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Determinants of Arrowhead Matrices over Finite Commutative Chain Rings

Somphong Jitman[1,*], Pornrudee Modjam[1]

[1] *Department of Mathematics, Faculty of Science, Silpakorn University, Nakhon Pathom 73000, Thailand*

Abstract. Arrowhead matrices have attracted attention due to their rich algebraic structures and numerous applications. In this paper, we focus on the enumeration of $n \times n$ arrowhead matrices with prescribed determinant over a finite field $\mathbb{F}_q$ and over a finite commutative chain ring R. The number of $n \times n$ arrowhead matrices over $\mathbb{F}_q$ of a fixed determinant a is determined for all positive integers n and for all elements $a \in \mathbb{F}_q$. As applications, this result is used in the enumeration of $n \times n$ non-singular arrowhead matrices with prescribed determinant over R. Subsequently, some bounds on the number of $n \times n$ singular arrowhead matrices over R of a fixed determinant are given. Finally, some open problems are presented.

2020 Mathematics Subject Classifications: 11C20, 15B33

Key Words and Phrases: Arrowhead matrices, Determinants, Finite fields, Finite commutative chain rings, Enumeration

1. Introduction

Matrices and their determinants have been known and extensively studied for their nice properties and wide applications (see, for example, [2], [9], and [10]). Singularity of matrices is useful in applications (see, for example, [2] and [11]). The number of $n \times n$ singular (resp., nonsingular) matrices over a finite field $\mathbb{F}_q$ has been determined in [13]. As a generalization of a prime field $\mathbb{Z}_p$, the number of $n \times n$ matrices over $\mathbb{Z}_m$ of a fixed determinant has been first studied in [1]. An alternative study of the problem in [1] has been given in [10] using a different and simpler approach. A finite commutative chain ring (FCCR) and a principal ideal ring are generalizations of the rings $\mathbb{Z}_p$ and $\mathbb{Z}_m$ that are useful in applications such as coding theory and cryptography. In [3], the techniques in [10] have been extended to matrices over FCCRs and principal ideal rings. Precisely, the number of $n \times n$ matrices over FCCRs and principal ideal rings of a fixed determinant has been completely determined. Diagonal matrices are interesting subfamilies of the ones

*Corresponding author.
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Email addresses: sjitman@gmail.com (S. Jitman), pornrudee.ole@gmail.com (P. Modjam)

in [3]. The enumeration of diagonal matrices over FCCRs of a fixed determinant are presented in [8] and applied in the study of the determinant of some circulant matrices over FCCRs.

For a commutative ring R and a positive integer n, an $n \times n$ *arrowhead matrix* over R is defined to be a square matrix containing zeros in all entries except for the first row, first column, and main diagonal. Precisely, the arrowhead matrix is in the form of

$$A = \begin{pmatrix} * & * & * & * & \cdots & * \\ * & * & 0 & 0 & \cdots & 0 \\ * & 0 & * & 0 & \cdots & 0 \\ * & 0 & 0 & * & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & 0 & 0 & 0 & \cdots & * \end{pmatrix},$$

where $*$'s are arbitrary elements in R and they are not necessarily the same. From the definition, an arrowhead matrix is a generalization of a diagonal matrix over R. It is easily seen that the 1×1 matrices, 2×2 matrices, and $n \times n$ diagonal matrices over R are arrowhead matrices for all positive integers n. Some properties of arrowhead matrices such as eigenvalues, eigenvectors, and inverses have been studied in [14], [15], and [16]. Arrowhead matrices have applications in various fields, e.g., wireless communications in [15], eigenvalue decompositions of some matrices in [16], the study of directed multigraphs and hub-directed multigraphs in[12], and the study of disordered quantum spins in [4].

As a generalization of [8], the enumeration of arrowhead matrices with prescribed determinant over a FCCR is investigated in the following set up. For a FCCR R, let $U(R)$ denote the set of units in R and let $Z(R)$ denote the set of zero-divisors in R. Let $\mathcal{A}_n(R)$ denote the set of $n \times n$ arrowhead matrices over R. It is not difficult to see that $\mathcal{A}_n(R)$ is a group under addition and

$$|\mathcal{A}_n(R)| = |R|^{3n-2}. \tag{1}$$

An $n \times n$ matrix A over R is said to be *non-singular* (or, *invertible*) if $\det(A) \in U(R)$. Otherwise, A is called a *singular matrix*. Let

$$\mathcal{IA}_n(R) = \{A \in \mathcal{A}_n(R) \mid \det(A) \in U(R)\}$$

be the set of $n \times n$ non-singular arrowhead matrices over R. For each $a \in R$, let

$$\mathcal{A}_n(R, a) = \{A \in \mathcal{A}_n(R) \mid \det(A) = a\}.$$

be the set of all $n \times n$ arrowhead matrices over R whose determinant is a. Clearly,

$$\mathcal{IA}_n(R) = \bigcup_{a \in U(R)} \mathcal{A}_n(R, a)$$

is a disjoint union.

The main focus of this paper is the enumeration of $n \times n$ arrowhead matrices with prescribed determinant over a finite field $\mathbb{F}_q$ and over a FCCR R. The paper is organized as follows. The number $|\mathcal{A}_n(\mathbb{F}_q, a)|$ of $n \times n$ arrowhead matrices over $\mathbb{F}_q$ of determinant a is determined for all positive integers n and for all elements $a \in \mathbb{F}_q$ in Section 2. As applications, these results are used in the enumeration of arrowhead matrices of a fixed determinant over R in Section 3. The number of $n \times n$ non-singular arrowhead matrices of a fixed determinant over R in Subsection 3.1. Subsequently, bounds on the number of $n \times n$ singular arrowhead matrices over R of some fixed determinant are discussed in Subsection 3.2. Some remarks and open problems are given in Section 4.

2. Determinants of Arrowhead Matrices over $\mathbb{F}_q$

In this section, we focus on the enumeration of arrowhead matrices of a fixed determinant over a finite field $\mathbb{F}_q$. For an element $a \in \mathbb{F}_q$, the formula for the number of $n \times n$ arrowhead matrices over $\mathbb{F}_q$ of determinant a is given for all prime powers q and positive integers n.

A recursive formula for the number $|\mathcal{IA}_n(\mathbb{F}_q)|$ of $n \times n$ non-singular arrowhead matrices over $\mathbb{F}_q$ is given in Proposition 1. Later, an explicit formula for $|\mathcal{IA}_n(\mathbb{F}_q)|$ is established in Theorem 1 based on Proposition 1.

Proposition 1. *Let q be a prime power. Then*

$$|\mathcal{IA}_1(\mathbb{F}_q)| = q - 1$$

and

$$|\mathcal{IA}_n(\mathbb{F}_q)| = q^{2n-3}(q-1)^n + q^2(q-1)|\mathcal{IA}_{n-1}(\mathbb{F}_q)|$$

for all integers $n \geq 2$.

Proof. Clearly, $|\mathcal{IA}_1(\mathbb{F}_q)| = |\mathbb{F}_q \setminus \{0\}| = q - 1$. Let $n \geq 2$ be an integer and let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & 0 & \cdots & 0 & 0 \\ a_{31} & 0 & a_{33} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & 0 & 0 & \cdots & a_{n-1,n-1} & 0 \\ a_{n1} & 0 & 0 & \cdots & 0 & a_{nn} \end{pmatrix} \in \mathcal{IA}_n(\mathbb{F}_q).$$

For each $i \in \{1, 2, \ldots, n\}$, let R_i (resp., C_i) denote the ith row (resp, ith column) of A. We consider the two cases.

Case 1: $a_{nn} \neq 0$. Applying the elementary row operation $R_1 - a_{1n}{a_{nn}}^{-1}R_n \to R_1$ and the elementary column operation $C_1 - a_{n1}{a_{nn}}^{-1}C_n \to C_1$, it follows that

$$A \sim \left(\begin{array}{ccc|c} & & & 0 \\ & C & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & a_{nn} \end{array}\right),$$

where

$$C = \begin{pmatrix} a_{11} - a_{1n}a_{n1}{a_{nn}}^{-1} & a_{12} & a_{13} & \cdots & a_{1,n-1} \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & 0 & 0 & \cdots & a_{n-1,n-1} \end{pmatrix}.$$

Then $\det(A) = (-1)^{n+n}a_{nn}\det(C) = a_{nn}\det(C)$.

Let

$$S = \left\{ \begin{pmatrix} s_{11} & s_{12} & s_{13} & \cdots & s_{1,n-1} \\ s_{21} & s_{22} & 0 & \cdots & 0 \\ s_{31} & 0 & s_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n-1,1} & 0 & 0 & \cdots & s_{n-1,n-1} \end{pmatrix} \in \mathcal{A}_{n-1}(\mathbb{F}_q) \,\middle|\, \det\left(\begin{pmatrix} s_{11} - a_{1n}a_{n1}{a_{nn}}^{-1} & s_{12} & s_{13} & \cdots & s_{1,n-1} \\ s_{21} & s_{22} & 0 & \cdots & 0 \\ s_{31} & 0 & s_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n-1,1} & 0 & 0 & \cdots & s_{n-1,n-1} \end{pmatrix} \right) \neq 0 \right\}.$$

It follows that

$$\begin{pmatrix} s_{11} & s_{12} & s_{13} & \cdots & s_{1,n-1} \\ s_{21} & s_{22} & 0 & \cdots & 0 \\ s_{31} & 0 & s_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n-1,1} & 0 & 0 & \cdots & s_{n-1,n-1} \end{pmatrix} \in S$$

if and only if

$$\begin{pmatrix} s_{11} - a_{1n}a_{n1}{a_{nn}}^{-1} & s_{12} & s_{13} & \cdots & s_{1,n-1} \\ s_{21} & s_{22} & 0 & \cdots & 0 \\ s_{31} & 0 & s_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n-1,1} & 0 & 0 & \cdots & s_{n-1,n-1} \end{pmatrix} \in \mathcal{IA}_{n-1}(\mathbb{F}_q).$$

Consequently, we have $|S| = |\mathcal{IA}_{n-1}(\mathbb{F}_q)|$. We note that $0 \neq \det(A) = a_{nn}\det(C)$ if and only if $\det(C) \neq 0$, or equivalently,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & 0 & 0 & \cdots & a_{n-1,n-1} \end{pmatrix} \in S.$$

Hence, there are $|S| = |\mathcal{IA}_{n-1}(\mathbb{F}_q)|$ possibilities for C. The number of choices of a_{1n} and a_{n1} are q^2 and the number of choices for a_{nn} is $q-1$. Hence, the number of arrowhead matrices A in $\mathcal{IA}_n(\mathbb{F}_q)$ is

$$q^2(q-1)|\mathcal{IA}_{n-1}(\mathbb{F}_q)|.$$

Case 2: $a_{nn} = 0$. Since $\det(A) \neq 0$, we have $a_{1n} \neq 0$ and $a_{n1} \neq 0$. Applying the elementary row operation $R_i - a_{i1}{a_{n1}}^{-1}R_n \to R_i$ for all $i \in \{1, 2, \ldots, n-1\}$ and the elementary column operation $C_1 \leftrightarrow C_n$, we have

$$A \sim \begin{pmatrix} a_{1n} & a_{12} & a_{13} & \cdots & a_{1,n-1} & 0 \\ 0 & a_{22} & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_{33} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & a_{n1} \end{pmatrix} =: A'.$$

Since $\det(A') = -\det(A) \neq 0$ if and only if $a_{1n}, a_{22}, \ldots, a_{n-1,n-1}, a_{n1}$ are non-zero, the number of $(a_{1n}, a_{22}, a_{33}, \ldots, a_{n-1,n-1}, a_{n1})$ is $(q-1)^n$, the number of $(a_{12}, a_{13}, a_{14}, \ldots, a_{1,n-1})$ is q^{n-1}, and the number of $(a_{21}, a_{31}, a_{41}, \ldots, a_{n-2,1})$ is q^{n-2} In this case, the number of A in $\mathcal{IA}_n(\mathbb{F}_q)$ is

$$q^{2n-3}(q-1)^n.$$

From the two cases, it can be deduced that

$$|\mathcal{IA}_n(\mathbb{F}_q)| = q^{2n-3}(q-1)^n + q^2(q-1)|\mathcal{IA}_{n-1}(\mathbb{F}_q)|$$

as desired. ■

An explicit expression for the number $|\mathcal{IA}_n(\mathbb{F}_q)|$ can be derived using the recursive formula given in Proposition 1 and the principle of mathematical induction.

Theorem 1. *Let q be a prime power. Then*

$$|\mathcal{IA}_n(\mathbb{F}_q)| = q^{2n-3}(q-1)^n(q+(n-1))$$

for all positive integers n.

Proof. For $n = 1$, we have

$$|\mathcal{IA}_1(\mathbb{F}_q)| = q-1 = q^{2(1)-3}(q-1)^1(q+(1-1)).$$

Let $k \geq 2$ be an integer. Assume that

$$|\mathcal{IA}_{k-1}(\mathbb{F}_q)| = q^{2(k-1)-3}(q-1)^{k-1}(q+((k-1)-1)).$$

Using the recurrent relation given in Proposition 1, we have

$$\begin{aligned} |\mathcal{IA}_k(\mathbb{F}_q)| &= q^{2k-3}(q-1)^k + q^2(q-1)|\mathcal{IA}_{k-1}(\mathbb{F}_q)| \\ &= q^{2k-3}(q-1)^k + q^2(q-1)(q^{2(k-1)-3}(q-1)^{k-1}(q+((k-1)-1))) \end{aligned}$$

$$= q^{2k-3}(q-1)^k + q^{2k-3}(q-1)^k(q+(k-2))$$
$$= q^{2k-3}(q-1)^k(q+(k-1)).$$

Therefore, it follows that

$$|\mathcal{IA}_n(\mathbb{F}_q)| = q^{2n-3}(q-1)^n(q+(n-1))$$

for all positive integers n. ■

In the following proposition, a relation between $|\mathcal{A}_n(\mathbb{F}_q,1)|$ and $|\mathcal{A}_n(\mathbb{F}_q,a)|$ for all $a \in \mathbb{F}_q \setminus \{0\}$ is key to study the enumeration of $|\mathcal{A}_n(\mathbb{F}_q,a)|$ in Corollary 1.

Proposition 2. *Let q a prime power and let n be a positive integer. Then*

$$|\mathcal{A}_n(\mathbb{F}_q,1)| = |\mathcal{A}_n(\mathbb{F}_q,a)|$$

for all $a \in \mathbb{F}_q \setminus \{0\}$.

Proof. Let $a \in \mathbb{F}_q \setminus \{0\}$ and let $f : \mathcal{A}_n(\mathbb{F}_q,1) \to \mathcal{A}_n(\mathbb{F}_q,a)$ be defined by

$$f(A) = \mathrm{diag}(a,1,1,\ldots,1)A.$$

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in \mathcal{A}_n(\mathbb{F}_q,1).$$

Then $\det(A) = 1$,

$$f(A) = \mathrm{diag}(a,1,1,\ldots,1)A = \begin{pmatrix} aa_{11} & aa_{12} & aa_{13} & \cdots & aa_{1n} \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in \mathcal{A}_n(\mathbb{F}_q), \tag{2}$$

and

$$\det(f(A)) = \det(\mathrm{diag}(a,1,1,\ldots,1)A) = \det(\mathrm{diag}(a,1,1,\ldots,1)) \cdot \det(A) = a \cdot 1 = a.$$

Hence, $f(A) \in \mathcal{A}_n(\mathbb{F}_q,a)$. Since $\mathrm{diag}(a,1,1,\ldots,1)$ is invertible, we have that f is injective.

Let $X \in \mathcal{A}_n(\mathbb{F}_q,a)$ and let $A = \mathrm{diag}(a^{-1},1,1,\ldots,1)X$. Then we have $A \in \mathcal{A}_n(\mathbb{F}_q)$ and $\det(A) = \det(\mathrm{diag}(a^{-1},1,1,\ldots,1)X) = a^{-1} \cdot a = 1$. It follows that $A \in \mathcal{A}_n(\mathbb{F}_q,1)$ and

$$f(A) = f(\mathrm{diag}(a^{-1},1,1,\ldots,1)X) = \mathrm{diag}(a,1,1,\ldots,1)\mathrm{diag}(a^{-1},1,1,\ldots,1)X = X.$$

Consequently, f is surjective.

It follows that f is a bijection from $\mathcal{A}_n(\mathbb{F}_q, 1)$ onto $\mathcal{A}_n(\mathbb{F}_q, a)$, and hence, $|\mathcal{A}_n(\mathbb{F}_q, 1)| = |\mathcal{A}_n(\mathbb{F}_q, a)|$. ■

From Proposition 2, we have

$$|\mathcal{A}_n(\mathbb{F}_q, a)| = |\mathcal{A}_n(\mathbb{F}_q, 1)| = |\mathcal{A}_n(\mathbb{F}_q, b)|$$

for all $a, b \in \mathbb{F}_q \setminus \{0\}$.

Based on Theorem 1 and Proposition 2, the next corollary can be derived.

Corollary 1. *Let q be a prime power and let n be positive integer. Then*

$$|\mathcal{A}_n(\mathbb{F}_q, a)| = q^{2n-3}(q-1)^{n-1}(q+(n-1))$$

for all $a \in \mathbb{F}_q \setminus \{0\}$.

Proof. From Proposition 2, it follows that $|\mathcal{A}_n(\mathbb{F}_q, a)| = |\mathcal{A}_n(\mathbb{F}_q, 1)|$ for all $a \in \mathbb{F}_q \setminus \{0\}$. Since

$$\mathcal{IA}_n(\mathbb{F}_q) = \bigcup_{a \in \mathbb{F}_q \setminus \{0\}} \mathcal{A}_n(\mathbb{F}_q, a)$$

is a disjoint union and $|\mathbb{F}_q \setminus \{0\}| = q - 1$, it follows that

$$|\mathcal{IA}_n(\mathbb{F}_q)| = |\mathbb{F}_q \setminus \{0\}||\mathcal{A}_n(\mathbb{F}_q, 1)| = (q-1)|\mathcal{A}_n(\mathbb{F}_q, 1)|.$$

By Theorem 1 and Proposition 2, we have

$$\begin{aligned}
|\mathcal{A}_n(\mathbb{F}_q, a)| &= |\mathcal{A}_n(\mathbb{F}_q, 1)| \\
&= \frac{|\mathcal{IA}_n(\mathbb{F}_q)|}{q-1} \\
&= \frac{q^{2n-3}(q-1)^n(q+(n-1))}{q-1} \\
&= q^{2n-3}(q-1)^{n-1}(q+(n-1)).
\end{aligned}$$

This completes the proof. ■

We note that $|\mathcal{A}_n(\mathbb{F}_q)| = q^{3n-2}$ and

$$|\mathcal{IA}_n(\mathbb{F}_q)| = q^{2n-3}(q-1)^n(q+(n-1))$$

given in (1) and Theorem 1. The number $|\mathcal{A}_n(\mathbb{F}_q, 0)| = |\mathcal{A}_n(\mathbb{F}_q)| - |\mathcal{IA}_n(\mathbb{F}_q)|$ of $n \times n$ singular arrowhead matrices over $\mathbb{F}_q$ follows in the next corollary.

Corollary 2. *Let q be a prime power. Then*

$$|\mathcal{A}_n(\mathbb{F}_q, 0)| = q^{3n-2} - q^{2n-3}(q-1)^n(q+(n-1))$$

for all positive integers n.

3. Determinants of Arrowhead Matrices over FCCRs

In this section, the enumeration of $n \times n$ arrowhead matrices with prescribed determinant over R is discussed. The number of $n \times n$ non-singular (resp., singular) arrowhead matrices over R is presented. For non-singular arrowhead matrices, the number of $n \times n$ arrowhead matrices over R with a given determinant is established. For singular arrowhead matrices, bounds on the number of $n \times n$ arrowhead matrices with a fixed determinant over R are presented in some cases.

To be self-contained, a brief information of a FCCR is recalled. The reader may refer to [5], [6], and [7] for more details. A ring R with identity $1 \neq 0$ is called a *finite commutative chain ring* (FCCR) if it is finite, commutative, and its ideals are linearly ordered by inclusion. Let R be a FCCR whose maximal ideal is generated by γ. Then the ideals in R are of the form

$$R \supsetneq \gamma R \supsetneq \gamma^2 R \supsetneq \cdots \supsetneq \gamma^{e-1} R \supsetneq \gamma^e R = \{0\},$$

for some positive integer e. The smallest positive integer e such that $\gamma^e = 0$ is called the *nilpotency index* of R. The quotient ring $R/\gamma R$ is a finite field and it is referred to as the *residue field* of R. From [6] and [7], useful properties of a FCCR (cf. [3]) are summarized in the next lemma.

Lemma 1. *Let R be a FCCR of nilpotency index e and let γ be a generator of its maximal ideal. Let $V \subseteq R$ be a set of representatives for the equivalence classes of R under congruence modulo γ. Assume that the residue field $R/\langle\gamma\rangle \cong \mathbb{F}_q$ for some prime power q. Then the following statements hold.*

1) *For each $r \in R$, there exist unique $a_0, a_1, \ldots a_{e-1} \in V$ such that*

$$r = a_0 + a_1\gamma + \cdots + a_{e-1}\gamma^{e-1}.$$

2) $|V| = q$.

3) $|\gamma^j R| = q^{e-j}$ *for all* $0 \leq j \leq e$.

4) $U(R) = \{a + \gamma b \mid a \in V \setminus \{0\}$ *and* $b \in R\}$.

5) $|U(R)| = (q-1)q^{e-1}$.

6) *For each $0 \leq i \leq e$, $R/\gamma^i R$ is a FCCR of nilpotency index i and residue field $\mathbb{F}_q$.*

3.1. Non-Singular Arrowhead Matrices over FCCRs

First, the number of $n \times n$ non-singular arrowhead matrices over a FCCRs R is presented. Then it is followed by the number of $n \times n$ arrowhead matrices over R with prescribed determinant in $U(R)$.

An explicit formula for the number $|\mathcal{IA}_n(R)|$ of $n \times n$ non-singular matrices is given in the following theorem.

Theorem 2. *Let R be a FCCR with residue field $\mathbb{F}_q$ and nilpotency index e. Then*

$$|\mathcal{IA}_n(R)| = q^{e(3n-2)-(n+1)}(q-1)^n(q+(n-1))$$

for all positive integers n.

Proof. Let γ be a generator of the maximal ideal of R and let $\varphi : R \to \mathbb{F}_q$ be the ring homomorphism defined by $a \mapsto a + \langle\gamma\rangle$. By considering $\mathcal{A}_n(R)$ and $\mathcal{A}_n(\mathbb{F}_q)$ as additive groups, let $\phi : \mathcal{A}_n(R) \to \mathcal{A}_n(\mathbb{F}_q)$ be the group homomorphism defined by

$$A = [a_{ij}] \mapsto [\varphi(a_{ij})].$$

It is not difficult to see that ϕ is a surjective homomorphism. By the First Isomorphism Theorem for groups, it follows that $\mathcal{A}_n(\mathbb{F}_q) \cong \mathcal{A}_n(R)/\ker(\phi)$. Hence,

$$|\ker(\phi)| = \frac{|\mathcal{A}_n(R)|}{|\mathcal{A}_n(\mathbb{F}_q)|} = \frac{q^{e(3n-2)}}{q^{3n-2}} = q^{(e-1)(3n-2)}.$$

For $A \in \mathcal{A}_n(R)$, we have $\det(\phi(A)) = \varphi(\det(A))$ which implies that $\det(A)$ is a unit in R if and only if $\det(\phi(A)) \neq 0$ in $\mathbb{F}_q$. Equivalently, A is invertible over R if and only if $\phi(A)$ is invertible over $\mathbb{F}_q$. Then the restriction map $\phi|_{\mathcal{IA}_n(R)} : \mathcal{IA}_n(R) \to \mathcal{IA}_n(\mathbb{F}_q)$ is surjective and it is $|\ker(\phi)|$ to one map. From Theorem 1, we have

$$|\mathcal{IA}_n(\mathbb{F}_q)| = q^{2n-3}(q-1)^n(q+(n-1)).$$

It follows that

$$\begin{aligned}
|\mathcal{IA}_n(R)| &= |\ker(\phi)||\mathcal{IA}_n(\mathbb{F}_q)| \\
&= q^{(e-1)(3n-2)}|\mathcal{IA}_n(\mathbb{F}_q)| \\
&= q^{(e-1)(3n-2)}q^{2n-3}(q-1)^n(q+(n-1)) \\
&= q^{e(3n-2)-(n+1)}(q-1)^n(q+(n-1))
\end{aligned}$$

as desired. ∎

For each $a \in U(R)$, the relation between $|\mathcal{A}_n(R,1)|$ and $|\mathcal{A}_n(R,a)|$ in the following proposition is key to determine the number $|\mathcal{A}_n(R,a)|$ in Corollary 3.

Proposition 3. *Let R be a FCCR and let n be a positive integer. Then*

$$|\mathcal{A}_n(R,a)| = |\mathcal{A}_n(R,1)|$$

for all $a \in U(R)$.

Proof. Let $a \in U(R)$ and let $\theta : \mathcal{A}_n(R,1) \to \mathcal{A}_n(R,a)$ be the map defined by

$$\theta(A) = \mathrm{diag}(a,1,1,\ldots,1)A.$$

Using arguments similar to those in the proof of Proposition 2, it can be deduced that θ is a bijection from $\mathcal{A}_n(R,1)$ onto $\mathcal{A}_n(R,a)$. As desired, $|\mathcal{A}_n(R,a)| = |\mathcal{A}_n(R,1)|$. ■

From Proposition 3, it follows that $|\mathcal{A}_n(R,a)| = |\mathcal{A}_n(R,1)| = |\mathcal{A}_n(R,b)|$ for all units $a, b \in U(R)$. For a fixed unit $a \in R$, the number of $n \times n$ arrowhead matrices over R whose determinant is a will be given later in Corollary 3.

Corollary 3. *Let R be a FCCR with residue field $\mathbb{F}_q$ and nilpotency index e and let n be a positive integer. Then*

$$|\mathcal{A}_n(R,a)| = q^{3e(n-1)-n}(q-1)^{n-1}(q+(n-1))$$

for all $a \in U(R)$.

Proof. First, we note that $\mathcal{IA}_n(R)$ is disjoint union of $\mathcal{A}_n(R,a)$ for all $a \in U(R)$. Precisely,

$$\mathcal{IA}_n(R) = \bigcup_{a \in U(R)} \mathcal{A}_n(R,a)$$

is a disjoint union. By Proposition 3, $\mathcal{A}_n(R,a)$ has the same number of elements as $\mathcal{A}_n(R,1)$, and hence,

$$\begin{aligned}
|\mathcal{IA}_n(R)| &= \left| \bigcup_{a \in U(R)} \mathcal{A}_n(R,a) \right| \\
&= \sum_{a \in U(R)} |\mathcal{A}_n(R,a)| \\
&= \sum_{a \in U(R)} |\mathcal{A}_n(R,1)| \\
&= |\mathcal{U}(R)||\mathcal{A}_n(R,1)|.
\end{aligned}$$

From Lemma 1, we have $|U(R)| = (q-1)q^{e-1}$. By Proposition 3, it can be deduced that

$$\begin{aligned}
|\mathcal{A}_n(R,a)| &= |\mathcal{A}_n(R,1)| \\
&= \frac{|\mathcal{IA}_n(R)|}{|\mathcal{U}(R)|} \\
&= \frac{q^{e(3n-2)-(n+1)}(q-1)^n(q+(n-1))}{(q-1)q^{e-1}} \\
&= q^{3e(n-1)-n}(q-1)^{n-1}(q+(n-1)).
\end{aligned}$$

The proof is completed. ■

3.2. Singular Arrowhead Matrices over FCCRs

In this subsection, the enumeration of singular arrowhead matrices with prescribed determinant over a FCCR R are studied. Unlike the previous subsection, only bounds on the number of singular $n \times n$ arrowhead matrices over R with prescribed determinant are given.

Since the number of $n \times n$ arrowhead matrices over R is $q^{e(3n-2)}$, the next corollary follow immediately from Theorem 2.

Corollary 4. *Let R be a FCCR with residue field $\mathbb{F}_q$ and nilpotency index e. Then the number of $n \times n$ singular arrowhead matrices over R is*

$$q^{e(3n-2)-(n+1)}\left(q^{n+1}-(q-1)^n(q+(n-1))\right)$$

for all positive integers n.

3.2.1. Singular Arrowhead Matrices over FCCRs with Zero Determinant

A general recursive lower bound on the number of $n \times n$ arrowhead matrices over R with zero determinant is given in the next proposition. For $e = 2$, a more specific bound is derived in Corollary 5.

Proposition 4. *Let R be a FCCR of nilpotency index e and residue field $\mathbb{F}_q$. If γ is a generator of the maximal ideal of R, then $|\mathcal{A}_1(R,0)| = 1$ and*

$$|\mathcal{A}_n(R,0)| \geq (q-1)q^{2(e-1)}(q^{e+1}+1)|\mathcal{A}_{n-1}(R,0)| + q^{3n-4}|\mathcal{A}_n(R/\gamma^{e-1}R, 0+\gamma^{e-1}R)|$$

for all integers $n \geq 2$.

Proof. Clearly, $|\mathcal{A}_1(R,0)| = 1$. Let $n \geq 2$ be an integer and let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & 0 & \cdots & 0 & 0 \\ a_{31} & 0 & a_{33} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & 0 & 0 & \cdots & a_{n-1,n-1} & 0 \\ a_{n1} & 0 & 0 & \cdots & 0 & a_{nn} \end{pmatrix} \in \mathcal{A}_n(R,0).$$

For convenience, for each $i \in \{1,2,\ldots,n\}$, denote by R_i (resp., C_i) the ith row (resp, ith column) of A. We consider the following two cases.

Case 1: $a_{1n} \in U(R)$ or $a_{nn} \in U(R)$.

Case 1.1: $a_{nn} \in U(R)$. Using the elementary row operation $R_1 - a_{1n}a_{nn}^{-1}R_n \to R_1$, we have that

$$A \sim \begin{pmatrix} & & & 0 \\ & C & & \vdots \\ & & & 0 \\ a_{n1} & 0 & \cdots & 0 & a_{nn} \end{pmatrix},$$

where

$$C = \begin{pmatrix} a_{11} - a_{1n}a_{n1}{a_{nn}}^{-1} & a_{12} & a_{13} & \cdots & a_{1,n-1} \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & 0 & 0 & \cdots & a_{n-1,n-1} \end{pmatrix}.$$

Then

$$\det(A) = (-1)^{n+n} a_{nn} \det(C) = a_{nn} \det(C). \tag{3}$$

Let

$$T = \left\{ \begin{pmatrix} t_{11} & t_{12} & t_{13} & \cdots & t_{1,n-1} \\ t_{21} & t_{22} & 0 & \cdots & 0 \\ t_{31} & 0 & t_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1,1} & 0 & 0 & \cdots & t_{n-1,n-1} \end{pmatrix} \in \mathcal{A}_{n-1}(R) \,\middle|\, \det\left(\begin{pmatrix} t_{11} - a_{1n}a_{n1}a_{nn}^{-1} & t_{12} & t_{13} & \cdots & t_{1,n-1} \\ t_{21} & t_{22} & 0 & \cdots & 0 \\ t_{31} & 0 & t_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1,1} & 0 & 0 & \cdots & t_{n-1,n-1} \end{pmatrix} \right) = 0 \right\}.$$

Since

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} & \cdots & t_{1,n-1} \\ t_{21} & t_{22} & 0 & \cdots & 0 \\ t_{31} & 0 & t_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1,1} & 0 & 0 & \cdots & t_{n-1,n-1} \end{pmatrix} \in T$$

if and only if

$$\begin{pmatrix} t_{11} - a_{1n}a_{n1}a_{nn}^{-1} & t_{12} & t_{13} & \cdots & t_{1,n-1} \\ t_{21} & t_{22} & 0 & \cdots & 0 \\ t_{31} & 0 & t_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1,1} & 0 & 0 & \cdots & t_{n-1,n-1} \end{pmatrix} \in \mathcal{A}_{n-1}(R, 0),$$

it follows that $|T| = |\mathcal{A}_{n-1}(R, 0)|$. From (3), $\det(A) = 0$ if and only if $\det(C) = 0$. The number of matrices C with determinant 0 is $|T| = |\mathcal{A}_{n-1}(R, 0)|$. The number of choices for a_{n1} is q^e, the number of choices for a_{1n} is q^e, and the number of choices for a_{nn} is $(q-1)q^{e-1}$. In this case, the possible choices for A is

$$(q-1)\, q^{3e-1} |\mathcal{A}_{n-1}(R, 0)|.$$

Case 1.2: $a_{1n} \in U(R)$ and $a_{nn} \notin U(R)$. Let

$$D = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & 0 & 0 & \cdots & a_{n-1,n-1} \end{pmatrix}.$$

Using the cofactor expansion through the last column of A, it follows that

$$\begin{aligned} \det(A) &= (-1)^{n+1}(-1)^{n-1+1}a_{1n}a_{n1}\mathrm{diag}(a_{22}, a_{33}, \ldots, a_{n-1,n-1}) + (-1)^{n+n}a_{nn}\det(D) \\ &= -a_{1n}a_{n1}\mathrm{diag}(a_{22}, a_{33}, \ldots, a_{n-1,n-1}) + a_{nn}\det(D). \end{aligned} \tag{4}$$

It is easily seen that $\det(A) = 0$ whenever $a_{n1} = 0$ and $D \in \mathcal{A}_{n-1}(R, 0)$. The number of choices for a_{1n} is $(q-1)q^{e-1}$, the number of choices for a_{nn} is q^{e-1}, and the number of choices for D is $|\mathcal{A}_{n-1}(R, 0)|$. In this case, the possible choices for A is at least

$$(q-1)\, q^{2(e-1)}|\mathcal{A}_{n-1}(R, 0)|.$$

Case 2: $a_{nn} \notin U(R)$ and $a_{1n} \notin U(R)$. Then the elements in the last column are in γR. Let $B = [b_{ij}]$ be the matrix in $\mathcal{A}_n(R)$ be defined by

$$b_{ij} = \begin{cases} w_{ij} & \text{if } (i,j) \in \{(1,n),(n,n)\} \\ a_{ij} & \text{otherwise,} \end{cases}$$

where $a_{1n} = \gamma w_{1n}$ and $a_{nn} = \gamma w_{nn}$ for some for some $w_{1n}, w_{nn} \in \sum\limits_{j=0}^{e-2} \gamma^j V$ and V is defined in Lemma 1. Let $C = [c_{ij}]$ be the matrix in $\mathcal{A}_n(R/\gamma^{e-1}R)$ defined by $c_{ij} = b_{ij} + \gamma^{e-1}R$. We note that $\det(A) = \gamma\det(B) \in R$. Then $\det(A) = 0$ in R if and only if $\det(B) \in \gamma^{e-1}R$ which is equivalent to $\det(C) = 0 + \gamma^{e-1}R$ in $R/\gamma^{e-1}R$. For each matrix $C \in \mathcal{A}_n(R/\gamma^{e-1}R, 0 + \gamma^{e-1}R)$, there are q^{3n-4} corresponding matrices $B \in \mathcal{A}_n(R, 0)$. Since the number of possible matrices C is $|\mathcal{A}_n(R/\gamma^{e-1}R, 0 + \gamma^{e-1}R)|$ and the matrix A is uniquely determined by B by multiplying the last column by γ, the number of choices for A is

$$q^{3n-4}|\mathcal{A}_n(R/\gamma^{e-1}R, 0 + \gamma^{e-1}R)|.$$

In summary, we have

$$|\mathcal{A}_n(R, 0)| \geq (q-1)\, q^{2(e-1)}(q^{e+1} + 1)|\mathcal{A}_{n-1}(R, 0)| + q^{3n-4}|\mathcal{A}_n(R/\gamma^{e-1}R, 0 + \gamma^{e-1}R)|$$

as desired. ■

For a FCCR of nilpotency index 2, we have the following bound.

Corollary 5. *Let R be a FCCR of nilpotency index 2 and residue field $\mathbb{F}_q$. If γ is a generator of the maximal ideal of R, then $|\mathcal{A}_1(R,0)| = 1$ and*

$$|\mathcal{A}_n(R,0)| \geq (q-1)q^2(q^3+1)|\mathcal{A}_{n-1}(R,0)| + q^{3n-4}\left(q^{3n-2} - q^{2n-3}(q-1)^n(q+(n-1))\right)$$

for all integers $n \geq 2$.

Proof. Clearly, $|\mathcal{A}_1(R,0)| = 1$. Let $n \geq 2$ be an integer. We note that $R/\gamma^{e-1}R \cong \mathbb{F}_q$. From Proposition 4 and Corollary 2, we have

$$\begin{aligned}|\mathcal{A}_n(R,0)| &\geq (q-1)q^2(q^3+1)|\mathcal{A}_{n-1}(R,0)| + q^{3n-4}|\mathcal{A}_n(\mathbb{F}_q,0)|\\ &= (q-1)q^2(q^3+1)|\mathcal{A}_{n-1}(R,0)| + q^{3n-4}\left(q^{3n-2} - q^{2n-3}(q-1)^n(q+(n-1))\right)\end{aligned}$$

as desired. ■

3.2.2. Singular Arrowhead Matrices over FCCRs with Non-Zero Determinant

In this subsection, an upper bound on the number of $n \times n$ singular arrowhead matrices over R with a fixed non-zero determinant is presented.

First, a relation between $|\mathcal{A}_n(R,\gamma^i)|$ and $|\mathcal{A}_n(R,b)|$ is derived for all $b \in \gamma^i R \setminus \gamma^{i+1}R$.

Proposition 5. *Let R be a FCCR with maximal ideal generated by γ, residue field $\mathbb{F}_q$, and nilpotency index e. Then*

$$|\mathcal{A}_n(R,\gamma^i)| = |\mathcal{A}_n(R,b)|$$

for all $b \in \gamma^i R \setminus \gamma^{i+1}R$ and $1 \leq i < e$.

Proof. Let $b \in \gamma^i R \setminus \gamma^{i+1}R$. Then $b = a\gamma^i$ for some $a \in U(R)$. Let $\psi : \mathcal{A}_n(R,\gamma^i) \to \mathcal{A}_n(R,a\gamma^i)$ be the function defined by

$$\psi(A) = \mathrm{diag}(a,1,1,\ldots,1)A.$$

Using the fact that a is convertible and arguments similar to those in the proof of Proposition 2, it can be deduced that ψ is a bijection from $\mathcal{A}_n(R,\gamma^i)$ onto $\mathcal{A}_n(R,a\gamma^i)$. As desired, $|\mathcal{A}_n(R,b)| = |\mathcal{A}_n(R,\gamma^i)|$. ■

Lemma 2. *Let R be a FCCR of nilpotency index $e \geq 3$ and residue field $\mathbb{F}_q$ and let n be a positive integer. If γ is a generator of the maximal ideal of R, then*

$$|\mathcal{A}_n(R,\gamma^s)| = q^{3(n-1)}|\mathcal{A}_n(R/\gamma^{e-1}R, \gamma^s + \gamma^{e-1}R)|$$

for all $1 \leq s < e-1$.

Proof. Let $1 \leq s < e-1$ be an integer and let $\beta : \mathcal{A}_n(R) \to \mathcal{A}_n(R/\gamma^{e-1}R)$ be an additive group homomorphism defined by

$$\beta(A) = \overline{A},$$

where $\overline{[a_{ij}]} := [a_{ij} + \gamma^{e-1}R]$ for all $[a_{ij}] \in \mathcal{A}_n(R)$. Note that, for each $A \in \mathcal{A}_n(R)$, $\det(\beta(A)) = \gamma^s + \gamma^{e-1}R$ if and only if $\det(A) = \gamma^s + \gamma^{e-1}b$ for some $b \in V$, where V is defined in Lemma 1. Since $1 \le e-s-1 < e-1$, it follows that $1+\gamma^{e-s-1}b$ is a unit in $U(R)$. Hence,

$$\begin{aligned}|\{A \in \mathcal{A}_n(R) \mid \det(A) = \gamma^s + \gamma^{e-1}b \text{ for some } b \in V\}| \\ &= |\{A \in \mathcal{A}_n(R) \mid \det(A) = \gamma^s(1+\gamma^{e-s-1}b) \text{ for some } b \in V\}| \\ &= |\{A \in \mathcal{A}_n(R) \mid \det(A) = \gamma^s\}| \\ &= |\mathcal{A}_n(R,\gamma^s)|.\end{aligned}$$

Equivalently,

$$|\{A \in \mathcal{A}_n(R) \mid \det(\beta(A)) = \gamma^s + \gamma^{e-1}R\}| = |V||\mathcal{A}_n(R,\gamma^s)| = q|\mathcal{A}_n(R,\gamma^s)|. \tag{5}$$

Since $|\ker(\beta)| = q^{3n-2}$, we have

$$\begin{aligned}|\{A \in \mathcal{A}_n(R) \mid \det(\beta(A)) = \gamma^s + \gamma^{e-1}R\}| \\ &= |\ker(\beta)||\{B \in \mathcal{A}_n(R/\gamma^{e-1}R) \mid \det(B) = \gamma^s + \gamma^{e-1}R\}| \\ &= q^{3n-2}|\mathcal{A}_n(R/\gamma^{e-1}R, \gamma^s + \gamma^{e-1}R)|. \end{aligned} \tag{6}$$

Combining (5) and (6), it can be concluded that

$$q|\mathcal{A}_n(R,\gamma^s)| = q^{3n-2}|\mathcal{A}_n(R/\gamma^{e-1}R, \gamma^s + \gamma^{e-1}R)|.$$

Therefore,

$$|\mathcal{A}_n(R,\gamma^s)| = q^{3(n-1)}|\mathcal{A}_n(R/\gamma^{e-1}R, \gamma^s + \gamma^{e-1}R)|$$

as desired. ∎

Applying Lemma 2 recursively, the next corollary follows.

Corollary 6. *Let R be a FCCR of nilpotency index $e+f$ and residue field $\mathbb{F}_q$, where $2 \le e$ and $1 \le f$ are integers. If the maximal ideal of R is generated by γ, then*

$$|\mathcal{A}_n(R,\gamma^s)| = q^{3f(n-1)}|\mathcal{A}_n(R/\gamma^e R, \gamma^s + \gamma^e R)|$$

for all $1 \le s < e$.

A general recursive formula for the number $\mathcal{A}_n(R,\gamma^s)$ is presented for all $s \ge 1$ in the next theorem.

Theorem 3. *Let R be a FCCR of nilpotency index e and residue field $\mathbb{F}_q$ and let n be a positive integer. If the maximal ideal of R is generated by γ, then*

$$|\mathcal{A}_n(R,\gamma^s)| = \frac{q^{3(e-s-1)(n-1)}}{q-1}\left(q^{3n-2}|\mathcal{A}_n(R/\gamma^s R, 0+\gamma^s R)| - |\mathcal{A}_n(R/\gamma^{s+1}R, 0+\gamma^{s+1}R)|\right).$$

for all integers $1 \le s < e$.

Proof. Let $1 \leq s < e$ be an integer and let $\mu : \mathcal{A}_n(R/\gamma^{s+1}R) \to \mathcal{A}_n(R/\gamma^s R)$ be an additive group homomorphism defined by

$$\mu(A) = \overline{A},$$

where $\overline{[a_{ij} + \gamma^{s+1}R]} := [a_{ij} + \gamma^s R]$ for all $[a_{ij} + \gamma^{s+1}R] \in \mathcal{A}_n(R/\gamma^{s+1}R)$. Then, for each $A \in \mathcal{A}_n(R/\gamma^{s+1}R)$, $\det(\mu(A)) = 0 + \gamma^s R$ if and only if $\det(A) = \gamma^s b + \gamma^{s+1}R$ for some $b \in V$, where V is defined in Lemma 1. Since $|\ker(\mu)| = q^{3n-2}$, we have

$$\begin{aligned} q^{3n-2}|\mathcal{A}_n(R/\gamma^s R, 0 + \gamma^s R)| &= |\ker(\mu)||\mathcal{A}_n(R/\gamma^s R, 0 + \gamma^s R)| \\ &= |\mathcal{A}_n(R/\gamma^{s+1}R, 0 + \gamma^{s+1}R)| \\ &\quad + \sum_{b \in V \setminus \{0\}} |\mathcal{A}_n(R/\gamma^{s+1}R, \gamma^s b + \gamma^{s+1}R)| \\ &= |\mathcal{A}_n(R/\gamma^{s+1}R, 0 + \gamma^{s+1}R)| \\ &\quad + (q-1)|\mathcal{A}_n(R/\gamma^{s+1}R, \gamma^s + \gamma^{s+1}R)| \end{aligned}$$

by Proposition 5. Hence, we have

$$\begin{aligned} &|\mathcal{A}_n(R/\gamma^{s+1}R, \gamma^s + \gamma^{s+1}R)| \\ &\quad = \frac{1}{q-1}\left(q^{3n-2}|\mathcal{A}_n(R/\gamma^s R, 0 + \gamma^s R)| - |\mathcal{A}_n(R/\gamma^{s+1}R, 0 + \gamma^{s+1}R)|\right). \end{aligned} \tag{7}$$

By Corollary 6, we have

$$\begin{aligned} |\mathcal{A}_n(R, \gamma^s)| &= |\mathcal{A}_n(R/\gamma^{e+1+(s-e-1)}R, \gamma^s + \gamma^{e+1+(s-e-1)}R)| \\ &= q^{3(e-s-1)(n-1)}|\mathcal{A}_n(R/\gamma^{s+1}R, \gamma^s + \gamma^{s+1}R)|. \end{aligned} \tag{8}$$

Combining (7) and (8), we therefore have

$$|\mathcal{A}_n(R, \gamma^s)| = \frac{q^{3(e-s-1)(n-1)}}{q-1}\left(q^{3n-2}|\mathcal{A}_n(R/\gamma^s R, 0 + \gamma^s R)| - |\mathcal{A}_n(R/\gamma^{s+1}R, 0 + \gamma^{s+1}R)|\right)$$

as desired. ■

For a FCCR of nilpotency index 2, the following bound on $|\mathcal{A}_n(R, a)|$ is derived for all $a \in R \setminus \mathbb{F}_q$ and positive integers n.

Corollary 7. *Let R be a FCCR of nilpotency index 2 and residue field $\mathbb{F}_q$. If the maximal ideal of R is generated by γ, then $|\mathcal{A}_1(R, a)| = 1$ and*

$$|\mathcal{A}_n(R, a)| \leq (q+1)q^{5n-7}\left(q^{n+1} - (q-1)^n(q + (n-1))\right) - q^2(q^3+1)|\mathcal{A}_{n-1}(R, 0)|$$

for all $a \in R \setminus \mathbb{F}_q$ and integers $n \geq 2$.

Proof. Clearly, $|\mathcal{A}_1(R, a)| = 1$. Let $n \geq 2$ be an integer. By setting $s = 1$ in (7), we have

$$|\mathcal{A}_n(R, a)| = |\mathcal{A}_n(R, \gamma)|$$

$$= \frac{1}{q-1}\left(q^{3n-2}|\mathcal{A}_n(R/\gamma R, 0+\gamma R)| - |\mathcal{A}_n(R,0)|\right)$$
$$= \frac{1}{q-1}\left(q^{3n-2}|\mathcal{A}_n(\mathbb{F}_q, 0)| - |\mathcal{A}_n(R,0)|\right).$$

Form the proof of Corollary 5, we have

$$|\mathcal{A}_n(R,0)| \geq (q-1)q^2(q^3+1)|\mathcal{A}_{n-1}(R,0)| + q^{3n-4}|\mathcal{A}_n(\mathbb{F}_q,0)|$$

which implies that

$$\begin{aligned}
|\mathcal{A}_n(R,a)| &\leq \frac{1}{q-1}\left(q^{3n-2}|\mathcal{A}_n(\mathbb{F}_q,0)| - \left((q-1)q^2(q^3+1)|\mathcal{A}_{n-1}(R,0)| + q^{3n-4}|\mathcal{A}_n(\mathbb{F}_q,0)|\right)\right) \\
&= \frac{1}{q-1}\left((q^{3n-2}-q^{3n-4})|\mathcal{A}_n(\mathbb{F}_q,0)| - (q-1)q^2(q^3+1)|\mathcal{A}_{n-1}(R,0)|\right) \\
&= \frac{1}{q-1}\left((q^2-1)q^{3n-4}|\mathcal{A}_n(\mathbb{F}_q,0)| - (q-1)q^2(q^3+1)|\mathcal{A}_{n-1}(R,0)|\right) \\
&= (q+1)q^{3n-4}|\mathcal{A}_n(\mathbb{F}_q,0)| - q^2(q^3+1)|\mathcal{A}_{n-1}(R,0)|.
\end{aligned}$$

By Corollary 2, we have

$$|\mathcal{A}_n(\mathbb{F}_q,0)| = q^{3n-2} - q^{2n-3}(q-1)^n(q+(n-1)),$$

and hence,

$$\begin{aligned}
|\mathcal{A}_n(R,a)| &\leq (q+1)q^{3n-4}\left(q^{3n-2} - q^{2n-3}(q-1)^n(q+(n-1))\right) - q^2(q^3+1)|\mathcal{A}_{n-1}(R,0)| \\
&= (q+1)q^{5n-7}\left(q^{n+1} - (q-1)^n(q+(n-1))\right) - q^2(q^3+1)|\mathcal{A}_{n-1}(R,0)|
\end{aligned}$$

as desired. ■

We note that, for a FCCR of nilpotency index $e = 2$, a bound on $|\mathcal{A}_{n-1}(R,0)|$ is determined recursively in Corollary 5.

4. Conclusion and Remarks

The enumeration of arrowhead matrices with prescribed determinant has been established over a finite field $\mathbb{F}_q$ and a finite commutative chain ring R. Over $\mathbb{F}_q$, the number of $n \times n$ arrowhead matrices with prescribed determinant has been completely determined for all positive integers n. Subsequently, the number of $n \times n$ non-singular arrowhead matrices with prescribed determinant over R has been given for all positive integers n. For singular arrowhead matrices over R, bounds on the number of $n \times n$ singular arrowhead matrices have been presented. A general set up for an upper bound for the number of $n \times n$ singular arrowhead matrices over R with zero determinant has been given as well as a lower bound for the number of $n \times n$ singular arrowhead matrices over R with a zero-divisor determinant. For $e = 2$, rigorous forms of such bounds have been presented.

It would be interesting to derive an explicit formula for the number of $n \times n$ singular arrowhead matrices of a fixed determinant in a FCCR R. In general, the study of $n \times n$ arrowhead matrices with prescribed determinant over more general finite commutative rings such as principal ideal rings, local rings, and Frobenius rings is another interesting problem.

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