EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 1, 2024, 135-146
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Common Terms of k-Pell and Tribonacci Numbers

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Abstract. Let T_m be a Tribonacci sequence, and let the k-Pell sequence be a generalization of the Pell sequence for $k \geq 2$. The first k terms are $0, 0, ..., 0, 1$, and each term after the forewords is defined by linear recurrence

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + ... + P_{n-k}^{(k)}.$$

We study the solution of the Diophantine equation $P_n^{(k)} = T_m$ for the positive integer (n, k, m) with $k \geq 2$. We use the lower bound for linear forms in logarithms of algebraic numbers with the theory of the continued fraction.

2020 Mathematics Subject Classifications: 11D61, 11J86, 11J70, 11B83

Key Words and Phrases: Exponential Diophantine equation, Linear forms in logarithms, k-Pell numbers, Tribonacci numbers.

1. Introduction

The Pell sequence is defined by $P_n = 2P_{n-1} + P_{n-2}$, for all $n \geq 3$, where $P_0 = 0$ and $P_1 = 1$.
Let an integer $k \geq 2$. The generalization of the Pell sequence is a k-Pell sequence, denoted by $\{P_n^{(k)}\}_{n \geq -(k-2)}$ given linear recurrence as:

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + ... + P_{n-k}^{(k)} \quad \text{for all } n \geq 2, \tag{1}$$

with the initial conditions $P_{-(k-2)}^{(k)} = P_{-(k-3)}^{(k)} = ... = P_0^{(k)} = 0$ and $P_1^{(k)} = 1$. If $k = 2$ in equation (1), it becomes a linear recurrence of the Pell sequence.
The Tribonacci sequence T_m is defined by

$$T_m = T_{m-1} + T_{m-2} + T_{m-3} \quad \text{for each } m \geq 3 \tag{2}$$

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DOI: https://doi.org/10.29020/nybg.ejpam.v17i1.4989

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with initial conditions $T_0 = 0$, $T_1 = T_2 = 1$. It's first few terms are

$$0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, ...$$

The Online Encyclopedia of Integer (OEIS) of Pell and Tribonacci sequences are A000129 and A000073, respectively. Presently, researchers are finding the intersection between two recurrences, and several studies have been published on k-Fibonacci, k-Pell, Tribonacci, Padovan, and Perrin sequences related to other sequences. One can cite [1, 3, 7, 9, 10, 13]. Our aim is to show that there are common terms between k-generalized Pell numbers and Tribonacci numbers. The earlier findings guided our completion of the investigation.

2. Auxiliary Results

2.1. Properties of Tribonacci sequence

The characteristic polynomial of the Tibonacci sequence is

$$f(x) = x^3 - x^2 - x - 1.$$

The Tribonacci sequence has one real root η_1 with two complex roots η_2 and η_3.

$$\eta_1 = \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3},$$
$$\eta_2 = \frac{1 + \omega \sqrt[3]{19 + 3\sqrt{33}} + \omega^2 \sqrt[3]{19 - 3\sqrt{33}}}{3},$$
$$\eta_3 = \frac{1 + \omega^2 \sqrt[3]{19 + 3\sqrt{33}} + \omega \sqrt[3]{19 - 3\sqrt{33}}}{3},$$

where $\omega = \frac{-1+i\sqrt{3}}{2}$. Spickerman [12] found the Binet formula of the Tribonacci numbers as

$$T_m = \frac{\eta_1^{m+1}}{(\eta_1 - \eta_2)(\eta_1 - \eta_3)} + \frac{\eta_2^{m+1}}{(\eta_2 - \eta_1)(\eta_2 - \eta_3)} + \frac{\eta_3^{m+1}}{(\eta_3 - \eta_1)(\eta_3 - \eta_2)}, \quad \text{for all } m \geq 0. \quad (3)$$

The generating function of the Tribonacci sequence is:

$$g(x) = \frac{x}{1 - x - x^2 - x^3} = \sum_{m=0}^{\infty} T_m x^m.$$

Note that we have the following identities

$$\eta_1 + \eta_2 + \eta_3 = 1,$$
$$\eta_1\eta_2 + \eta_2\eta_3 + \eta_1\eta_3 = -1,$$
$$\eta_1\eta_2\eta_3 = 1.$$

Furthermore, Dresden and Du [6] presented a Binet-style formula for generating k-generalized Fibonacci numbers. If $k = 3$, it follows that:

$$T_m = \frac{(\eta_1 - 1)\eta_1^{m-1}}{2 + 4(\eta_1 - 2)} + \frac{(\eta_2 - 1)\eta_2^{m-1}}{2 + 4(\eta_2 - 2)} + \frac{(\eta_3 - 1)\eta_3^{m-1}}{2 + 4(\eta_3 - 2)}, \quad \text{for all } m \geq 0. \tag{4}$$

Moreover, Dresden and Du [6, Lemma 5] found that the Tribonacci numbers can be written as

$$T_m = c\eta_1^{m-1} + d_m \quad \text{with } |d_m| < \frac{1}{2}, \quad \text{for all } m \geq 1, \tag{5}$$

where $c = (\eta_1 - 1)/(4\eta_1 - 6) \approx 0.61$. For $m \geq 1$, the inequality

$$\eta_1^{m-2} \leq T_m \leq \eta_1^{m-1}, \tag{6}$$

hold.

2.2. Properties of $k-$generalized Pell sequence

We are aware that the characteristic polynomial of the k-generalized Pell sequence is

$$\Psi_k(x) = x^k - 2x^{k-1} - x^{k-2} - ... - x - 1.$$

Bravo, Herrera and Luca [4] showed that $\Psi_k(x)$ is irreducible over $\mathbb{Q}[x]$ and has one positive real root $\alpha(k)$ outside the unit circle. The other roots were inside the unit circle. Moreover, they showed the following:

$$\phi^2(1 - \phi^{-k}) < \alpha(k) < \phi^2, \quad \text{for all } k \geq 2, \tag{7}$$

where $\phi = ((1 + \sqrt{5})/2)$. To simplify the notation, we omit the dependence on k of α. The authors found that the Binet formula for $P_n^{(k)}$ is

$$P_n^{(k)} = \sum_{i=1}^{k} g_k(\alpha_i)(\alpha_i)^n, \tag{8}$$

where a_i represents the root of the characteristic polynomial $\Psi_k(x)$ and g_k is given by

$$g_k(x) = \frac{x - 1}{(k + 1)x^2 - 3kx + k - 1}, \quad \text{for all } k \geq 2.$$

Bravo and Herrera [2, Lemma 1] proved that

$$0.276 < g_k(\alpha) < 0.5 \quad \text{and } |g_k(\alpha_i)| < 1, \quad 2 \leq i \leq k,$$

where $g_k(\alpha)$ is not an algebraic integer. Furthermore, they proved that the logarithmic height of g_k is

$$h(g_k) < 4k\log(\phi) + k\log(k + 1), \quad \text{for all } k \geq 2. \tag{9}$$

According to the above notation, Bravo, Herrera and Luca [4] showed that formula (8), given by the approximation

$$\left|P_n^{(k)} - g_k(\alpha)\alpha^n\right| < \frac{1}{2}, \text{ for all } n \geq 2-k.$$

Therefore, for $n \geq 1$ and $k \geq 2$, we have

$$P_n^{(k)} = g_k(\alpha)\alpha^n + e_k(n), \quad \text{where } |e_k(n)| \leq \frac{1}{2}. \tag{10}$$

Moreover, the inequality

$$\alpha^{n-2} \leq P_n^{(k)} \leq \alpha^{n-1} \tag{11}$$

holds for all $n \geq 1$ and $k \geq 2$.

Lemma 1. *([2, Lemma 2]) If $k \geq 30$ and $n \geq 1$ are integers satisfying $n < \phi^{k/2}$, then*

$$g_k(\alpha)\alpha^n = \frac{\phi^{2n}}{\phi+2}\left(1+\zeta\right), \quad \textit{where } |\zeta| < \frac{4}{\phi^{k/2}},\ \phi = \frac{1+\sqrt{5}}{2}. \tag{12}$$

Lemma 2. *([14, Lemma 2.2]) Let $v, x \in \mathbb{R}$ and $0 < v < 1$. If $|x| < v$, then*

$$|\log(1+x)| < \frac{-\log(1-v)}{v}|x|.$$

2.3. Linear forms in logarithms

Let γ be an algebraic number of degree d with minimal polynomial

$$c_0x^d + c_1x^{d-1} + \ldots + c_d = c_0\prod_{i=1}^{d}\left(x - \gamma^{(i)}\right) \in \mathbb{Z}[x],$$

where the $\gamma^{(i)}$'s are conjugates of γ, and the c_i's are relative primes to each other with $c_0 > 0$. Then the logarithmic height of γ is given by

$$h(\gamma) = \frac{1}{d}\left(\log c_0 + \sum_{i=1}^{d}\log\left(\max\left\{\left|\gamma^{(i)}\right|, 1\right\}\right)\right). \tag{13}$$

If $\gamma = \frac{a}{b}$ is rational number with $gcd(a, b) = 1$ and $b > 0$, then $h(\gamma) = \log(\max\{|a|, b\})$. Some properties of the logarithmic height function are listed below, which will be used in the next parts of this paper:

$$h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2, \tag{14}$$

$$h\left(\eta\gamma^{\pm 1}\right) \leq h(\eta) + h(\gamma), \tag{15}$$

$$h\left(\eta^k\right) = |k|h(\eta). \tag{16}$$

We use the following [5, Theorem 9.4], which is a modified version of the Matveev result [8]

Theorem 1. *Let $\mathbb{L}$ be a real algebraic number field of degree D over $\mathbb{Q}$. Let $\gamma_1, \ldots, \gamma_t \in \mathbb{L}$ be a positive real algebraic number, and $b_1, b_2, \ldots, b_t$ be nonzero integers such that*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1,$$

is not zero. Then

$$\log|\Lambda| > (-1.4)\left(30^{t+3}\right)\left(t^{4.5}\right)\left(D^2\right)(A_1 \ldots A_t)(1 + \log D)(1 + \log B),$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$A_i \geq \max\{Dh(\gamma_i), |\log(\gamma_i)|, 0.16\}, 1 \leq i \leq t.$$

2.4. De Weger reduction method

To reduce the upper bound, we present a variant of Baker and Davenport's reduction method [14]. Let $\vartheta_1, \vartheta_2, \beta \in \mathbb{R}$ be given, and let $x_1, x_2 \in \mathbb{Z}$ be unknowns. Let

$$\Lambda = \beta + x_1\vartheta_1 + x_2\vartheta_2. \tag{17}$$

Let c, δ be positive constants. Set $X = \max\{|x_1|, |x_2|\}$. Let X_0, Y be positive. Assume that

$$|\Lambda| < c \cdot \exp(-\delta \cdot Y), \tag{18}$$

$$Y \leq X \leq X_0. \tag{19}$$

When $\beta = 0$ in (17), we get

$$\Lambda = x_1\vartheta_1 + x_2\vartheta_2.$$

Put $\vartheta = -\vartheta_1/\vartheta_2$. We assume that x_1 and x_2 are coprime. Let the continued fraction expansion of ϑ be given by

$$[a_0, a_1, a_2, \ldots],$$

and let the k-th convergent of ϑ be p_k/q_k for $k = 0, 1, 2, \ldots$ We may assume without loss of generality that $|\vartheta_1| < |\vartheta_2|$ and that $x_1 > 0$. We obtain the following results.

Lemma 3. *([14, Lemma 3.2]) Let*

$$A = \max_{0 \leq k \leq Y_0} a_{k+1},$$

where

$$Y_0 = -1 + \frac{\log\left(\sqrt{5}X_0 + 1\right)}{\log\left(\frac{1+\sqrt{5}}{2}\right)}.$$

If (18) and (19) hold for x_1, x_2 and $\beta = 0$, then

$$Y < \frac{1}{\delta}\log\left(\frac{c(A+2)X_0}{|\vartheta_2|}\right). \tag{20}$$

When $\beta \neq 0$ in (17), put $\vartheta = -\vartheta_1/\vartheta_2$ and $\psi = \beta/\vartheta_2$. Then we have $\frac{\Lambda}{\vartheta_2} = \psi - x_1\vartheta + x_2$. Let p/q be a convergent of ϑ with $q > X_0$. Tthe distance between real number T and the closest integer is expressed as $\|T\| = \min\{|T-n| : n \in \mathbb{Z}\}$. We obtain the following result.

Lemma 4. *([14, Lemma 3.3]) Suppose that*

$$\|q\psi\| > \frac{2X_0}{q}.$$

Then, the solutions of (18) and (19) satisfy

$$Y < \frac{1}{\delta}\log\left(\frac{q^2c}{|\vartheta_2|\, X_0}\right). \tag{21}$$

We need the following discovery to prove our theorem.

Lemma 5. *([11, Lemma 7]) If $r \geq 1$ and $S \geq (4r^2)^r$, and $\frac{L}{(\log L)^r} < S$, then*

$$L < 2^r S(\log S)^r.$$

3. Main Results

Theorem 2. *The positive integer solutions of the Diophantine equation*

$$P_n^{(k)} = T_m, \tag{22}$$

where $k \geq 2$ are $P_1^{(k)} = T_1 = T_2, P_2^{(k)} = T_3$, and $P_4^{(k)} = T_6$.

To prove Theorem 2 will be done in four steps.

3.1. Relation between n and m

For the Diophantine equation (22) in the range $1 \leq n \leq k+1$, we have $P_n^{(k)} = F_{2n-1}$, where F_n is a Fibonacci number, and we obtain the set of solutions in Theorem 2. For the remaining possibility, we assumed that $n \geq k+2$ and $k \geq 2$. By combining inequalities (6) and (11) with equation (22), we obtain:

$$\alpha^{n-2} \leq P_n^{(k)} = T_m \leq \eta_1^{m-1} \quad \text{and } \eta_1^{m-2} \leq T_m = P_n^{(k)} \leq \alpha^{n-1},$$

we conclude that

$$(n-2)\frac{\log(\alpha)}{\log(\eta_1)} \leq m-1 \quad \text{and } m \leq (n-1)\frac{\log(\alpha)}{\log(\eta_1)} + 2,$$

we obtain

$$0.79n - 1.58 < m - 1 < m < 1.58n + 0.42,$$

because $\phi^2(1-\phi^{-k}) < \alpha(k) < \phi^2$ for all $k \geq 2$. We consider the following

$$0.79n - 1.58 < m - 1 < m < 2n. \tag{23}$$

3.2. Bounding n in terms of k

In this step, we prove the following lemma to find an upper bound for n in terms of k.

Lemma 6. *If (m, n, k) is a positive integers solution of equation (22) with $k \geq 2$ and $n \geq k+2$, then the inequalities*

$$0.63m < n < 7.6 \cdot 10^{16} k^5 (\log(k))^3$$

hold.

Proof. Combining equation (22), (5), and (10), we obtain:

$$g_k(\alpha)\alpha^n + e_k(n) = c\eta_1^{m-1} + d_m.$$

Taking absolute values for both sides, we get

$$\left| g_k(\alpha)\alpha^n - c\eta_1^{m-1} \right| < \frac{1}{2} + |d_m| < 1. \tag{24}$$

Dividing both sides by $c\eta_1^{m-1}$, we deduce that

$$\left| (c^{-1} g_k(\alpha))\alpha^n \eta_1^{-(m-1)} - 1 \right| < \frac{1.6}{\eta_1^{m-1}}. \tag{25}$$

We apply Theorem 1 to the left-hand side inequality (25) with parameters $t := 3$, where $\gamma_1 := c^{-1} g_k(\alpha), \gamma_2 := \alpha, \gamma_3 := \eta_1$, and $b_1 := 1, b_2 := n, b_3 = -(m-1)$. So $\mathbb{L} := \mathbb{Q}(\gamma_1, \gamma_2, \gamma_3)$. Thus, $D := [\mathbb{L}, \mathbb{Q}] = 3k$. To show that Λ is nonzero, it is assumed that $\Lambda = 0$, which implies that $g_k(\alpha) = c\eta_1^{(m-1)}\theta_1^{-n}$, we obtain $g_k(\alpha)$ as an algebraic integer, which is a contradiction. Hence $\Lambda \neq 0$. Not that

$$h(\gamma_1) < h(c) + h(g_k(\alpha)) < \frac{\log(44)}{3} + 4k\log(\phi) + k\log(k+1) < 5.3k\log(k),$$

which holds for all $k \geq 2$ and a minimal polynomial $44x^3 - 44x^2 + 12x - 1$ of c. Therefore, $h(\gamma_2) = \frac{\log(\alpha)}{k} < \frac{2\log(\phi)}{k}$ and $h(\gamma_3) = \frac{\log(\eta_1)}{3}$. Thus, we obtained $A_1 := 15.9k^2\log(k), A_2 := 6\log(\phi)$, and $A_3 := k\log(\eta_1)$. In addition, taking $B := 2n$, since $\max\{|1|, |n|, |-(m-1)|\} \leq 2n$. Thus, by Theorem 1, we get that

$$\frac{1.6}{\eta_1^{m-1}} > |\Lambda| > \exp\{-G(1+\log(2n))(15.9k^2\log(k))(6\log(\phi))(k\log(\eta_1))\},$$

where $G = (1.4)\left(30^6\right)\left(3^{4.5}\right)(3k)^2(1+\log(3k))$. We get

$$(m-1)\log(\eta_1) - \log(1.6) < 3.61 \cdot 10^{13} k^5 \log(k)(1+\log(3k))(1+\log(2n)). \tag{26}$$

Using the facts that $(1+\log(3k)) < 4.1\log(k)$ for all $k \geq 2$ and $(1+\log(2n)) < 2.3\log(n)$ for all $n \geq 4$. Simplifying the calculation, we obtain

$$m - 1 < 5.6 \cdot 10^{14} k^5 (\log(k))^2 \log(n),$$

By inequality (23), we deduce that

$$\frac{n}{\log(n)} < 7.1 \cdot 10^{14} k^5 (\log(k))^2,$$

Now we apply Lemma 5 take $S := 7.1 \cdot 10^{14} k^5 (\log(k))^2, L := n, r := 1$ with $34.2 + 5\log(k) + 2\log(\log(k)) < 53.5\log(k)$ for all $k \geq 2$, we get

$$\begin{aligned} n &< 2(7.1 \cdot 10^{14} k^5 (\log(k))^2)(\log(7.1 \cdot 10^{14} k^5 (\log(k))^2)) \\ &< (1.42 \cdot 10^{15} k^5 (\log(k))^2)(34.2 + 5\log(k) + 2\log(\log(k))) \\ &< 7.6 \cdot 10^{16} k^5 (\log(k))^3. \end{aligned} \tag{27}$$

3.3. The case $2 \leq k \leq 350$

In the previous, we obtained a very large upper bound of n. We apply Lemma 4 to reduce the upper bound. In this case, we will prove the following lemma.

Lemma 7. *The only solution of the Diophantine equation* (22) *is* $P_4^{(k)} = T_6$ *where* $n \geq k+2$ *and* $2 \leq k \leq 350$

Proof. To apply Lemma 4, let

$$v_1 := n\log(\alpha) - (m-1)\log(\eta_1) + \log(c^{-1} g_k(\alpha)).$$

Then we have, by inequality (25),

$$|e^{v_1} - 1| < \frac{1.6}{\eta_1^{m-1}}.$$

We know $v_1 \neq 0$, since $\Lambda \neq 0$. If $m \geq 2$, we have

$$\frac{1.6}{\eta_1^{m-1}} < 0.87.$$

By Lemma 2, we get

$$|v_1| = |\log(\Lambda + 1)| = -\frac{\log(1 - 0.87)}{0.87} \cdot \frac{1.6}{\eta_1^{m-1}} < \frac{3.75}{\eta_1^{m-1}},$$

and

$$0 < \left|(m-1)(-\log(\eta_1)) + n\log(\alpha) + \log(c^{-1} g_k(\alpha))\right| < 3.75 \cdot \exp(-(m-1)\log(\eta_1)). \tag{28}$$

According to Lemma 4, we obtain

$$c := 3.75, \quad \delta := \log(\eta_1), \quad \psi := \frac{\log(c^{-1} g_k(\alpha))}{\log(\alpha)},$$

$$\vartheta := \frac{\log(\eta_1)}{\log(\alpha)}, \quad \vartheta_1 := -\log(\eta_1), \quad \vartheta_2 := \log(\alpha), \quad \beta := \log(c^{-1}g_k(\alpha)).$$

We are aware that ϑ is an irrational number. Taking $X_0 := 1.5 \cdot 10^{17}k^5(\log(k))^3$, which is an upper bound of $m-1$ and n. Using Maple program inspection, the maximum value of $\frac{1}{\delta}\log\left(\frac{q^2c}{|\vartheta_2|X_0}\right)$ for $k \in [2, 350]$ is 143 . We get $1 \leq m-1 \leq 143$ and discover the possible values of the Diophantine equation (22) for which $k \in [2, 350]$ have $2 \leq m \leq 144$, and by inequality (23), we obtain $4 \leq n \leq 181$. The only possible solution in this range was $P_4^{(k)} = T_6$.

3.4. The case $k > 350$

In this case, we prove the following lemma

Lemma 8. *The Diophantine equation* (22) *has no solution for* $n \geq k+2$ *and* $k > 350$

Proof. For $k > 350$, as a result of Lemma 1, we have

$$n < 7.6 \cdot 10^{16}k^5(\log(k))^3 < \phi^{k/2}.$$

From (12),(22) and (24), we get

$$\left|\frac{\phi^{2n}}{\phi+2} - c\eta_1^{m-1}\right| < \left|g_k(\alpha)\alpha^n - c\eta_1^{m-1}\right| + \frac{\phi^{2n}}{\phi+2}|\zeta| < 1 + \frac{4\phi^{2n}}{(\phi+2)\phi^{k/2}}.$$

Dividing both sides by $\frac{\phi^{2n}}{\phi+2}$, it becomes

$$|\Lambda_1| < \frac{7.6}{\phi^{k/2}}, \text{ where } \Lambda_1 := c(\phi+2)\phi^{-2n}\eta_1^{m-1} - 1. \tag{29}$$

Using the fact that $\frac{1}{\phi^{2n}} < \frac{1}{\phi^{k/2}}$ yield for $n \geq k+2$. It is known that Λ_1 is nonzero. If Λ_1 is zero, then $\frac{\phi^{2n}}{\eta_1^{m-1}} = c(\phi+2)$, and we get the left-hand side as an algebraic integer, but the right-hand side is not an algebraic integer, which is impossible, hence, $\Lambda_1 \neq 0$. We apply Theorem 1, we take parameters $t := 3$, and $\gamma_1 := c(\phi+2), \gamma_2 := \phi, \gamma_3 := \eta_1$, and $b_1 := 1, b_2 := -2n, b_3 = (m-1)$. So $\mathbb{L} := \mathbb{Q}(\gamma_1, \gamma_2, \gamma_3)$. Thus $D := [\mathbb{L}, \mathbb{Q}] = 6$. Moreover, $h(\eta_2) = \frac{\log(\phi)}{2}$, $h(\eta_3) = \frac{\log(\eta_1)}{3}$ and

$$h(\eta_1) \leq h(c) + h(\phi) + 2\log(2) < 2.9,$$

it follows that $A_1 := 17.4$, $A_2 := 1.45$ and $A_3 := 1.22$. Since $\max\{|1|, |-2n|, |(m-1)|\} \leq 2n$, we can take $B := 2n$. Thus, by Theorem 6 , we get

$$\frac{k}{2}\log(\phi) - \log(7.6) < 4.43 \cdot 10^{14} \cdot (1 + \log(2n)).$$

Using fact that $1 + \log(2n) < 1.3\log(n)$ for all $n \geq k + 2 > 352$, which implies that

$$k < 2.4 \cdot 10^{15}\log(n).$$

We have an upper bound of n in inequality(27), then $38.87 + 5\log(k) + 3\log(\log(k)) < 13\log(k)$ for all $k > 350$, we get

$$\begin{aligned} k &< 2.4 \cdot 10^{15}\log(7.6 \cdot 10^{16}k^5(\log(k))^3) \\ &< 2.4 \cdot 10^{15}(38.87 + 5\log(k) + 3\log(\log(k))) \\ &< 3.12 \cdot 10^{16}\log(k). \end{aligned}$$

The above inequality gives

$$k < 1.3 \cdot 10^{18}.$$

Thus, we get

$$\begin{aligned} n &< 7.6 \cdot 10^{16}(1.3 \cdot 10^{18})^5(\log(1.3 \cdot 10^{18}))^3 < 2.1 \cdot 10^{112} \\ m &< 2(2.1 \cdot 10^{112}) < 4.2 \cdot 10^{112}. \end{aligned}$$

Let

$$v_2 := (m-1)\log(\eta_1) - (2n)\log(\alpha) + \log(c(\phi+2)).$$

Then we have, by inequality (29),

$$|e^{v_2} - 1| < \frac{7.6}{\phi^{k/2}}.$$

We know $v_2 \neq 0$, since $\Lambda_1 \neq 0$. If $k \geq 350$, we get

$$\frac{7.6}{\phi^{k/2}} < 0.1.$$

By Lemma 2, we obtain the inequality

$$|v_2| = |\log(\Lambda_1 + 1)| = -\frac{\log(1-0.1)}{0.1} \cdot \frac{7.6}{\phi^{k/2}} < \frac{8.1}{\phi^{k/2}}.$$

Thus, we get

$$0 < |(m-1)\log(\eta_1) - 2n\log(\phi) + \log(c(\phi+2))| < 8.1 \cdot \exp(-0.24 \cdot k). \tag{30}$$

Applying lemma 4, we can take

$$c := 8.1, \quad \delta := 0.24, \quad \psi := -\frac{\log(c(\phi+2))}{\log(\phi)},$$

$$\vartheta := \frac{\log(\eta_1)}{\log(\phi)}, \quad \vartheta_1 := \log(\eta_1), \quad \vartheta_2 := -\log(\phi), \quad \beta := \log(c(\phi+2)).$$

We take $M := 4.2 \cdot 10^{112}$, which is the upper bound for $m-1$. A quick inspection with the help of Maple programming found that q_{211} is convergent of ϑ. By Lemma 4, we obtain

$$k < \frac{1}{0.24}\left(\frac{q_{211}^2 \cdot 8.1}{4.2 \cdot 10^{122} \cdot |-\log(\phi)|}\right) < 1105. \tag{31}$$

By inequalities of (27) and (23) we have

$$n < 4.3 \cdot 10^{34} \text{ and } m < 8.6 \cdot 10^{34}.$$

Again we apply Lemma 4 for (30) with $M := 8.6 \cdot 10^{34}$, we found that q_{72} is a convergent of ϑ, and $k < 389$. Hence

$$n < 1.4 \cdot 10^{32} \text{ and } m < 2.8 \cdot 10^{32}.$$

Third time applying Lemma 4 for (30) with $M := 2.8 \cdot 10^{32}$, we found that q_{65} is a convergent of ϑ, and $k < 342$, we get contradiction by our assumption that $k > 350$. Theorem 2 is proved.

4. Conclusion

We found all solutions of the Diophantine equation (22), where $P_n^{(k)}$ is a k-generalized Pell number and T_m is a Tribonacci number, for each positive integer n, m and k. We used a lower bound for linear forms in logarithms of algebraic numbers to get an upper bound for n. Then, we used a variation of the Baker-Davenport reduction method called the De Weger reduction method to reduce the upper bound.

Acknowledgements

The authors express their gratitude to the anonymous reviewers for the instructive suggestions.

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