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Direct summand of serial modules

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Abstract. Let R be an associative ring and M a unitary left R-module. An R-module M is said to be uniserial if its submodules are linearly ordered by inclusion. A serial module is a direct sum of uniserial modules. In this paper, we bring our modest contribution to the open problem listed in the book of Alberto Facchini "Module Theory" which states that: is any direct summand of a serial module serial? The answer is yes for particular rings and R-modules.

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1. Introduction

Let R be an associative ring and M a unitary left R-module. An R-module M is said to be uniserial if its submodules are linearly ordered by inclusion. A serial module is a direct sum of uniserial modules. The target of this paper comes from the following statement. Is any direct summand of serial module serial? This is an open problem listed in the Module theory book of Alberto Facchini. Some results has been obtained if the base ring is commutative or noetherian ... Other results are obtained in this paper for particular rings and modules. A module M is said to be a prime module if for every submodule N of M, $Ann(M) = Ann(N)$. A module is M is said to be faithful if $Ann(M) = 0$. A module M is said to be finitely cogenerated if its socle is essential in N and finitely generated

Lemma 1: Let M be an uniserial module over a ring R. M is said to be of type 1 if at least one of the following hold.
(1) M is projective;
(2) M is injective;
(3) M is artinian;

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(4) M is noetherian;
(5) R is commutative;
(6) R is a right noetherian.

Proof:
To see the proof refer from example 2.3 of [1].

Proposition 1: Let R be ring and M be a left local module over R, then $End(M)$ is local.
Proof.
Let M be an hollow module. Hence M is finitely generated. Let $f : R \longrightarrow M$ an homomorphism which is an epimorphism. Therefore $R/Ann(M)$ is isomorphic to M. Hence $R/Ann(M)$ is hollow. It follows from theorem 4.1 of [5] that $End(R/Ann(M))$ is local. Thus $End(M)$ is local.

Theorem 1:
Let $M_1, ..., M_n$ be uniserial local modules. Let $M = \bigoplus_{i\in I} M_i$ a serial module. Then every summand of M is serial.
Proof.
It results from proposition 1 that $End(M_i)$ is local. Therefore every direct summand of M is serial.

Proposition 2: Let R be ring and M a finitely cogenerated, prime and faithful R-module. Then R as a left R-module is uniserial. Moreover $End(R)$ is a local ring.
Proof.
Let M be a finitely cogenerated module over R. It is well known that any finitely cogenerated module has a small submodule. Let K be its small submodule. Then $f : R \longrightarrow K$ is an epimorphism.

$$\begin{array}{c} f : R \longrightarrow K \\ \downarrow \quad \swarrow \\ R/Ann(K) \end{array}$$

By the first isomorphism theorem, $R/Ann(K)$ is isomorphic to K. As M is a prime and faithful module then $Ann(K) = Ann(M) = 0$. Therefore R is simple as a left R-module. Hence R is uniserial because $\{0\} \subset R$. Since R is simple then for ever endomorphism of R is an automorphism. Hence for every endomorphism $g : R \longrightarrow R$ there exists always a endomorphism $g : R \longrightarrow R$ such that $g \circ h = Id$ and $h \circ g = Id$. That implies $End(R)$ is a division ring. It is well know that any division ring is a local ring.

Corollary 1:
Let $R = \bigoplus_{i\in I} R_i$ where $(R_i)_{i\in I}$ is family of rings such that there exists a finitely cogenerated, prime and faithful R_{i_0}-module with $i_0 \in I$. Then the following conditions are verified:

(1) Each R_i is uniserial as a left R_i-module for every $i \in I$.
(2) Every summand of R is serial.
Proof.
(1) Let M be a left finitely cogenerated prime and faithful R_{i_0}-module. Then M is a module over every R_i with $i \in I$ by the following homomorphism:

$$\begin{aligned} f : R_i &\longrightarrow R_{i_0} \times M \longrightarrow M \\ r &\longmapsto (f(r), m) \longmapsto f(r)m \end{aligned}$$

It follows from proposition 2 that R_i is simple hence uniserial for every $i \in I$.
(2) It results from proposition 2 that $End(R_i)$ is a local ring for every $i \in I$. Thus every summand of R is serial.
In the following corollary we show that a finitely generated module M is serial under certain conditions and every summand of M is serial.

Corollary 2:
Let R be a ring and $M = \bigoplus_{i=1}^{n} M_i$ a finitely generated and prime module Such that M has a small submodule. Then M is serial and so is every summand of M.
Proof.
Let M a finitely generated prime module. Let $f : R \longrightarrow K$ be an epimorphism where K is a small submodule.

$$\begin{array}{c} f : R \longrightarrow K \\ \downarrow \quad \swarrow \\ R/Ann(K) \end{array}$$

By the first isomorphic theorem $R/Ann(K) \simeq K$. Let $g : R \longrightarrow M_i$ another epimorphism for every $1 \leq i \leq n$

$$\begin{array}{c} f : R \longrightarrow M_i \\ \downarrow \quad \swarrow \\ R/Ann(M_i) \end{array}$$

We have also $R/Ann(M_i) \simeq M_i$. Since M is a prime module $Ann(K) = Ann(M_i)$, therefore $R/Ann(K) = R/Ann(M_i) \simeq M_i$ is simple for $1 \leq i \leq n$. Therefore $M = \bigoplus_{i=1}^{n} M_i$ is semisimple.

Proposition 3:
Let R be a ring and M an uniserial R-module, then $End(M)$ is local if,
(1) M is self-projective;
(2) M is self-injective;
(3) M is a free module.
Proof :
(1) As M is uniserial, hence it is uniform and indecomposable. Let $f \in End(M)$ therefore $\ker f \cap \ker(1 - f) = 0$. Since M is uniform, then f or $1 - f$ is a monomorphism. If M

is self-projective then any sequence $0 \longrightarrow N \longrightarrow M \longrightarrow M \longrightarrow 0$ is split. Hence an endomorphism of M is surjective. Therefore $End(M)$ is local.
(2) Let $f \in End(M)$ therefore $\ker f \cap \ker(1-f) = 0$. Then f is injective. Since M is self-injective, then $f(M)$ is a direct summand of M that is $M = f(M) \bigoplus N$. But M is uniserial, hence M is indecomposable. Therefore $f(M) = M$ which states that f is an epimorphism. Thus f is an automorphism. $End(M)$ is a division ring
(3) Since any free module is projective then $End(M)$ is local by lemma 1.

Theorem 2: Let R be a ring and $M = \oplus_{i=1}^{n} M_i$ a serial module with M_i self-projective(resp. self-injective or free). Then any direct summand of M is serial.

Proof.
Let R be a ring and $M = \oplus_{i=1}^{n} M_i$ a serial module with M_i self-projective(resp. self-injective or free. It results from proposition 3 that the endomorphism ring of any self-projective(resp. self-injective or free) module is local. It follows from proposition 2.2 of [4] that the direct summand of any direct sum of uniserial modules with local endomorphism rings is serial.

Proposition 4:
Let R be a ring.
(1) If R is semisimple ring, then every a left module M over R is serial. Moreover, every direct summand of M is serial.
(2) If R is principal ring and M a left serial R-module, then every direct summand of M is serial.

Proof:
(1) Let M be a left R-module. Since R is semisimple hence, M is semisimple. Let $M = \bigoplus_{i\in I} M_i$ with M_i simple. It is well know that any simple module is uniserial. Hence, M is serial. Let $S = End(M_i)$ an endomorphism ring of simple module M_i, by Schur's Lemma S is a division ring. Therefore S is a local ring. Thus every direct summand of M is serial.
(2) If R is principal ring, then every ideal over R is principal is cyclic(finitely generated). Thus by the definition of noetherian ring, R is noetherian.

Theorem 3:
Let R be a semisimple or principal ring and $M = \oplus_{i=1}^{n} M_i$ a serial module. Then every direct summand of M is serial.

Proof:
Let R be a semisimple $M = \oplus_{i=1}^{n} M_i$ a serial module. It results from proposition 4 that $End(M_i)$ is a local endomorphisme ring for any $1 \leq i \leq n$. By the proposition 2.2 of [4] that any direct summand of M is a direct sum of serial module.

Assume R is a principal ring. It results from proposition 4 that R is a noetherian ring. It follows from example 2.3 of [1] that any direct summand of M is serial.

Proposition 5:
Let $R = \bigoplus_{i=1}^{n} R_i$ be a serial semiperfect ring then
(1) $End(R_i)$ is local,
(2) every direct summand of R is serial.

Proof:
As R is semiperfect then R is a sum of is local ring. Hence R_i is local for any $1 \leq i \leq n$. It results from proposition 1, that $End(R_i)$ is local. Therefore any direct summand of R is serial.

Theorem 4:
Let R be a local ring and $M = \oplus_{i=1}^{n} M_i$ a finitely generated serial module. Then every direct summand of M is serial.

Proof.
Let R be a local ring and $M = \oplus_{i=1}^{n} M_i$ a finitely generated serial module left R-module. Assume M_i a cyclic module. By the following diagram,

$$\begin{array}{c} f : R \longrightarrow M_i \\ \downarrow \quad \swarrow \\ R/Ann(M_i) \end{array}$$

$R/Ann(M_i)$ is isomorphic to M_i. As R is local, it has a unique maximal ideal, J. Let $\bar{I}$ be an ideal of $R/Ann(M_i)$ then $\bar{I} = I/Ann(M_i)$ with I an ideal of R and $Ann(M_i) \subseteq J$. Therefore $J/Ann(M_i)$ is the unique maximal ideal of $R/Ann(M_i)$. Hence, $R/Ann(M_i)$ is a local ring. Thus M_i is local.

References

[1] A. Facchini, *Krull-Schmidt fails for serial modules, Trans. Amer. Math. Soc. 348 (1996), 4561-4575*

[2] A. Facchini, *Modules Theory: Endomorphism rings and direct sum decompositions in some classes of modules, Reprint of 1998 edition*

[3] F.W. Anderson and K. Fuller, *Rings and categories of modules, Springer-Verlag (1974)*

[4] N. V. Dung and A. Facchini, *Direct summands of serial modules, J. Pure Appl. Algebra 133 (1998), 93-106*

[5] P. Fleury, *Hollow modules and local endomorphism rings, Pac.J.Math. 53, 379-385 (1974)*

[6] A. M. Kaidi and M. Sangharé, *Une caractérisation des anneaux artiniens à idéaux principaux. L.Notes in Math.Springer Verlag (1988) pp.245 - 254*

[7] R. Wisbauer, *foundation of modules and ring theory, Gordon and Breach Science Publishers(1991).*