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Oscillatory Behavior of Higher-Order DEs with Delay Terms

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Abstract. The aim of this research is to study the oscillatory properties of higher -order delay half linear differential equations with non-canonical operators. Two techniques for establishing new oscillation conditions for all solutions of higher-order differential equations will be presented. The first method to employ the Riccati transformations, which differ from those described in some published works. The second method employs comparison principles with first-order delay differential equations, from which it is straightforward to deduce oscillation for all studied equation solutions. In addition to improving, extending, and significantly simplifying the previously established criteria, the newly proposed criteria have the potential to serve as a benchmark for the theory of delay differential equations of higher order, which is still in its infancy of development. We were able to determine three fundamental theorems regarding the oscillation of this equation. Some examples will be provided to illustrate the findings.

2020 Mathematics Subject Classifications: 34K10, 34K11

Key Words and Phrases: Delay, Differential equations, Oscillation, Higher-order comparison method, Riccati transformation

1. Introduction

Delay differential equations (DDEs) are a mathematical tool employed to address a wide range of challenges in various fields such as physics, medicine, engineering, aviation, and biology. In addition, they are utilized for the purpose of producing cardiac rhythms

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and facilitating the oscillations of bridges. Symmetric properties can have an impact on the Euler equation in certain variational problems. The selection of an optimal solution methodology for the given equation is facilitated, as evidenced in references [9, 16]. The findings of [2, 10, 11, 22] as well as [25] were extended with respect to a wide range of classes of 2nd order non-linear equations. In 2016, some new oscillatory behaviors of a class of non-linear second-order neutral DEs were achieved, given by:

$$\left(\mu(u)\left((x(u)+p(u)x(\tau(u)))'\right)^{\alpha}\right)'+q(u)x^{\alpha}(\sigma(u))=0.$$

Utilizing the generalised Riccati transformations and comparison method with the first order DDE, some additional oscillatory criteria were discovered. A class of second order DDE solutions oscillations is addressed in the findings, which extend and enhance numerous previous findings in the field. Examples are used to demonstrate how well the proposed criteria work. The manner in which the study's findings are presented is both fundamentally novel and highly generic (**(author?)** [15]).

In order to solve a class of second-order half-linear neutral DEs having delayed arguments of the type, **(author?)** [19] developed new oscillation criteria:

$$\left(\mu(u)\left(\varpi'(u)\right)^{\alpha}\right)'+q(u)x^{\alpha}(\sigma(u))=0, \quad u\geq u_0.$$

Here, including those for non-neutral DEs, it substantially enhanced the renowned findings given in the work. By accounting for the portion of the delay's total impact that was overlooked in the previous findings, the method adopted improves the traditional Riccati transformation technique. The oscillation of second-order DDEs was researched by **(author?)** [5], given as follows

$$\left[a(y)w'(y)\right]'+q(y)f(w(\tau(y)))=0, \quad y\geq y_0.$$

Using the generalized Riccati substitution, new oscillation criterion was developed. However, it is not obvious how symmetry considerations aid in choosing the most appropriate method of inquiry. **(author?)** [12] presented oscillation criteria for the third-order non-linear DDE

$$\left[c_2(u)\left\{\left(c_1(u)\left(x'(u)\right)^{\alpha_1}\right)'\right\}^{\alpha_2}\right]'+q(u)g(x(f(u)))=0.$$

They rely on novel comparison concepts that make it possible to determine the characteristics of the first-order non-linear DDE's oscillation in the third-order non-linear DE. The solutions' oscillation to a particular class of third-order non-linear DDE of the type was examined by **(author?)** [21], given by

$$\varpi'''(u)+p(u)\varpi'(u)+q(u)f(\varpi(\tau(u)))=0.$$

The newly presented technique could perhaps act as a benchmark in the less studied theory of non-canonical equations of higher order. The criteria not only improved but also extended and greatly simplified the current ones. The significance of the results is demonstrated by the Euler-type equations, which are crucial to the oscillation theory because they are typically used to compare the merit of various criteria. The study of non-canonical equations was significantly streamlined by the recently developed method. It is still unclear how to apply these findings to higher-order non-canonical equations.

Less attention has been paid in the literature to the determination of the qualitative behavior of fourth-order DE, particularly the fourth-order DDE. Nevertheless, some fourth-order DE findings are well known and have some applications in physics and biology mathematical modelling. The fourth-order DE's oscillatory behavior has been examined by **(author?)** [20], given by

$$\left(a(u)\left(x'(u)\right)^{\alpha}\right)''' + q(u)f(x(g(u))) = 0.$$

Furthermore, **(author?)** [13] studied the asymptotic behavior with respect to the solutions of higher-order DE and derived a new oscillation criterion expressed by

$$\left(\mu(v)\left(w^{(m-1)}(v)\right)^{\alpha}\right)' + p(v)f\left(w^{(m-1)}(v)\right) + q(v)g(w(\sigma(v))) = 0.$$

In order to create new oscillation conditions for a specific even order DDE, **(author?)** [26] used the generalized Riccati approach and the integral averaging technique expressed by

$$\left(\left|x^{(n-1)}(h)\right|^{e-1} x^{(n-1)}(h)\right)' + F(h, x[g(h)]) = 0, \quad (n \text{ even}).$$

(author?) [7] established new criteria with regard to the oscillatory behaviour of even order DDEs containing the neutral component by using the comparison approach, the Riccati transformation, and the integral averaging method. All three of these methods are statistical in nature.

$$\left(\gamma(u)\varpi^{(r-1)}(u)\right)' + \sum_{i=1}^{j} a_i(u)\varphi\left(h\left(w_i(u)\right)\right) = 0.$$

The researchers utilized the Riccati transformation and integral averaging technique to extend the results presented in the study under

$$\int_{t_0}^{\infty} \frac{1}{\gamma(a)}\,\mathrm{d}a = \infty.$$

(author?) [1] presented several oscillatory properties of higher-order non-linear DE with a middle term

$$\left(\alpha_1(\varepsilon)\left(w^{(j-1)}(\varepsilon)\right)^{\gamma}\right)' + \alpha_2(\varepsilon)\left(w^{(j-1)}(\varepsilon)\right)^{\gamma} + \sum_{i=1}^{n} \sigma_i(\varepsilon) w^{\gamma}\left(\beta_i(\varepsilon)\right) = 0.$$

The subsequent condition is met:

$$\int_{\varepsilon_0}^{\infty}\left(\frac{1}{\alpha_1(\varrho)}\exp\left(-\int_{z_0}^{s}\frac{\alpha_2(x)}{\alpha_1(x)}dx\right)\right)^{1/Y} d\varrho = \infty.$$

They produced several novel oscillation results that expanded upon and enhanced existing findings in the literature. Their findings do not necessitate that $\tau'(u) \geq 0$ in order to guarantee that all solutions to the equation is oscillatory or that it approaches zero as u tends to ∞, where some necessary conditions were created. In 2020 **(author?)** [14] established new oscillation results of solutions to a class of even-order advanced differential equations with a p-Laplacian like operator.

$$\left(a(v)\left|y^{(\kappa-1)}(v)\right|^{p-2} y^{(\kappa-1)}(v)\right)' + \sum_{i=1}^{j} q_i(v) g\left(y\left(\eta_i(v)\right)\right) = 0, v \geq v_0.$$

Riccati transformation and the theory of comparison with first and second-order delay equation had been used. In addition, their results were continue by **(author?)** [3] discussed the properties of non-oscillatory solutions of neutral differential equations related to p-Laplacian operators

$$\left(\varphi(1)\left(y'''(1)\right)^{p-1}\right)' + \omega_1(1) w^{p-1}\left(\omega_2(1)\right) = 0,$$

by applying the comparison method. Recently, the G$alpha$-transform in **(author?)** [24] was used to study, solutions of higher-order differential equations with polynomial coefficients (HODEPCs) and based on some characterizations,the solutions of HODEPCs were investigated. **(author?)** [4] studied n-th order neutral nonlinear differential equation

$$\left[r(t)[x(t)-p(t)x(t-\tau)]^{(n-1)}\right]' + (-1)^n\left[f_1\left(t, x\left(\sigma_1(t)\right)\right) - f_2\left(t, x\left(\sigma_2(t)\right)\right) - g(t)\right] = 0,$$

They used the Banach contraction principle and some sufficient conditions are established for the existence of nonoscillatory solutions.

The theory of higher-order differential equations has many connections with various branches of mathematics and applied sciences (we recall for example the models of suspension bridge and noise removal). The present investigation aims to establish specific oscillation and asymptotic criteria for delay terms of higher order in the structure of half-linear equations. The oscillatory behaviour of the following is the subject of our study,

$$\left(r(u)\left(\varpi^{(n-1)}(u)\right)^{\alpha}\right)' + \sum_{i=1}^{m} q_i(u) \varpi^{\beta}\left(\tau_i(u)\right) = 0, \quad u \geq u_0, \tag{1}$$

under the conditions

$$\int_{u_0}^{\infty} \frac{1}{r^{1/\alpha}(u)} \mathrm{d}u < \infty, \text{ and } r'(u) \geq 0. \tag{2}$$

In this work we suppose that

- α, β, where $\beta \leq \alpha$, are the ratios of odd positive integers,
- $r(u) \in C^1\left[u_0, \infty\right),\ r(u) > 0$,
- $q_i(u)$, $\tau_i(u) \in C\left[u_0, \infty\right), q_i(u) > 0$, $\tau_i(u) < u$ and $\lim_{u \to \infty} \tau_i(u) = \infty$, $i = 1, \ldots, m$.

A solution of equation (1) is defined as a function that exhibits the property $r(u)\left(\varpi^{(n-1)}(u)\right)^{\alpha} \in C^{1}\left[T_z, \infty\right)$ and satisfies the equation on the interval $[T_z, \infty)$. The solutions of (1) that satisfy $\sup\{|\varpi(u)| : u \geq T\} > 0$ for all $T \geq T_z$ are the only ones that we consider. It is postulated that a viable solution exists for equation (1).

Definition 1. *A solution $\varpi(u)$ of (1) is called oscillatory if it has arbitrary large zeros on $[T_z, \infty)$, and otherwise, it is said to be nonoscillatory.*

We point out that there are only two cases in the investigation of the asymptotic behaviour of the positive solutions of (1):

$$\text{Case 1}: \quad \varpi(u) > 0,\ \varpi^{(n-1)}(u) > 0,\ \varpi^{(n)}(u) < 0,\ \left(r(u)\left(\varpi^{(n-1)}(u)\right)^{\alpha}\right)' < 0,$$

$$\text{Case 2}: \quad \varpi(u) > 0,\ \varpi^{(n-2)}(u) > 0,\ \varpi^{(n-1)}(u) < 0,\ \left(r(u)\left(\varpi^{(n-1)}(u)\right)^{\alpha}\right)' < 0.$$

The following lemma will serve as our starting point.

Lemma 1. *[17]. Let $g \in C^{m}\left([h_0, \infty), \mathbb{R}^{+}\right)$ such that $g^{(m-1)}(h)g^{(m)}(h) \leq 0$ for all $h \geq h_1$. If $\lim_{h\to\infty} g(h) \neq 0$, $\forall\lambda \in (0,1)$, there exists $h_\lambda \in [h_1, \infty)$ such that*

$$g \geq \frac{\lambda}{(m-1)!} h^{m-1}\left|g^{(m-1)}\right|. \tag{3}$$

For convenience, we write

$$\begin{aligned} \delta(\eta) &= \int_t^\infty \left(\int_v^\infty \left[\frac{1}{r(x)}\int_x^\infty \sum_{i=1}^{m} q_i(s)\left(\frac{\tau_i(s)}{s}\right)^{\beta} \mathrm{d}s\right]^{1/\alpha} \mathrm{d}x\right) \mathrm{d}v, \\ \phi(\sigma) &= \frac{\int_t^\infty (\eta-u)^{n-4}\delta(\eta)\,\mathrm{d}\eta}{(n-4)!}. \end{aligned}$$

2. Oscillation criteria

In this section, we will determine some oscillation criteria for (1).

Theorem 1. *Assume that $n \geq 2$ and (2) holds.If*

$$w'(u) + \sum_{i=1}^{m} q_i(u)\left(\frac{\lambda_0 \tau_i^{n-1}(u)}{(n-1)! r^{\frac{1}{\alpha}}\left(\tau_i(u)\right)}\right)^{\beta} w^{\frac{\beta}{\alpha}}\left(\tau_i(u)\right) = 0, \tag{4}$$

is oscillatory for $\lambda_0 \in (0,1)$, then (1) is oscillatory.

Proof. Assume that the non-oscillatory solution ϖ to equation (1) exists. We can make the assumption that ϖ will eventually be positive without losing generality. Let

$$w(u) := r(u)\left(\varpi^{(n-1)}(u)\right)^{\alpha},$$

which when combined with (1) yields

$$w'(\mathrm{u})+\sum_{i=1}^{m} q_i(u)\varpi^{\beta}\left(\tau_i(u)\right)=0. \tag{5}$$

Since $\lim_{u\to\infty}\varpi(u)\neq 0$ and by Lemma (1), we get

$$\left(\varpi\left(\tau_i(u)\right)\right)^{\beta}\geq \frac{\lambda^{\beta}\tau_i^{\beta n-\beta}}{((n-1)!)^{\beta} r^{\frac{\beta}{\alpha}}\left(\tau_i(u)\right)}\left(r^{\frac{1}{\alpha}}\left(\tau_i(\mathrm{u})\right)\varpi^{(n-1)}\left(\tau_i(\mathrm{u})\right)\right)^{\beta},\quad \forall\lambda\in(0,1) \tag{6}$$

From (5) and (6), it can be observed that

$$w'(\mathrm{u})+\sum_{i=1}^{m} q_i(u)\frac{\lambda^{\beta}\tau_i^{\beta n-\beta}}{((n-1)!)^{\beta} r^{\frac{\beta}{\alpha}}\left(\tau_i(u)\right)}\left(r^{\frac{1}{\alpha}}\left(\tau_i(\mathrm{u})\right)\varpi^{(n-1)}\left(\tau_i(\mathrm{u})\right)\right)^{\beta}\leq 0.$$

So, we obtain $w(u)>0$ and

$$w'(u)+\sum_{i=1}^{m} q_i(u)\left(\frac{\lambda\tau_i^{n-1}(u)}{(n-1)!r^{1/\alpha}\left(\tau_i(u)\right)}\right)^{\beta} w^{\beta/\alpha}\left(\tau_i(u)\right)\leq 0.$$

By applying Corollary 1 from reference [8], it can be observed that (4) possesses a solution that is positively valued. This leads to a clear contradiction, thereby establishing the completion of the proof.

Theorem 2. *Let $n\geq 2$, and assume (2) holds and there exists a constant $\lambda_0\in(0,1)$. If*

$$\limsup_{u\to\infty}\int_{u_0}^{u}\left[H^{\beta-\alpha}\sum_{i=1}^{m} q_i(s)\left(\frac{\lambda_1}{(n-2)!}\tau_i^{n-2}(s)\right)^{\beta}\theta^{\alpha}(s)-\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\frac{1}{\theta(s)r^{1/\alpha}(s)}\right]\mathrm{d}s=\infty \tag{7}$$

for some $\lambda_1\in(0,1)$ and $\forall H>0$, then each solution to (1) is oscillatory or converges to zero, i.e.

$$\theta(u):=\int_{\mathrm{u}}^{\infty}\frac{1}{r^{1/\alpha}(s)}\,\mathrm{d}s.$$

Proof. Let

$$\omega(u)=\frac{\left(\varpi^{(n-1)}(u)\right)^{\alpha}}{\left(r^{(-1/\alpha)}(u)\varpi^{(n-2)}(u)\right)^{\alpha}}. \tag{8}$$

$\omega(u)<0$ for $u\geq u_1$.
Divided (8)by $r^{1/\alpha}(s)$ and integrated from u to ζ yields

$$\varpi^{(n-2)}(\zeta)\leq\varpi^{(n-2)}(u)+r^{1/\alpha}(u)\varpi^{(n-1)}(u)\int_{t}^{\zeta}\frac{1}{r^{1/\alpha}(s)}\mathrm{d}s.$$

Given that the function $r(u)\left(\varpi^{(n-1)}(u)\right)^{\alpha}$ has a decreasing trend, it can be concluded that

$$r^{1/\alpha}(s)\varpi^{(n-1)}(s) \leq r^{1/\alpha}(u)\varpi^{(n-1)}(u),\ s \geq u \geq u_1.$$

Hence we obtain

$$0 \leq r^{1/\alpha}(u)\varpi^{(n-1)}(u)\theta(u) + \varpi^{(n-2)}(u),$$

when $\zeta \to \infty$. This leads to

$$-\frac{r^{1/\alpha}(u)\varpi^{(n-1)}(u)}{\varpi^{(n-2)}(u)}\theta(u) \leq 1.$$

Therefore, based on (8), it can be observed that

$$-\omega(u)\theta^{\alpha}(u) \leq 1. \tag{9}$$

According to (8)

$$\omega'(u) = \frac{\left(r(u)\left(\varpi^{(n-1)}(u)\right)^{\alpha}\right)'}{\left(\varpi^{(n-2)}(u)\right)^{\alpha}} - \alpha\frac{r(u)\left(\varpi^{(n-1)}(u)\right)^{\alpha+1}}{\left(\varpi^{(n-2)}(u)\right)^{\alpha+1}}. \tag{10}$$

Employing Lemma 1 yields

$$\frac{\varpi(u)}{\varpi^{(n-2)}(u)} \geq \frac{\lambda}{(n-2)!}u^{n-2}, \quad \forall\lambda \in (0,1).$$

Then a constant $H > 0$ exists so that

$$\begin{aligned}\omega'(u) &= -\sum_{i=1}^{m} q_i(u)\left(\varpi^{(n-2)}\left(\tau_i(u)\right)\right)^{\beta-\alpha}\frac{\varpi^{\beta}\left(\tau_i(u)\right)}{\left(\varpi^{(n-2)}\left(\tau_i(u)\right)\right)^{\beta}}\frac{\left(\varpi^{(n-2)}\left(\tau_i(u)\right)\right)^{\alpha}}{\left(\varpi^{(n-2)}(u)\right)^{\alpha}} \\ &\quad -\alpha\frac{\omega^{(\alpha+1)/\alpha}(u)}{r^{1/\alpha}(u)}, \\ &\leq -H^{\beta-\alpha}\sum_{i=1}^{m} q_i(u)\left(\frac{\lambda}{(n-2)!}\tau_i^{n-2}(u)\right)^{\beta} - \alpha\frac{\omega^{(\alpha+1)/\alpha}(u)}{r^{1/\alpha}(u)}.\end{aligned} \tag{11}$$

Multiplying this inequality by $\theta^{\alpha}(u)$ and integrating it, we obtain

$$\begin{aligned}&\theta^{\alpha}(u)\omega(u) - \theta^{\alpha}(u_1)\,\omega(u_1) + \alpha\int_{t_1}^{u} r^{-\frac{1}{\alpha}}(s)\theta^{\alpha-1}(s)\omega(s)\,\mathrm{d}s \\ &+ \int_{u_1}^{u} H^{\beta-\alpha}\sum_{i=1}^{m} q_i(s)\left(\frac{\lambda}{(n-2)!}\tau_i^{n-2}(s)\right)^{\beta}\theta^{\alpha}(s)\,\mathrm{d}s + \int_{u_1}^{u}\alpha\frac{\omega^{(\alpha+1)/\alpha}(s)}{r^{1/\alpha}(s)}\theta^{\alpha}(s)\,\mathrm{d}s \leq 0.\end{aligned}$$

Now let $B := r^{-1/\alpha}(s)\theta^{\alpha-1}(s), A := \theta^{\alpha}(s)/r^{1/\alpha}(s)$ and $\varepsilon := -\omega(s)$. When we use the inequality

$$A\varepsilon^{(\alpha+1)/\alpha} \geq -\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1} + A^{\alpha}B}{A^{\alpha}}\varepsilon,$$

we have

$$\int_{u_1}^{u}\left[H^{\beta-\alpha}\sum_{i=1}^{m}q_i(s)\left(\frac{\lambda}{(n-2)!}{\tau_i}^{n-2}(s)\right)^{\beta}\theta^{\alpha}(s)-\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\frac{1}{\theta(s)r^{1/\alpha}(s)}\right]\mathrm{d}s\le\theta^{\alpha}\left(u_1\right)\omega\left(u_1\right)+1,$$

due to (9), which contradicts (7). This completes the proof.

The next Corollary presented is derived from the oscillation of (4) and Theorem 2.1.1 of [23].

Corollary 1. *Let $n \geq 2$. Suppose that (2) holds and $\alpha = \beta$. If*

$$\liminf_{u\to\infty}\int_{\tau(u)}^{u}\sum_{i=1}^{m}q_i(s)\frac{\left(\tau_i^{n-1}(s)\right)^{\alpha}}{r\left(\tau_i(s)\right)}\,\mathrm{d}s>\frac{((n-1)!)^{\alpha}}{\mathrm{e}},\tag{12}$$

and

$$\limsup\int_{u_0}^{u}\left[\sum_{i=1}^{m}q_i(s)\left(\frac{\lambda_1}{(n-2)!}\tau_i^{n-2}(s)\right)^{\alpha}\theta^{\alpha}(s)-\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\frac{1}{\theta(s)r^{1/\alpha}(s)}\right]\mathrm{d}s=\infty,\tag{13}$$

for $\lambda \in (0, 1)$, then all solutions of (1) are oscillatory or tend to zero.

The following Corollary is derived from (4) and Theorem 1 of [6].

Corollary 2. *Let $n \geq 2$. Suppose that (2) holds and*

$$\limsup_{u\to\infty}\int_{\tau(u)}^{u}\sum_{i=1}^{m}q_i(s)\frac{\left(\tau_i^{n-1}(s)\right)^{\beta}}{r^{\beta/\alpha}\left(\tau_i(s)\right)}\,\mathrm{d}s>0.\tag{14}$$

for $\alpha > \beta$ and τ is an increasing function. If (7) holds for some $\lambda \in (0, 1)$ and $\forall H > 0$, then all solutions of (1) are oscillatory or tend to zero.

Theorem 3. *If all solutions of*

$$\varphi'(u)+H^{\beta-\alpha}\sum_{i=1}^{m}q_i(u)+\alpha\frac{\int_t^{\infty}(\eta-u)^{n-4}\delta(\eta)\,\mathrm{d}\eta}{(n-4)!}\varphi^{\frac{\alpha+1}{\alpha}}(u)=0\tag{15}$$

are oscillatory, then (1) is oscillatory.

Proof. By Lemma 1 and Case 2, and integrating (1) from $u \to a$, we have

$$r(a)\left(\varpi^{(\mathrm{n}-1)}(a)\right)^{\alpha}=r(u)\left(\varpi^{(n-1)}(u)\right)^{\kappa}-\int_{u}^{a}\sum_{i=1}^{m}q_i(s)\varpi^{\beta}\left(\tau_i(s)\right)\mathrm{d}s.\tag{16}$$

By Lemma 1 we get

$$\frac{\varpi\left(\tau_i(u)\right)}{\varpi(u)}\ge\lambda\frac{\tau_i(u)}{u},$$

which with (16) gives

$$r(a)\left(\varpi^{(\mathrm{n}-1)}(a)\right)^{\alpha}-r(u)\left(\varpi^{(n-1)}(u)\right)^{\alpha}+\lambda^{\beta}\int_{u}^{a}\sum_{i=1}^{m}q_i(s)\left(\frac{\tau_i(s)}{s}\right)^{\beta}\varpi^{\beta}(s)\,\mathrm{d}s\leq 0. \quad (17)$$

Since $\varpi'>0$, we find

$$r(a)\left(\varpi^{(\mathrm{n}-1)}(a)\right)^{\alpha}-r(u)\left(\varpi^{(n-1)}(u)\right)^{\alpha}+\lambda^{\beta}\varpi^{\beta}(s)\int_{u}^{a}\sum_{i=1}^{m}q_i(s)\left(\frac{\tau_i(s)}{s}\right)^{\beta}\mathrm{d}s\leq 0. \quad (18)$$

Taking $a\to\infty$, we obtain

$$-r(u)\left(\varpi^{(n-1)}(u)\right)^{\alpha}+\lambda^{\beta}\varpi^{\beta}(s)\int_{t}^{\infty}\sum_{i=1}^{m}q_i(s)\left(\frac{\tau_i(s)}{s}\right)^{\beta}\mathrm{d}s\leq 0,$$

that is

$$\varpi^{(n-1)}(u)\geq\frac{\lambda^{\beta/\alpha}}{r^{1/\alpha}(u)}\varpi^{\beta/\alpha}(u)\left(\int_{u}^{\infty}\sum_{i=1}^{m}q_i(s)\left(\frac{\tau_i(s)}{s}\right)^{\beta}\mathrm{d}s\right)^{1/\alpha}.$$

Integrating from u to ∞,

$$-\varpi^{(\mathrm{n}-2)}(u)\geq\lambda^{\beta/\alpha}\varpi^{\beta/\alpha}(u)\int_{u}^{\infty}\left(\frac{1}{r(x)}\int_{x}^{\infty}\sum_{i=1}^{m}q_i(s)\left(\frac{\tau_i(s)}{s}\right)^{\beta}\mathrm{d}s\right)^{\frac{1}{\alpha}}\mathrm{d}x.$$

Integrating from u to ∞, we find

$$-\varpi^{(\mathrm{n}-3)}(u)\geq\lambda^{\beta/\alpha}\varpi^{\beta/\alpha}(u)\int_{u}^{\infty}\left(\int_{v}^{\infty}\left(\frac{1}{r(x)}\int_{x}^{\infty}\sum_{i=1}^{m}q_i(s)\left(\frac{\tau_i(s)}{s}\right)^{\beta}\mathrm{d}s\right)^{\frac{1}{\alpha}}\mathrm{d}x\right)\mathrm{d}v.$$

Integrating the above inequality $(n-4)$ times from u to ∞, we get

$$-\varpi'(u)\geq\frac{-r^{1/\alpha}\lambda^{\beta/\alpha}(u)\varpi^{(n-1)}(u)}{(n-4)!}\int_{u}^{\infty}(\eta-u)^{n-4}\delta(\eta)\,\mathrm{d}\eta. \quad (19)$$

In the same way, integrating (19) from u to ∞ implies that

$$\varpi(u)\geq\frac{-r^{\frac{1}{\alpha}}(u)\lambda^{\beta/\alpha}(u)\varpi^{(n-1)}(u)}{(n-3)!}\int_{u}^{\infty}(\eta-u)^{n-3}\delta(\eta)\,\mathrm{d}\eta.$$

Define φ by

$$\varphi(u):=\frac{r(u)\left(\varpi^{(n-1)}(u)\right)^{\alpha}}{(\varpi(u))^{\alpha}},\ u\geq u_1. \quad (20)$$

Then $\varphi(u)<0$ for $u\geq u_1$. Differentiating (20), we have

$$\varphi'(u)=\frac{\left(r(u)\left(\varpi^{(n-1)}(u)\right)^{\alpha}\right)'}{(\varpi(u))^{\alpha}}-\alpha\frac{r(u)\left(\varpi^{(n-1)}(u)\right)^{\alpha}\varpi'(u)}{(\varpi(u))^{\alpha+1}}.$$

It can be concluded from equations (1) and (19) that

$$\varphi'(u) \leq -\sum_{i=1}^{m} q_i(u) \frac{\varpi^{\beta}(\tau_i(u))}{(\varpi(u))^{\alpha}} - \alpha \frac{\int_u^{\infty} (\eta - u)^{n-4} \delta(\eta)\, \mathrm{d}\eta}{(n-4)!} \varphi^{(\alpha+1)/\alpha}(u).$$

Recalling $\tau(u) < u$ and $\varpi' < 0$, then there exists a constant $H > 0$ such that

$$\varphi'(u) \leq -\sum_{i=1}^{m} q_i(u) \frac{\varpi^{\alpha}(\tau_i(u))}{(\varpi(u))^{\alpha}} \varpi^{\beta-\alpha}(\tau_i(u)) - \alpha \frac{\int_u^{\infty} (\eta - u)^{n-4} \delta(\eta)\, \mathrm{d}\eta}{(n-4)!} \varphi^{(\alpha+1)/\alpha}(u),$$

we get

$$\varphi'(u) + H^{\beta-\alpha} \sum_{i=1}^{m} q_i(u) + \alpha \frac{\int_u^{\infty} (\eta - u)^{n-4} \delta(\eta) d\eta}{(n-4)!} \varphi^{\frac{\alpha+1}{\alpha}}(u) \leq 0.$$

From [18] Theorem 2.6, we obtain (15) is non-oscillatory, which is a contradiction, so the proof of this theorem is complete.

3. Examples

This section presents examples that are intended to demonstrate the validity of the findings stated in the previous part.

Example 1. *Consider the following differential equation*

$$\left(u^2 \varpi'''(u)\right)' + \left(\sqrt{10} e^u \left(2 e^{\arcsin \frac{\sqrt{10}}{10}} - 1\right) + \sqrt{10} e^u\right) \varpi \left(u - \arcsin \frac{\sqrt{10}}{10}\right) = 0, \quad u \geq 1, \tag{21}$$

where $\alpha = 1$, $\beta = 1$, $q(u) = \sqrt{10} e^u \left(2 e^{\arcsin \frac{\sqrt{10}}{10}} - 1\right) + \sqrt{10} e^u$, $\tau = u - \arcsin \frac{\sqrt{10}}{10}$.

Using Corollary 1, we have

$$\begin{aligned} &\liminf_{u\to\infty} \int_{\tau(u)}^{u} \left(\sqrt{10} e^s \left(2 e^{\arcsin \frac{\sqrt{10}}{10}} - 1\right) + \sqrt{10} e^s\right) \left(s - \arcsin \frac{\sqrt{10}}{10}\right) \mathrm{d}s \\ &= \liminf_{u\to\infty} 2\sqrt{10} \int_{\tau(u)}^{u} \left(\mathrm{s}\, e^{s + \arcsin \frac{\sqrt{10}}{10}} - \arcsin \frac{\sqrt{10}}{10} e^{s + \arcsin \frac{\sqrt{10}}{10}}\right) \mathrm{d}s \\ &= \infty > \frac{6}{e} \end{aligned}$$

and condition (13) becomes

$$\limsup_{u\to\infty} \int_{u_0}^{u} \left[2\sqrt{10} e^{s + \arcsin \frac{\sqrt{10}}{10}} \left[\frac{\lambda_1}{2\ \mathrm{s}} \left(s - \arcsin \frac{\sqrt{10}}{10}\right)^2\right] - \frac{1}{4s}\right] \mathrm{d}s = \infty.$$

It is clear to notice that all conditions of Corollary (1) hold. Hence every solution of (21) is oscillatory or tends to zero.

Example 2. *Consider the following differential equation*

$$\left(\mathrm{u}^6(\varpi(u))'''\right)' + \left(\frac{\eta}{u}\left(u^2+u+1\right)(u-1) + \frac{\eta}{u}\right)\varpi\left(\frac{u}{2}\right) = 0, \tag{22}$$

where $u \geq 1$ *and* $\eta > 0$.
We observe that $\alpha = 3$, $\beta = 1$, $r(u) = u^6$, $\tau(u) = u/2$ *and* $q(u) = \frac{\eta}{u}(u^2+u+1)(u-1) + \frac{\eta}{u}$. *Thus, it is easy to verify Condition (14). Obviously, all conditions for Corollary (2) are achieved. Thus, all solutions of (22) are oscillatory or tend to zero.*

4. Conclusion

As explained in the Introduction, the theory of higher-order differential equations has connections with many different fields of mathematics and the applied sciences. The topic of oscillation in relation to (1) is heavily emphasised in the present study. Utilising Riccati transformation and comparison strategies involving first order differential equations has resulted in the discovery of new oscillatory properties. The previously mentioned criteria serve as a supplement to the documented outcomes in the currently available collection of literature.

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Conflict of interest

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Materials and data availability

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References

[1] Almutairi A., Bazighifan O., and Raffoul Y. N. Oscillation results for nonlinear higher-order differential equations with delay term. *Symmetry*, 13(3):446, 2021.

[2] Wintner A. A criterion of oscillatory stability. *Quarterly of Applied Mathematics*, 7(1):115–117, 1949.

[3] Almarri Barakah, Ali Ali Hasan, Al-Ghafri Khalil S, Almutairi Alanoud, Bazighifan Omar, and Awrejcewicz Jan. Symmetric and non-oscillatory characteristics of the neutral differential equations solutions related to p-laplacian operators. *Symmetry*, 14(3):566, 2022.

[4] Cina Bengu, Candan Tuncay, and Senel M Tamer. Existence of nonoscillatory solutions of higher order nonlinear neutral differential equations. *European Journal of Pure and Applied Mathematics*, 16(2):713–723, 2023.

[5] Cesarano C. and Bazighifan O. Qualitative behavior of solutions of second order differential equations. *Symmetry*, 11(6):777, 2019.

[6] Shreve W. E. Oscillation in first order nonlinear retarded argument differential equations. *Proceedings of the American Mathematical Society*, 41(2):565–568, 1973.

[7] Mofarreh F., Almutairi A., Bazighifan O., Aiyashi M. A., and Vîlcu A. D. On the oscillation of solutions of differential equations with neutral term. *Mathematics*, 9(21):2709, 2021.

[8] Philos C. G. On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays. *Archiv der Mathematik*, 36:168–178, 1981.

[9] Hale J. K. Global theory. *Theory of Functional Differential Equations*, pages 320–335, 1977.

[10] Elabbasy E. M. On the oscillation of nonlinear second order differential equations. *PanAmerican Mathematical Journal*, 6:69–84, 1996.

[11] Elabbasy E. M. and Elsharabasy M. A. Oscillation properties for second order nonlinear differential equations. *Kyungpook Mathematical Journal*, 37(2):211–211, 1997.

[12] Elabbasy E. M., Hassan T., and Elmatary B. M. Oscillation criteria for third order delay nonlinear differential equations. *Electronic Journal of Qualitative Theory of Differential Equations*, 2012(5):1–9, 2012.

[13] Bazighifan O. and Ramos H. On the asymptotic and oscillatory behavior of the solutions of a class of higher-order differential equations with middle term. *Applied Mathematics Letters*, 107:106431, 2020.

[14] Bazighifan Omar and Kumam Poom. Oscillation theorems for advanced differential equations with p-laplacian like operators. *Mathematics*, 8(5):821, 2020.

[15] Agarwal R. P., Zhang C., and Li T. Some remarks on oscillation of second order neutral differential equations. *Applied Mathematics and Computation*, 274:178–181, 2016.

[16] Agarwal R. P., Grace S. R., and O'Regan D. *Oscillation theory for difference and functional differential equations.* Springer Science & Business Media, 2000.

[17] Agarwal R. P., Grace S. R., and O'Regan D. Oscillation criteria for certain nth order differential equations with deviating arguments. *Journal of Mathematical Analysis and Applications*, 262(2):601–622, 2001.

[18] Agarwal R. P., Shieh S.-L., and Yeh C.-C. Oscillation criteria for second-order retarded differential equations. *Mathematical and Computer Modelling*, 26(4):1–11, 1997.

[19] Grace S. R., Dzurina J., Jadlovska I., and Li T. An improved approach for studying oscillation of second-order neutral delay differential equations. *Journal of Inequalities and Applications*, 2018(1):1–13, 2018.

[20] Grace S. R., Agarwal R. P., and Graef J. R. Oscillation theorems for fourth order functional differential equations. *Journal of Applied mathematics and Computing*, 30(1-2):75–88, 2009.

[21] Graef J. R. and Saker S. H. Oscillation theory of third-order nonlinear functional differential equations. *Hiroshima Mathematical Journal*, 43(1):49–72, 2013.

[22] Yan J. R. Oscillation theorems for second order linear differential equations with damping. *Proceedings of the American Mathematical Society*, 98(2):276–282, 1986.

[23] Ladde G. S., Lakshmikantham V., and Zhang B. G. *Oscillation theory of differential equations with deviating arguments*. M. Dekker New York, 1987.

[24] Khan Sana Ullah, Khan Asif, Ullah Aman, Ahmad Shabir, Awwad Fuad A, Ismail Emad AA, Maitama Shehu, Umar Huzaifa, and Ahmad Hijaz. Solving nth-order integro-differential equations by novel generalized hybrid transform. *European Journal of Pure and Applied Mathematics*, 16(3):1940–1955, 2023.

[25] Nagabuchi Y. and Yamamoto M. Some oscillation criteria for second order nonlinear ordinary differential equations with damping. *Proc. Japan Acad. Ser. A Math. Sci.*, 64(8):282–285, 1988.

[26] Xu Z. and Xia Y. Integral averaging technique and oscillation of certain even order delay differential equations. *Journal of Mathematical Analysis and Applications*, 292(1):292, 2004.