
(b, c)	$\{0.2, 0.9\}, \{0.5, 0.9\}$
(b, d)	$\{0.4, 0.3, 0.9\}, \{0.1, 0.8, 0.1\}$
(c, a)	$\{0.7, 0.9\}, \{0.8, 0.7, 0.9\}$
(c, b)	$\{0.8, 0.5, 0.6\}, \{0.7, 0.8\}$
(c, c)	$\{0.6, 0.4, 0.9\}, \{0.8, 0.7, 0.9\}$
(c, d)	$\{0.7, 0.5, 0.9\}, \{0.8, 0.7, 0.6\}$

Thus, μ_{Φ_i} is an m-polar Q-hesitant anti-fuzzy subalgebra.

Proposition 1. *If every m-polar Q-hesitant anti-fuzzy subalgebra of B satisfies the following inequality:*

$$\forall \chi, \omega \in B, q \in Q : \mu_{\Phi_i}(\chi * \omega, q) \supseteq \mu_{\Phi_i}(\omega, q) \tag{22}$$

then we have

$$\mu_{\Phi_i}(\chi, q) = \mu_{\Phi_i}(0, q) \tag{23}$$

for all $i = 1, 2, \ldots, m$.

Proof. From the BCK/BCI-algebra's definition, we have $\forall\chi \in B, (\chi * 0) = \chi$. Then, $\forall\chi \in B, q \in Q$, we have

$$\mu_{\Phi_i}(\chi, q) = \mu_{\Phi_i}(\chi * 0, q) \subseteq \mu_{\Phi_i}(0, q),$$

and it follows from Proposition 1 that

$$\mu_{\Phi_i}(\chi, q) = \mu_{\Phi_i}(0, q).$$

Proposition 2. *Every m-polar Q-hesitant anti-fuzzy subalgebra of B satisfies the following:*

$$\forall\chi \in B, q \in Q : \mu_{\Phi_i}(\chi, q) \supseteq \mu_{\Phi_i}(0, q) \tag{24}$$

For all $i = 1, 2, \ldots, m$.

Proof. For any $\chi \in B$ and $q \in Q$, we have:

$$\mu_{\Phi_i}(0, q) = \mu_{\Phi_i}(\chi * \chi, q) \subseteq \mu_{\Phi_i}(\chi, q) \cup \mu_{\Phi_i}(\chi, q) = \mu_{\Phi_i}(\chi, q),$$

which completes the proof.

4. m-polar Q-hesitant anti-fuzzy ideal

In this section, we will delve into the fascinating concept of an m-polar Q-hesitant anti-fuzzy ideal. This concept adds a layer of complexity to algebraic structures, allowing us to tackle real-world uncertainties and vagueness. We will begin by defining what an m-polar Q-hesitant anti-fuzzy ideal.

Definition 9. *Let*

$$\Phi_i = \{(\chi, q); \mu_{\Phi_i}(\chi, q)\ j\ \nu \in B, q \in Q\} \tag{25}$$

be a Q-hesitant anti-fuzzy set in B. Then, μ_{Φ_i} *is called an m-polar Q-hesitant anti-fuzzy ideal of B if it satisfies the following conditions:*

(i)

$$\mu_{\Phi_i}(0, q) \subseteq \mu_{\Phi_i}(\chi, q) \tag{26}$$

(ii)

$$\mu_{\Phi_i}(\chi, q) \subseteq \mu_{\Phi_i}(\chi * \omega, q) \cup \mu_{\Phi_i}(\omega, q) \tag{27}$$

for all $(\chi, \omega) \in B; q \in Q$ *and* $i = 1, 2, \ldots, m$.

Example 2. *Let $B = \{a, b, c, d\}$ be a set with a binary operation "$*$," which is given in the following Cayley table:*

$*$	a	b	c	d
a	c	a	a	a
b	b	c	b	b
c	c	c	c	c
d	d	d	d	c

Then $(B, *, c)$ is a BCK-algebra. Define a set $Q = \{a, b, c\}$ and a 3-polar fuzzy set μ_{Φ_i} on B as follows:

(a, a)	$\{0.6, 0.9\}, \{0.8, 0.5\}, \{0.3, 0.8, 0.4\}$
(a, b)	$\{0.5, 0.9, 0.6\}, \{0.9, 0.4\}, \{0.5, 0.9, 0.4, 0.5\}$
(a, c)	$\{0.8, 0.5\}, \{0.4, 0.3, 0.9\}, \{0.8, 0.7\}$
(b, a)	$\{0.3, 0.2, 0.9\}, \{0.4, 0.8\}, \{0.2, 0.1, 0.9\}$
(b, b)	$\{0.4, 0.3, 0.9\}, \{0.3, 0.9\}, \{0.2, 0.9, 0.1\}$
(b, c)	$\{0.1, 0.8\}, \{0.9\}, \{0.5, 0.8\}$
(c, a)	$\{0.9, 0.9\}, \{0.8, 0.7\}; \{0.8\}$
(c, b)	$\{0.9, 0.8\}, \{0.9\}, \{0.7, 0.9\}$
(c, c)	$\{0.8\}, \{0.8, 0.9\}, \{0.7, 0.8\}$
(d, a)	$\{0.7, 0.9\}, \{0.7, 0.8\}, \{0.8\}$
(d, b)	$\{0.7, 0.6, 0.9\}, \{0.9, 0.6\}, \{0.6, 0.9\}$
(d, c)	$\{0.8, 0.7\}, \{0.5, 0.9, 0.5\}, \{0.8, 0.7\}$

It is routine to check that μ_{Φ_i} is a 3-polar Q-hesitant anti-fuzzy ideal.

Theorem 1. *In BCK-algebra B, every m-polar Q-hesitant anti-fuzzy ideal is every m-polar Q-hesitant antifuzzy subalgebra.*

Proof. Let $q \in Q$, and μ_{Φ_i} is a m-polar Q-hesitant fuzzy ideal over B. Then,

$$\begin{aligned}\mu_{\Phi_i}(\chi * \omega, q) &\subseteq \mu_{\Phi_i}((\chi * \omega) * \chi, q) \cup \mu_{\Phi_i}(\chi, q)\\ &= \mu_{\Phi_i}((\chi * \chi) * \omega, q) \cup \mu_{\Phi_i}(\chi, q)\\ &= \mu_{\Phi_i}(0 * \omega, q) \cup \mu_{\Phi_i}(\chi, q)\\ &= \mu_{\Phi_i}(0, q) \cup \mu_{\Phi_i}(\chi, q) \subseteq \mu_{\Phi_i}(\omega, q) \cup \mu_{\Phi_i}(\chi, q)\end{aligned}$$

for $\chi, \omega \in B$, $q \in Q$, and for all $i = 1, 2, \ldots, m$. Thus, μ_{Φ_i} is a m-polar Q-hesitant anti-fuzzy subalgebra over B. This completes the proof.

Proposition 3. *Every m-polar Q-hesitant anti-fuzzy ideal satisfies the following conditions:*

(1) If

$$\chi \leq \omega, \text{ then } \mu_{\Phi_i}(\chi, q) \subseteq \mu_{\Phi_i}(\omega, q) \tag{28}$$

(2)

$$\mu_{\Phi_i}(\chi * \omega, q) \subseteq \mu_{\Phi_i}(\chi * \tau, q) \cup \mu_{\Phi_i}(\tau * \omega, q) \tag{29}$$

(3) If

$$\mu_{\Phi_i}(\chi * \omega, q) = \mu_{\Phi_i}(0, q), \text{ then } \mu_{\Phi_i}(\chi, q) \subseteq \mu_{\Phi_i}(\omega, q) \tag{30}$$

Proof. Let $q \in Q$ and $\chi, \omega, \tau \in B$.

(1) If $\chi \leq \omega$, then $\chi * \omega = 0$ since μ_{Φ_i} is a m-polar Q-hesitant anti-fuzzy ideal of B:

$$\begin{aligned}\mu_{\Phi_i}(\chi, q) &\subseteq \mu_{\Phi_i}(\chi * \omega, q) \cup \mu_{\Phi_i}(\omega, q)\\ &= \mu_{\Phi_i}(0, q) \cup \mu_{\Phi_i}(\omega, q)\\ &= \mu_{\Phi_i}(\omega, q).\end{aligned}$$

(2) Since $(\chi * \omega) * (\chi * \tau) \nleq \tau * \omega$ from (1), we have

$$\begin{aligned}\mu_{\Phi_i}(\chi * \omega, q) &\subseteq \mu_{\Phi_i}((\chi * \omega) * (\chi * \tau), q)\\ &\subseteq \mu_{\Phi_i}(\chi * \tau, q) \cup \mu_{\Phi_i}(\tau * \omega, q)\\ &\subseteq \mu_{\Phi_i}(\chi * \tau, q) \cup \mu_{\Phi_i}(\tau * \omega, q).\end{aligned}$$

(3) If $\mu_{\Phi_i}(\chi * \omega, q) = \mu_{\Phi_i}(0, q)$, then

$$\begin{aligned}\mu_{\Phi_i}(\chi, q) &\subseteq \mu_{\Phi_i}(\chi * \omega, q) \cup \mu_{\Phi_i}(\omega, q)\\ &= \mu_{\Phi_i}(0, q) \cup \mu_{\Phi_i}(\omega, q)\\ &= \mu_{\Phi_i}(\omega, q).\end{aligned}$$

Proposition 4. *All m-polar Q-hesitant anti-fuzzy ideals over B satisfy the following condition:*

$$\forall \chi, \omega, \tau \in B, q \in Q : (\chi, \omega \leq \tau) \Rightarrow \mu_{\Phi_i}(\chi, q) \subseteq \mu_{\Phi_i}(\omega, q) \cup \mu_{\Phi_i}(\tau, q). \tag{31}$$

Proof. Let $\chi, \omega, \tau \in B$ and $q \in Q$ such that $\chi, \omega \leq \tau$. Then, $(\chi * \omega) * \tau = 0$, so:

$$\begin{aligned}\mu_{\Phi_i}(\chi * \omega, q) &\subseteq \mu_{\Phi_i}((\chi * \omega) * \tau, q)\\ &\subseteq \mu_{\Phi_i}(\tau, q).\end{aligned}$$

It follows that:

$$\begin{aligned}\mu_{\Phi_i}(\chi, q) &\subseteq \mu_{\Phi_i}(\chi * \omega, q) \cup \mu_{\Phi_i}(\omega, q)\\ &\subseteq \mu_{\Phi_i}(\omega, q) \cup \mu_{\Phi_i}(\tau, q).\end{aligned}$$

This completes the proof.

Proposition 5. *Every m-polar Q-hesitant anti-fuzzy ideal over BCI-algebra B satisfies the following inequality:*

$$\forall \chi \in B, q \in Q : \mu_{\Phi_i}(0 * (0 * \chi), q) \subseteq \mu_{\Phi_i}(\chi, q). \tag{32}$$

Proof. Let μ_{Φ_i} be an m-polar Q-hesitant anti-fuzzy ideal. Then, for $q \in Q$ and $\chi \in B$:

$$\begin{aligned}\mu_{\Phi_i}(0 * (0 * \chi), q) &\subseteq \mu_{\Phi_i}((0 * (0 * \chi)) * \nu, q) \\ &\subseteq \mu_{\Phi_i}(\chi, q).\end{aligned}$$

This holds for all $i = 1, 2, \ldots, m$.

5. Closed M-polar Q-hesitant Anti-fuzzy Ideal

In this section, we'll explore the concept of a closed m-polar Q-hesitant Anti-fuzzy Ideal. This intriguing concept extends the traditional ideals in algebra to accommodate uncertainty and vagueness, making it a valuable tool for tackling real-world problems. We'll start by defining what a closed m-polar Q-hesitant Anti-fuzzy Ideal.

Definition 10. *A m-polar Q-hesitant anti-fuzzy ideal Φ_i defined as:*

$$\Phi_i = \{(\nu, q); \mu_{\Phi_i}(\nu, q) \ j \ \nu \in B, q \in Q\} \tag{33}$$

of a BCI-algebra is said to be closed if it satisfies the following condition:

$$\mu_{\Phi_i}(\nu, q) \supseteq \mu_{\Phi_i}(0 * \nu, q) \tag{34}$$

for all $\nu \in B$, $q \in Q$, and $i = 1, 2, \ldots, m$.

Example 3. *The m-polar Q-hesitant anti-fuzzy ideal described in Example 4.2 is a closed m-polar Q-hesitant anti-fuzzy ideal.*

Theorem 2. *A m-polar Q-hesitant anti-fuzzy ideal μ_{Φ_i} is said to be closed if and only if it satisfies the following condition:*

$$\forall \chi, \omega \in B, q \in Q : \mu_{\Phi_i}(\chi * \omega, q) \subseteq \mu_{\Phi_i}(\chi, q) \cup \mu_{\Phi_i}(\omega, q) \tag{35}$$

Proof. ($\Rightarrow$) Assume that μ_{Φ_i} is a closed m-polar Q-hesitant anti-fuzzy ideal over a BCI-algebra B. Since $(\chi * \omega) * \chi \leq 0 * \omega$ for all $\chi, \omega \in B$:

$$\begin{aligned}\mu_{\Phi_i}(\chi * \omega, q) &\subseteq \mu_{\Phi_i}((\chi * \omega) * \chi, q) \\ &\subseteq \mu_{\Phi_i}(\chi, q) \cup \mu_{\Phi_i}(0 * \omega, q) \\ &\subseteq \mu_{\Phi_i}(\chi, q) \cup \mu_{\Phi_i}(\omega, q).\end{aligned}$$

for all $\chi \in B$, $q \in Q$, and $i = 1, 2, \ldots, m$. ($\Leftarrow$) Conversely, let μ_{Φ_i} be a m-polar Q-hesitant anti-fuzzy ideal over a BCI-algebra B. Since $\mu_{\Phi_i}(\chi, q) \supseteq \mu_{\Phi_i}(0, q)$ for all $\chi \in B$, $q \in Q$:

$$\begin{aligned}\mu_{\Phi_i}(0 * \chi, q) &\subseteq \mu_{\Phi_i}(0, q) \cup \mu_{\Phi_i}(\chi, q) \\ &= \mu_{\Phi_i}(\chi, q).\end{aligned}$$

Therefore, μ_{Φ_i} is a closed m-polar Q-hesitant anti-fuzzy ideal over a BCI-algebra B for all $i = 1, 2, \ldots, m$.

6. m-polar Q-hesitant Anti-fuzzy Commutative Ideal

In this section, we'll explore a unique algebraic concept the m-polar Q-hesitant Anti-fuzzy Commutative Ideal. This concept is designed to address uncertainties and hesitations that often arise in real-world scenarios, making it a valuable tool for solving complex problems. We'll start by defining what a m-polar Q-hesitant Anti-fuzzy Commutative Ideal.

Definition 11. *A m-polar Q-hesitant anti-fuzzy set Φ_i defined as:*

$$\Phi_i = \{(\chi, q); \mu_{\Phi_i}(\chi, q) \ j \ \nu \in B, q \in Q\} \tag{36}$$

in a BCK-algebra is called a m-polar Q-hesitant anti-fuzzy commutative ideal of B if it satisfies the following conditions:

(i)

$$\mu_{\Phi_i}(0, q) \subseteq \mu_{\Phi_i}(\chi, q) \ \ for \ \ all \ \ \chi \in B. \tag{37}$$

(ii)

$$\mu_{\Phi_i}(\chi*(\omega*(\omega*\chi)), q) \subseteq \mu_{\Phi_i}((\chi*\omega)*\tau, q) \cup \mu_{\Phi_i}(\tau, q) \ \ for \ \ all \ \ \chi, \omega, \tau \in B \ \ and \ \ i = 1, 2, \ldots, m. \tag{38}$$

Example 4. *The m-polar Q-hesitant anti-fuzzy subalgebra described in Example 3.2 is a m-polar Q-hesitant anti-fuzzy commutative ideal.*

Theorem 3. *Every m-polar Q-hesitant anti-fuzzy commutative ideal is a m-polar Q-hesitant anti-fuzzy ideal of B.*

Proof. Let $\chi, \omega, \tau \in B$, $q \in Q$. Let μ_{Φ_i} be a m-polar Q-hesitant anti-fuzzy commutative ideal of B. Then:

$$\begin{aligned} \mu_{\Phi_i}(\chi, q) &= \mu_{\Phi_i}(\chi * (0 * (0 * \chi)), q) \\ &\subseteq \mu_{\Phi_i}((\chi * 0), q) \cup \mu_{\Phi_i}(\tau, q) \\ &= \mu_{\Phi_i}(\tau * \tau, q) \cup \mu_{\Phi_i}(\tau, q) \end{aligned}$$

for all $\chi, \tau \in B$, $q \in Q$. Hence, μ_{Φ_i} is a m-polar Q-hesitant anti-fuzzy ideal.

Theorem 4. *A m-polar Q-hesitant anti-fuzzy ideal of a BCK-algebra is a m-polar Q-hesitant anti-fuzzy commutative ideal if and only if it satisfies the following condition:*

$$\mu_{\Phi_i}(\chi * (\omega * (\chi * \omega)), q) \subseteq \mu_{\Phi_i}(\chi * \omega, q) \tag{39}$$

for all $\chi, \omega \in B$ and $q \in Q$.

Proof. Let $\chi, \omega \in B$ and $q \in Q$.

(i) Suppose that μ_{Φ_i} is a m-polar Q-hesitant anti-fuzzy commutative ideal of B.Taking $\tau = 0$, we have:

$$\begin{aligned}\mu_{\Phi_i}(\chi * (\omega * (\chi * \omega)), q) &\subseteq \mu_{\Phi_i}(\chi * \omega * 0, q) \cup \mu_{\Phi_i}(0, q) \\ &= \mu_{\Phi_i}(\chi * \omega, q)\end{aligned} \tag{i}$$

(ii) Conversely, assume that μ_{Φ_i} satisfies:

$$\mu_{\Phi_i}(\chi * (\omega * (\chi * \omega)), q) \subseteq \mu_{\Phi_i}(\chi * \omega, q) \tag{ii}$$

Combining (i) and (ii), we obtain:

$$\mu_{\Phi_i}(\chi * (\omega * (\chi * \omega)), q) \subseteq \mu_{\Phi_i}((\chi * \omega) * \tau, q) \cup \mu_{\Phi_i}(\tau, q) \tag{39}$$

for all $\chi, \omega, \tau \in B$ and $q \in Q$. Hence, μ_{Φ_i} is a m-polar Q-hesitant anti-fuzzy commutative ideal.

7. m-polar Q-hesitant anti-fuzzy implicative ideal

In this section, we will introduce a unique algebraic concept, the m-polar Q-hesitant Anti-fuzzy Implicative Ideal. This concept is designed to handle complex uncertainties and hesitations that often arise in practical situations. We will explore what this concept entails and how it can be applied to solve real-world problems. This section is essential for those interested in using algebraic structures to address complex, uncertain scenarios.

Definition 12. *A m-polar Q-hesitant anti-fuzzy set*

$$\Phi_i = \{(\chi, q), \mu_{\Phi_i}(\chi, q) \; j \; \nu \in B, q \in Q \; in \; B\} \tag{40}$$

is called a m-polar Q-hesitant anti-fuzzy implicative ideal if it satisfies the following conditions:

(i)

$$\mu_{\Phi_i}(0, q) \subseteq \mu_{\Phi_i}(\chi, q) \tag{41}$$

(ii)

$$\mu_{\Phi_i}(\chi, q) \subseteq \mu_{\Phi_i}((\chi * (\omega * \chi)) * \tau, q) \cup \mu_{\Phi_i}(\tau, q) \tag{42}$$

for all $i = 1, 2, \ldots, m$.

Example 5. *Let* $B = \{\alpha', \beta', \gamma'\}$ *be a BCK-algebra with a binary operation "∗," which is given in the following Cayley table:*

$*$	α'	β'	γ'
α'	α'	α'	α'
β'	β'	α'	β'
γ'	γ'	γ'	α'

Define the set $Q = \{\gamma\}$ and a 3-polar anti-fuzzy set on B as follows:

(α', γ)	$\{0.8, 0.9, 0.8\}, \{0.7, 0.6, 0.9\}, \{0.8\}$
(β', γ)	$\{0.7, 0.8, 0.9\}, \{0.5, 0.4, 0.9\}, \{0.6, 0.8\}$
(γ', γ)	$\{0.6, 0.9, 0.5\}, \{0.9, 0.2, 0.1\}, \{0.8, 0.1, 0.2\}$

Thus, μ_{Φ_i} is a 3-polar Q-hesitant anti-fuzzy implicative ideal.

Theorem 5. *Let B be an implicative BCK-algebra, then every m-polar Q-hesitant anti-fuzzy ideal over B is a m-polar Q-hesitant anti-fuzzy implicative ideal.*

Proof. Let B be an implicative BCK-algebra. It follows that $\chi = \chi * (\omega * \chi)$ for all $\chi, \omega \in$B. Let μ_{Φ_i} be a m-polar Q-hesitant anti-fuzzy ideal, then we have:

$$\mu_{\Phi_i}(\chi, q) \subseteq \mu_{\Phi_i}((\chi * \tau, q) \cup \mu_{\Phi_i}(\tau, q)$$

for all $\chi, \omega, \tau \in$B. Hence, it is a m-polar Q-hesitant anti-fuzzy implicative ideal of B. That is, μ_{Φ_i} is a m-polar Q-hesitant anti-fuzzy implicative ideal of B.

Proposition 6. *In BCK-algebra B, every m-polar Q-hesitant anti-fuzzy implicative ideal is a m-polar Q-hesitant anti-fuzzy ideal.*

Proof. Let μ_{Φ_i} be a m-polar Q-hesitant anti-fuzzy implicative ideal over B. Let $\chi, \omega, \tau \in B$, then:

$$\mu_{\Phi_i}(\chi, q) \subseteq \mu_{\Phi_i}((\chi * (\omega * \chi)) * \tau, q) \cup \mu_{\Phi_i}(\tau, q)$$

Replace $\omega = \chi$, and using $\chi * \chi = 0$, we get:

$$\mu_{\Phi_i}(\chi, q) \subseteq \mu_{\Phi_i}((\chi * (\chi * \chi)) * \tau, q) \cup \mu_{\Phi_i}(\tau, q)$$

for all $\chi, \tau \in B$. Thus, μ_{Φ_i} is a m-polar Q-hesitant anti-fuzzy ideal.

Theorem 6. *Let μ_{Φ_i} be a m-polar Q-hesitant anti-fuzzy ideal of a BCK-algebra B. Then μ_{Φ_i} is a m-polar Q-hesitant anti-fuzzy implicative ideal of B if and only if it satisfies the condition:*

$$\mu_{\Phi_i}(\chi, q) \subseteq \mu_{\Phi_i}(\chi * (\omega * \chi), q) \quad (43)$$

for all $\chi, \omega \in B$, $q \in Q$, and $i = 1, 2, \ldots, m$.

Proof. Assume that μ_{Φ_i} is a m-polar Q-hesitant anti-fuzzy implicative ideal of B. Take $\tau = 0$ in:

$$\mu_{\Phi_i}(\chi, q) \subseteq \mu_{\Phi_i}((\chi * (\omega * \chi)) * \eta, q) \cup \mu_{\Phi_i}(\tau, q)$$
$$= \mu_{\Phi_i}((\chi * (\omega * \chi)) * 0, q) \cup \mu_{\Phi_i}(0, q)$$
$$= \mu_{\Phi_i}((\chi * (\omega * \chi)), q)$$

Conversely, suppose that μ_{Φ_i} satisfies the condition. As μ_{Φ_i} is a m-polar Q-hesitant anti-fuzzy ideal of B, we have:

$$\mu_{\Phi_i}(\chi, q) \subseteq \mu_{\Phi_i}((\chi * (\omega * \chi)), q)$$
$$\subseteq \mu_{\Phi_i}((\chi * (\omega * \chi)), q) \cup \mu_{\Phi_i}(\tau, q)$$

Then μ_{Φ_i} is a m-polar Q-hesitant anti-fuzzy implicative ideal of B, and the proof is completed.

8. m-polar Q-hesitant anti-fuzzy positive implicative ideal

In this section, we will introduce m-polar Q-hesitant Anti-fuzzy positive Implicative Ideals. These mathematical constructs provide a versatile framework to address uncertainty and hesitation in algebraic structures. We will briefly explore the concept, its properties.

Definition 13. *A m-polar Q-hesitant anti-fuzzy set*

$$\Phi_i = \{(\chi, q), \mu_{\Phi_i}(\chi, q) \ j \ \nu \in B, q \in Q\} \tag{44}$$

in a BCK-algebra B is called a m-polar Q-hesitant anti-fuzzy positive implicative ideal of B if it satisfies the following conditions:

(i)

$$\mu_{\Phi_i}(0, q) \subseteq \mu_{\Phi_i}(\chi, q) \tag{45}$$

(ii)

$$\mu_{\Phi_i}(\chi * \tau, q) \subseteq \mu_{\Phi_i}((\chi * \omega) * \tau, q) \cup \mu_{\Phi_i}(\omega * \tau, q) \tag{46}$$

for all $\chi, \omega, \tau \in B$, $q \in Q$, and $i = 1, 2, \ldots, m$.

Example 6. *Let $B = \{i, d, e, a, l\}$ be a BCK-algebra with a binary operation "$*$," which is given in the following Cayley table:*

$*$	i	d	e	a	l
i	i	i	i	i	i
d	d	i	i	i	i
e	e	e	i	i	e
a	a	a	a	i	a
l	l	l	l	l	i

Define the set $Q = \{a', b'\}$ and a 2-polar fuzzy set on B as follows:

(i, a')	$\{0.8, 0.9, 0.8\}, \{0.9\}$
(i, b')	$\{(0.8, 0.9), (0.8, 0.8)\}$
(d, a')	$[0.5, 0.9], [0.7, 0.9]$
(d, b')	$(0.7, 0.9), (0.7, 0.6, 0.8]$
(e, a')	$(0.3, 0.9), [0.9]$
(e, b')	$(0.1, 0.3, 0.2, 0.9), (0.2, 0.8)$
(a, a')	$(0.2, 0.9), [0.3, 0.9]$
(a, b')	$(0.3, 0.9), (0.2, 0.8)$
(l, a')	$(0.4, 0.9), [0.5, 0.9]$
(l, b')	$(0.6, 0.9), [0.8]$

Thus, μ_{Φ_i} is a 2-polar Q-hesitant anti-fuzzy positive implicative ideal.

Proposition 7. *In BCK-algebra B, every m-polar Q-hesitant anti-fuzzy positive implicative ideal is an m-polar Q-hesitant anti-fuzzy ideal.*

Proof. Let μ_{Φ_i} be an m-polar Q-hesitant anti-fuzzy positive implicative ideal of BCK-algebra B. For all $\chi, \omega, \tau \in B$, and $q \in Q$, we have:

$$\mu_{\Phi_i}(\chi * \tau, q) \subseteq \mu_{\Phi_i}((\chi * \omega) * \tau, q) \cup \mu_{\Phi_i}(\omega * \tau, q)$$

Now, put $\tau = 0$:

$$\mu_{\Phi_i}(\chi, q) \subseteq \mu_{\Phi_i}((\chi * \omega) * \eta, q) \cup \mu_{\Phi_i}(\omega, q)$$

Therefore, μ_{Φ_i} is an m-polar Q-hesitant anti-fuzzy ideal. This concludes the proof.

Proposition 8. *If B is a positive implicative BCK-algebra, then every m-polar Q-hesitant anti-fuzzy ideal of B is an m-polar Q-hesitant anti-fuzzy positive implicative ideal of B.*

Proof. Assume that μ_{Φ_i} is an m-polar Q-hesitant anti-fuzzy positive implicative ideal of B. For all $\chi, \omega \in B$, and $q \in Q$, we have:

$$\mu_{\Phi_i}(\chi, q) \subseteq \mu_{\Phi_i}(\chi * \omega, q) \cup \mu_{\Phi_i}(\omega; q)$$

By replacing χ with $\chi * \tau$ and ω with $\omega * \tau$, we get:

$$\mu_{\Phi_i}(\chi * \tau, q) \subseteq \mu_{\Phi_i}((\chi * \tau) * (\omega * \tau), q) \cup \mu_{\Phi_i}(\omega * \tau, q)$$

Since B is a positive implicative BCK-algebra, $(\chi * \tau) * (\omega * \tau) = (\chi * \omega) * \tau$ for all $\chi, \omega, \tau \in B$. Hence, we have:

$$\mu_{\Phi_i}(\chi * \tau, q) \subseteq \mu_{\Phi_i}((\chi * \omega) * \tau, q) \cup \mu_{\Phi_i}(\omega * \tau, q)$$

Thus, μ_{Φ_i} is an m-polar Q-hesitant anti-fuzzy positive implicative ideal of B for all $i = 1, 2, \ldots, m$.

Theorem 7. *Let μ_{Φ_i} be an m-polar Q-hesitant anti-fuzzy ideal over B. Then μ_{Φ_i} is an m-polar Q-hesitant anti-fuzzy positive implicative ideal if and only if:*

$$\mu_{\Phi_i}(\chi * \omega, q) \subseteq \mu_{\Phi_i}((\chi * \omega) * \omega, q) \tag{47}$$

for all $\chi, \omega, \tau \in B$, $q \in Q$, and $i = 1, 2, \ldots, m$.

Proof. (⇒) Suppose that the m-polar Q-hesitant anti-fuzzy ideal μ_{Φ_i} of B is a m-polar Q-hesitant anti-fuzzy positive implicative ideal. So, we have:

$$\mu_{\Phi_i}(\chi * \tau, q) \subseteq \mu_{\Phi_i}((\chi * \omega) * \tau, q) \cup \mu_{\Phi_i}(\omega * \tau, q)$$

If we put $\tau = \omega$, we have:

$$\mu_{\Phi_i}(\chi * \omega, q) \subseteq \mu_{\Phi_i}((\chi * \omega) * \omega, q) \cup \mu_{\Phi_i}(\omega * \omega, q)$$

But, in a BCK-algebra, $\omega * \omega = 0$. So:

$$\mu_{\Phi_i}(\chi * \omega, q) \subseteq \mu_{\Phi_i}((\chi * \omega) * \omega, q)$$

For all $\chi, \omega \in B$, $q \in Q$, and $i = 1, 2, \ldots, m$.

(⇐) Conversely Suppose that μ_{Φ_i} is an m-polar Q-hesitant anti-fuzzy ideal over B and satisfies the inequality:

$$\mu_{\Phi_i}(\chi * \omega, q) \subseteq \mu_{\Phi_i}((\chi * \omega) * \omega, q)$$

Since $\mu_{\Phi_i}(\chi', \omega', q) \subseteq \mu_{\Phi_i}(\chi, q)$, we can now prove that for all $\chi, \omega, \tau \in B$:

$$\mu_{\Phi_i}(\chi * \tau, q) \subseteq \mu_{\Phi_i}((\chi * \omega) * \tau, q) \cup \mu_{\Phi_i}(\omega * \tau, q)$$

In contrast, if there exist $\chi^{'}, \omega^{'} \in B$ such that:

$$\mu_{\Phi_i}(\chi', \omega', q) \subseteq \mu_{\Phi_i}((\chi', \omega') * \omega', q)$$

It implies that:

$$\mu_{\Phi_i}(\chi', \omega', q) \subseteq \mu_{\Phi_i}(0, q)$$

Which is a contradiction. Therefore,

$$\mu_{\Phi_i}(\chi * \tau, q) \subseteq \mu_{\Phi_i}((\chi * \omega) * \tau, q) \cup \mu_{\Phi_i}(\omega * \tau, q)$$

For all $\chi, \omega, \tau \in B$ and $q \in Q$. Thus, μ_{Φ_i} is an m-polar Q-hesitant anti-fuzzy positive implicative ideal of B for all $i = 1, 2, \ldots, m$.

9. Conclusion

The exploration of m-polar Q-hesitant anti-fuzzy sets in the context of BCK-algebras represents a substantial stride in uniting the realms of algebraic structures and fuzzy set theories. This endeavor expands the boundaries of fuzzy set concepts within abstract algebra, particularly within the framework of BCK-algebras.

As we look ahead to future research in this direction, several intriguing questions arise:

- Can we formulate the notion of BCK-algebra ideal spaces within the domain of general topology?
- Is it feasible to extend the concept of m-polar Q-hesitant anti-fuzzy sets to BCK-algebra ideal spaces, thereby establishing connections between algebraic structures and topological spaces?
- Might we introduce the concept of BCK-algebra topological spaces, creating a bridge between abstract algebra and general topology?
- Could the powerful concept of m-polar Q-hesitant anti-fuzzy sets find applications in the domain of BCK-algebra topological spaces, leading to innovative approaches to algebraic structures in the context of topological settings?

These questions open numerous things for further exploration and research, offering exciting possibilities for interdisciplinary investigations that integrate algebraic structures, fuzzy set theories, and topological spaces. The continuous evolution and interplay of these theories hold the potential for groundbreaking insights and practical applications.

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