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Looking at Two Ways of Constructing Quotient Hyper BN-algebras and Some Notes on Hyper BN-ideals

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Abstract. A hyper BN-algebra is a nonempty set H together with a hyperoperation "$\circledast$" and a constant 0 such that for all $x, y, z \in H$: $x \ll x$, $x \circledast 0 = \{x\}$, and $(x \circledast y) \circledast z = (0 \circledast z) \circledast (y \circledast x)$, where $x \ll y$ if and only if $0 \in x \circledast y$. We investigated the structures of ideals in the Hyper BN-algebra setting. We established equivalency of weak hyper BN-ideals and hyper subBN-algebras. Also, we found a condition when a strong hyper BN-ideal become a hyper BN-ideal. Finally, we looked at two ways in constructing the quotient hyper BN-algebras and investigated the relationship between the two constructions.

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1. Introduction

In classical algebraic theory, groups are sets equipped with an operation that combines any two elements to produce a third element. They are often used to study symmetry and transformations. Rings, on the other hand, are sets with two operations, usually addition and multiplication, and they are used to study arithmetic properties. Fields are algebraic structures that have both addition and multiplication operations, and they are fundamental in areas like number theory and geometry.

The concept of the algebraic hyperstructure theory was brought by F. Marty [9] at the 8th Congress of Scandinavian Mathematicians in 1934. One of the main point of this

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introduction is to generalized groups. A binary operation was generalized using hyperoperation in this setting. If we have a set H, then a hyperoperation is a mapping from $H \times H$ to the set of nonempty subsets of H.

After quite some time, researchers explore this concept and formulated counterparts of some classical algebraic structures. This led to various introduction of algebraic hyperstructures: hyper BCI-algebras [10], hyper BCC-algebras [1], hyper GR-algebras [7], hyper B-algebras [5], etc. In 2022, we applied this concept to BN-algebras [8]. We called them hyper BN-algebras [3].

In mathematics, an ideal is a fundamental concept in the study of algebraic structures, particularly in the field of abstract algebra. Ideals are subsets of algebraic structures that possess special properties. They are a powerful tool in abstract algebra, allowing mathematicians to study the structure and properties of algebraic structures in a more general and systematic way. In [10], various ideals of a hyper BCI-algebra was introduced and some relationship were established from among these ideals. A more specific properties involving weak and strong hyper BCI-ideals was dealt in [2]. Ideals were also investigated in other hyper algebras.

On the other hand, quotient structures of algebras are a concept in abstract algebra that allow us to create new algebraic structures by "modding out" or "factoring out" certain elements or subsets of an existing algebraic structure. This process involves defining an equivalence relation on the original structure and then forming equivalence classes based on this relation. The significance of quotient structures lies in their ability to simplify the study of algebraic structures by focusing on the essential properties and relationships. They provide a way to abstract away certain elements or subsets that may not be of immediate interest, allowing mathematicians to analyze the structure in a more manageable and structured manner.

In this paper, we will introduce the notion of ideals on hyper BN-algebras and look at two ways of constructing quotient hyper BN-algebras.

2. Preliminaries

This section provides some preliminary concepts and results needed for this paper.

Definition 1. [6] A *binary relation* or simply a *relation* $\sim$ from a set A into a set B is a subset of $A \times B$. If $\sim$ is a relation from A to B, we denote $(a, b) \in \sim$ as $a \sim b$. If $A = B$, we say that $\sim$ is a relation on A.

Definition 2. [6] Let $\sim$ be a binary relation on a set A. Then $\sim$ is called

(i) *reflexive* if for all $x \in A$, $x \sim x$;

(ii) *symmetric* if for all $x, y \in A$, $x \sim y$ implies $y \sim x$; and

(iii) *transitive* if for all $x, y, z \in A$, $x \sim y$ and $y \sim z$ imply $x \sim z$.

If $\sim$ is reflexive, symmetric, and transitive, then $\sim$ is called an *equivalence relation* on A.

Definition 3. [6] Let $\sim$ be an equivalence relation on a set A. For all $x \in A$, the set $\{y \in A : y \sim x\}$ is called the *equivalence class* determined by x, denoted by $[x]_\sim$.

Definition 4. [4] Define $\mathcal{P}(H)$ to be the power set of H and $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\varnothing\}$. A *hyperoperation* on a nonempty set H is a function $\circledast : H \times H \to \mathcal{P}^*(H)$. The value $(x, y) \in H \times H$ under $\circledast$ is defined by $x \circledast y$. If $x \in H$ and $\varnothing \neq A, B \subseteq H$, then

(i) $A \circledast B = \displaystyle\bigcup_{a \in A, b \in B} a \circledast b$; and

(ii) $A \circledast x = A \circledast \{x\}$ and $x \circledast B = \{x\} \circledast B$.

In what follows, the concepts and results are taken from [3] as this is the main reference of this paper.

Definition 5. Let H be a nonempty set and $\circledast$ be a hyperoperation on H. Then $(H, \circledast, 0)$ is called a *hyper* BN*-algebra*, if $0 \in H$ and the following conditions hold: for all $x, y, z \in H$,

(i) $x \ll x$;

(ii) $x \circledast 0 = \{x\}$; and

(iii) $(x \circledast y) \circledast z = (0 \circledast z) \circledast (y \circledast x)$,

where $x \ll y$ if and only if $0 \in x \circledast y$.

Example 1. Let $H = \{0, a, b\}$ be a set. If we define a hyperoperation "$\circledast$" on H as follows:

$\circledast$	0	a	b
0	$\{0\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{0, a\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

then by routinary calculations $(H, \circledast, 0)$ is a hyper BN-algebra.

Example 2. Let $H = \{0, 1, 2\}$ be a set. If we define a hyperoperation "$\circledast$" on H as follows:

$\circledast$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0, 2\}$	$\{0, 1\}$
2	$\{2\}$	$\{0, 1\}$	$\{0, 1\}$

then by routinary calculations $(H, \circledast, 0)$ is a hyper BN-algebra.

Example 3. Let $H = \{0, 1, 2, 3\}$ be a set. If we define a hyperoperation "$\circledast$" on H as follows:

$\circledast$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{3\}$	$\{2\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,3\}$
2	$\{2\}$	$\{0,1,3\}$	$\{0,1,2,3\}$	$\{0,2\}$
3	$\{3\}$	$\{0,1,2\}$	$\{0,3\}$	$\{0,1,2,3\}$

then by routinary calculations $(H, \circledast, 0)$ is a hyper BN-algebra.

Example 4. Let $\mathbb{Z}$ be the set of integers. Define a hyperoperation "$\circledast$" on $\mathbb{Z}$ by:

$$x \circledast y = \begin{cases} \{x\}, & \text{if } y = 0 \\ \{y\}, & \text{if } x = 0 \\ \{x-y, y-x, x+y\}, & \text{otherwise.} \end{cases}$$

Then, we can show that $(\mathbb{Z}, \circledast, 0)$ is a hyper BN-algebra. Note that, the same holds when $\mathbb{Z}$ is replaced by $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

Theorem 1. *In any hyper BN-algebra H, the following hold: for any $x, y, z \in H$ and $\varnothing \neq A, B, C \subseteq H$,*

(i) $x \circledast x = \{x\} \Leftrightarrow x = 0$;

(ii) $x \ll 0 \Rightarrow x = 0$;

(iii) $0 \circledast (0 \circledast x) = \{x\}$;

(iv) $0 \circledast (x \circledast y) = y \circledast x$;

(v) $x \circledast y = (0 \circledast y) \circledast (0 \circledast x)$;

(vi) $(0 \circledast x) \circledast y = (0 \circledast y) \circledast x$;

(vii) $x \ll y \Rightarrow y \ll x$;

(viii) $0 \circledast x = 0 \circledast y \Rightarrow x = y$;

(ix) $(x \circledast z) \circledast (y \circledast z) = (z \circledast y) \circledast (z \circledast x)$;

(x) $A \ll A$;

(xi) $A \subseteq B \Rightarrow A \ll B$;

(xii) $A \subseteq B$ *and* $B \ll C$ *imply* $A \ll C$;

(xiii) $A \ll \{0\} \Rightarrow A = \{0\}$;

(xiv) $A \circledast \{0\} = \{0\} \Rightarrow A = \{0\}$; *and*

(xv) $(A \circledast B) \circledast C = (0 \circledast C) \circledast (B \circledast A)$.

Definition 6. A hyper BN-algebra H is said to be *commutative* if for all $x, y \in H$, $x \circledast y = y \circledast x$.

Example 5. The hyper BN-algebras in Example 1 and Example 2 are commutative while the hyper BN-algebra in Example 3 is not because $2 \circledast 0 = \{2\} \neq \{3\} = 0 \circledast 2$.

Theorem 2. *Let H be a hyper BN-algebra. Then H is commutative if and only if $0 \circledast x = \{x\}$ for all $x \in H$.*

We will provide some basic concepts and results related to hyper BN-algebras. These are taken again from [3].

Definition 7. Let $(H, \circledast, 0)$ be a hyper BN-algebra and let S be a subset of H containing 0. If S is a hyper BN-algebra with respect to the hyperoperation "$\circledast$" on H, we say that S is a *hyper subBN-algebra* of H.

Example 6. Consider the hyper BN-algebra H in Example 1. Let $S = \{0, a\}$ and $T = \{0, b\}$. By routine calculations, both S and T are hyper subBN-algebra of H. If we consider the hyper BN-algebra H in Example 2, then the sets $L = \{0, 1\}$ and $M = \{0, 2\}$ are not hyper subBN-algebra of H.

Theorem 3. *Let S be a nonempty subset of a hyper BN-algebra. Then S is a hyper subBN-algebra if and only if $x \circledast y \subseteq S$, for all $x, y \in S$.*

Definition 8. Let N be a nonempty subset of a hyper BN-algebra. Then N is called *normal* if $(x \circledast a) \circledast (y \circledast b) \subseteq N$ whenever $x \circledast y, a \circledast b \subseteq N$.

Example 7. Consider the hyper BN-algebra $H = \{0, a, b\}$ in Example 1. Let $N_1 = \{0, a\}$ and $N_2 = \{0, b\}$. Then it can be shown that N_1 is normal. However, N_2 is not normal because $0 \circledast b = \{b\} \subseteq N_2$ and $a \circledast b = \{b\} \subseteq N_2$ but $(0 \circledast a) \circledast (b \circledast b) = \{a, b\} \nsubseteq N_2$.

Definition 9. A nonempty subset I of a hyper BN-algebra H is said to be *reflexive* if $x \circledast x \subseteq I$ for all $x \in H$.

Example 8. Let $H = \{0, 1, 2\}$ with hyperoperation $\circledast$ defined by the following Cayley table:

$\circledast$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0, 1\}$	$\{2\}$
2	$\{2\}$	$\{2\}$	$\{0, 1\}$

H is a hyper BN-algebra by routine calculations. Let $I = \{0, 1\}$. Then it is reflexive because $x \circledast x \subseteq I$ for $x = 0, 1, 2$. Let $J = \{0, 2\}$. Then J is not reflexive because $1 \circledast 1 \nsubseteq J$.

Theorem 4. *Every normal subset N of a hyper BN-algebra H is a hyper subBN-algebra of H.*

Definition 10. A hyper subBN-algebra S of a hyper BN-algebra H is called *reflexive* (resp. *normal*) *hyper subBN-algebra* if it is reflexive (resp. normal). S is called a *reflexive normal hyper subBN-algebra* if it is both reflexive and normal.

Example 9. Consider the set $H = \{0, 1, 2, 3, 4\}$. Define the hyperoperation "$\circledast$" by the following Cayley table:

$\circledast$	0	1	2	3	4
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$
1	$\{1\}$	$\{0, 3\}$	$\{3\}$	$\{1\}$	$\{4\}$
2	$\{2\}$	$\{3\}$	$\{0, 3\}$	$\{3\}$	$\{4\}$
3	$\{3\}$	$\{1\}$	$\{3\}$	$\{0, 3\}$	$\{4\}$
4	$\{4\}$	$\{4\}$	$\{4\}$	$\{4\}$	$\{0, 3\}$

By routine calculations, H is a hyper BN-algebra. Let $I = \{0, 3\}$. Then I is a reflexive normal hyper subBN-algebra of H.

Theorem 5. *The intersection of family of reflexive normal hyper subBN-algebras of a hyper BN-algebra H is a reflexive normal hyper subBN-algebra of H.*

3. Hyper BN-ideals

In this section, we introduce hyper BN-ideals and reflexive normal hyper BN-ideals. We also give a weaker and stronger version of this concept. We will investigate the nature of relationships between these ideals and also gave some conditions where equivalency of some of these ideals are achieved.

3.1. (Weak, Strong) Hyper BN-ideals

In what follows, we will introduce the concepts of hyper BN-ideals, weak hyper BN-ideals, and strong hyper BN-ideals. We will also investigate their general relationship. Finally, we will investigate the relationship between hyper BN-ideals and hyper subBN-algebras.

Definition 11. Let I be a nonempty subset of a hyper BN-algebra H such that $0 \in I$.

(*i*) I is a *hyper BN-ideal* if for all $x, y \in H$, $x \circledast y \ll I$ and $y \in I$ imply that $x \in I$.

(*ii*) I is a *weak hyper BN-ideal* if for all $x, y \in H$, $x \circledast y \subseteq I$ and $y \in I$ imply that $x \in I$.

(*iii*) I is a *strong hyper BN-ideal* if for all $x, y \in H$, $(x \circledast y) \cap I \neq \varnothing$ and $y \in I$ imply that $x \in I$.

Example 10. Consider the hyper BN-algebra H in Example 1. Let $I = \{0\}$ and $I_1 = \{0, a\}$. By routine calculations, I, and I_1, are hyper BN-ideals of H. If we consider the hyper BN-algebra H in Example 2, then the set $J = \{0\}$ is a hyper BN-ideal of H. However, $J_1 = \{0, 1\}$ is not a hyper BN-ideal of H because $2 \circledast 1 = \{0, 1\} \ll J_1$ and $1 \in J_1$ but $2 \notin J_1$. Also, $J_2 = \{0, 2\}$ is not a hyper BN-ideal of H because $1 \circledast 2 = \{0, 1\} \ll J_2$ and $2 \in J_2$ but $1 \notin J_2$.

Example 11. Consider the hyper BN-algebra $(\mathbb{Z}, \circledast, 0)$ in Example 4. Let $I_x = \{0, x\}$. By routine calculations, I_x is a hyper BN-ideal of H for all $x \in \mathbb{Z}$.

Example 12. Consider the hyper BN-algebra $H = \{0, a, b\}$ in Example 1. Let $I = \{0\}$ and $I_1 = \{0, a\}$. By routine calculations, I and I_1 are strong hyper BN-ideals of H. Furthermore, I and I_1 are weak hyper BN-ideals of H.

Example 13. Consider the hyper BN-algebra $H = \{0, 1, 2\}$ in Example 2. Let $I_1 = \{0, 1\}$. I_1 is not a strong hyper BN-ideal of H because $(2 \circledast 1) \cap I_1 \neq \varnothing$ and $1 \in I_1$ but $2 \notin I_1$. It is not also a weak hyper BN-ideal of H because $2 \circledast 1 \subseteq I_1$ and $1 \in I_1$ but $2 \notin I_1$.

Proposition 1. *Let H be a hyper BN-algebra. Then*

(i) *every hyper BN-ideal of H is a weak hyper BN-ideal of H; and*

(ii) *every strong hyper BN-ideal of H is a hyper BN-ideal of H.*

Proof. Let H be a hyper BN-algebra.

(i) Let I be a hyper BN-ideal of H. Thus, $0 \in I$. Suppose that $x, y \in H$ such that $x \circledast y \subseteq I$ and $y \in I$. By Theorem 1(xi), $x \circledast y \ll I$. Since I is a hyper BN-ideal of H, it follows that $x \in I$. Thus, I is a weak hyper BN-ideal of H.

(ii) Let I be a strong hyper BN-ideal of H. Thus, $0 \in I$. Suppose that $x, y \in H$ such that $x \circledast y \ll I$ and $y \in I$. Then for each $a \in x \circledast y$, there exists $b \in I$ such that $a \ll b$, that is, $0 \in a \circledast b$. Since $0 \in I$, $(a \circledast b) \cap I \neq \varnothing$. I is a strong ideal with $b \in I$ implies that $a \in I$. Thus, $x \circledast y \subseteq I$. Hence, $(x \circledast y) \cap I \neq \varnothing$ and so we have $x \in I$. Therefore, I is a hyper BN-ideal of H. □

The following example will show that the converse of Proposition 1(i) is not necessarily true.

Example 14. Consider the hyper BN-algebra H in Example 2 and let $J_2 = \{0, 2\}$. J_2 is a weak hyper BN-ideal of H by routine calculations. However, it is not a hyper BN-ideal of H as shown in Example 10.

The following example will show that the converse of Proposition 1(ii) is not necessarily true.

Example 15. Let $H = \{0, 1, 2, 3\}$ be a set with hyperoperation $\circledast$ defined by the following Cayley table:

$\circledast$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{0,1\}$	$\{1,2\}$	$\{1,3\}$
2	$\{2\}$	$\{1,2\}$	$\{0,2\}$	$\{2,3\}$
3	$\{3\}$	$\{1,3\}$	$\{2,3\}$	$\{0,3\}$

By routine calculations, H is a hyper BN-algebra. Let $I = \{0, 1\}$. Then I is a hyper BN-ideal of H. However, I is not a strong hyper BN-ideal of H because $(2 \circledast 1) \cap I = \{1\} \neq \varnothing$ and $1 \in I$ but $2 \notin I$.

Theorem 6. *If H is a hyper BN-algebra, then $\{0\}$ is a strong hyper BN-ideal. Moreover, it is a hyper BN-ideal and a weak hyper BN-ideal.*

Proof. Let $x, y \in H$ and suppose that $(x \circledast y) \cap \{0\} \neq \varnothing$ and $y \in \{0\}$. Then $y = 0$ and $(x \circledast 0) \cap \{0\} \neq \varnothing$. This implies that $0 \in x \circledast 0$, that is, $x \ll 0$. By Theorem 1(ii), $x = 0$ and so $x \in \{0\}$. Thus, $\{0\}$ is a strong hyper BN-ideal of H. By Propositions 1(ii) and (i), $\{0\}$ is also a hyper BN-ideal and a weak hyper BN-ideal of H. □

Lemma 1. *Let A, B, and C be nonempty subsets of a hyper BN-algebra. If $A \ll B$ and $B \subseteq C$, then $A \ll C$.*

Proof. Let $a \in A$. Since $A \ll B$, there exists $b \in B$ such that $a \ll b$. Since $B \subseteq C$, $b \in C$ with $a \ll b$. Therefore, $A \ll C$. □

Theorem 7. *Let $\{A_i : i \in I\}$ be a nonempty collection of subsets of a hyper BN-algebra H.*

(i) If A_i is a hyper BN-ideal of H for all $i \in I$, then so is $\bigcap_{i\in I} A_i$.

(ii) If A_i is a weak hyper BN-ideal of H for all $i \in I$, then so is $\bigcap_{i\in I} A_i$.

(iii) If A_i is a strong hyper BN-ideal of H for all $i \in I$, then so is $\bigcap_{i\in I} A_i$.

Proof. Let $\{A_i : i \in I\}$ be a nonempty collection of subsets of a hyper BN-algebra H.

(i) Suppose that A_i is a hyper BN-ideal of H for all $i \in I$. Thus, $0 \in A_i$ for all $i \in I$. And so, $0 \in \bigcap_{i\in I} A_i$. Assume $x, y \in H$ such that $x \circledast y \ll \bigcap_{i\in I} A_i$ and $y \in \bigcap_{i\in I} A_i$. Since $\bigcap_{i\in I} A_i \subseteq A_i$ for all $i \in I$, it follows from Lemma 1 that $x \circledast y \ll A_i$ for all $i \in I$. Also, $y \in A_i$ for all $i \in I$. Since A_i is a hyper BN-ideal of H for all $i \in I$, we have $x \in A_i$ for all $i \in I$. Therefore, $x \in \bigcap_{i\in I} A_i$, and so $\bigcap_{i\in I} A_i$ is a hyper BN-ideal of H.

(ii) Suppose that A_i is a weak hyper BN-ideal of H for all $i \in I$. Thus, $0 \in A_i$ for all $i \in I$. And so, $0 \in \bigcap_{i\in I} A_i$. Assume $x, y \in H$ such that $x \circledast y \subseteq \bigcap_{i\in I} A_i$ and $y \in \bigcap_{i\in I} A_i$. Since $\bigcap_{i\in I} A_i \subseteq A_i$ for all $i \in I$, it follows that $x \circledast y \subseteq A_i$ for all $i \in I$. Also, $y \in A_i$ for all $i \in I$. Since A_i is a weak hyper BN-ideal of H for all $i \in I$, we have $x \in A_i$ for all $i \in I$. Therefore, $x \in \bigcap_{i\in I} A_i$, and so $\bigcap_{i\in I} A_i$ is a weak hyper BN-ideal of H.

(iii) Suppose that A_i is a strong hyper BN-ideal of H for all $i \in I$. Thus, $0 \in A_i$ for all $i \in I$. And so, $0 \in \bigcap_{i\in I} A_i$. Assume $x, y \in H$ such that $(x \circledast y) \cap \left(\bigcap_{i\in I} A_i\right) \neq \varnothing$ and $y \in \bigcap_{i\in I} A_i$. Since $\bigcap_{i\in I} A_i \subseteq A_i$ for all $i \in I$, it follows that $(x \circledast y) \cap A_i \neq \varnothing$ for all $i \in I$. Also, $y \in A_i$ for all $i \in I$. Since A_i is a strong hyper BN-ideal of H for all $i \in I$, we have $x \in A_i$ for all $i \in I$. Therefore, $x \in \bigcap_{i\in I} A_i$, and so $\bigcap_{i\in I} A_i$ is a strong hyper BN-ideal of H. □

The following examples will show the relationship between hyper subBN-algebras and hyper BN-ideals of hyper BN-algebras.

Example 16. Consider the set $H = \{0, 1, 2, 3\}$ with hyperoperation $\circledast$ defined by the following Cayley table:

$\circledast$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{0\}$	$\{1\}$	$\{1, 3\}$
2	$\{2\}$	$\{1\}$	$\{0\}$	$\{2, 3\}$
3	$\{3\}$	$\{1, 3\}$	$\{2, 3\}$	$\{0, 3\}$

By routine calculations, we can show that H is a hyper BN-algebra. Let $S = \{0, 1\}$. In view of Theorem 3, S is a hyper subBN-algebra of H. However, it is not a hyper BN-ideal because $2 \circledast 1 \ll S$ and $1 \in S$ but $2 \notin S$.

A hyper subBN-algebra of a hyper BN-algebra H may not be a hyper BN-ideal of H and a hyper BN-ideal of H may not be a hyper subBN-algebra as shown in the next example

Example 17. Consider the set $H = \{0, a, b\}$ with hyperoperation $\circledast$ defined by the following Cayley table:

$\circledast$	0	a	b
0	$\{0\}$	$\{b\}$	$\{a\}$
a	$\{a\}$	$\{0, a, b\}$	$\{a, b\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

By routine calculations, we can show that H is a hyper BN-algebra. Let $I = \{0, a\}$. It can be shown that I is a hyper BN-ideal of H. However, in view of Theorem 3, I is not a hyper subBN-algebra because $0, a \in I$ but $0 \circledast a = \{b\} \nsubseteq I$.

3.2. Reflexive Normal Hyper BN-ideals

At this point, we will give additional conditions to the underlying set of a hyper BN-ideals. With regards to this, we will establish the equivalency of weak hyper BN-ideals and hyper subBN-algebras. Also, these conditions will be the key to the equivalency of strong hyper BN-ideals and hyper BN-ideals.

Definition 12. A hyper BN-ideal I of a hyper BN-algebra H is called *reflexive* (resp. *normal*) *hyper BN-ideal* if it is reflexive (resp. normal). I is called a *reflexive normal hyper BN-ideal* if it is both reflexive and normal.

In the definition above, we can replace a hyper BN-ideal to a weak or strong hyper BN-ideal.

Example 18. In Example 8, we can show that I and J are hyper BN-ideals. Since I is reflexive, I is a reflexive hyper BN-ideal of H. Furthermore, we can show that I is normal. Then I is a normal hyper BN-ideal. Thus, I is a reflexive normal hyper BN-ideal of H. J is not a reflexive hyper BN-ideal because J is not reflexive since $2 \circledast 2 = \{0, 1\} \nsubseteq J$. Further, J is not a normal hyper BN-ideal because J is not normal since $1 \circledast 2 \subseteq J$ and $2 \circledast 1 \subseteq J$ but $(1 \circledast 2) \circledast (2 \circledast 1) = \{0, 1\} \nsubseteq J$. Also, I and J are weak hyper BN-ideals of H. Thus, I is a reflexive normal weak hyper BN-ideal of H while J is not because it is not reflexive nor normal as shown above. Furthermore, I is a strong hyper BN-ideal of H. Hence, I is a reflexive normal strong hyper BN-ideal of H while J is not because it is not even a strong hyper BN-ideal of H since $(1 \circledast 2) \cap J \neq \varnothing$ and $2 \in J$ but $1 \notin J$.

The next result is a special case of Theorem 4.

Corollary 1. *If I is a normal weak hyper BN-ideal of a hyper BN-algebra H, then I is a hyper subBN-algebra of H.*

Proposition 2. *Let H be a hyper BN-algebra and let $S \subseteq H$. Then S is a normal hyper subBN-algebra of H if and only if S is a normal weak hyper BN-ideal of H.*

Proof. Let S be a normal hyper subBN-algebra of a hyper BN-algebra H. Thus, $0 \in S$. Now, let $x \circledast y \subseteq S$ and $y \in S$. Since S is a hyper subBN-algebra, we have $0 \circledast y \subseteq S$. By normality of S, we have $\{x\} = x \circledast 0 = (x \circledast 0) \circledast 0 \subseteq (x \circledast 0) \circledast (y \circledast y) \subseteq S$. Thus, $x \in S$. Hence, S is a normal weak hyper BN-ideal of H. The converse follows from Corollary 1. □

Corollary 2. *Let H be a hyper BN-algebra and let $S \subseteq H$. Then S is a reflexive normal hyper subBN-algebra of H if and only if S is a reflexive normal weak hyper BN-ideal of H.*

Theorem 8. *Let $\{A_i | i \in \mathscr{I}\}$ be a family of reflexive normal weak hyper BN-ideals of a hyper BN-algebra H. Then $\bigcap_{i \in \mathscr{I}} A_i$ is also a reflexive normal weak hyper BN-ideal of H.*

Proof. Since reflexive normal weak hyper BN-ideals are reflexive normal hyper subBN-algebra by Corollary 2, the conclusion follows from Theorem 5. □

Lemma 2. *Let A, B, C and I be subsets of a hyper BN-algebra H.*

(*i*) *If $A \circledast x \ll I$ for all $x \in H$, then $a \circledast x \ll I$ for all $a \in A$.*

(*ii*) *If I is a hyper BN-ideal of H and if $A \circledast x \ll I$ for all $x \in I$, then $A \ll I$.*

Proof. Let A, B, C and I be subsets of a hyper BN-algebra H.

(i) Suppose that $A \circledast x \ll I$ for all $x \in H$. Assume that there exists $a' \in A$ with $a' \circledast x \not\ll I$. Then there is an element $d \in a' \circledast x \subseteq \bigcup_{a \in A} a \circledast x = A \circledast x$ such that $d \not\ll k$ for all $k \in I$, which is a contradiction. Thus, $a \circledast x \ll I$ for all $a \in A$.

(ii) Assume that I is a hyper BN-ideal of H and $A \circledast x \ll I$ for all $x \in I$. Then by (i), $a \circledast x \ll I$ for all $a \in A$. Since I is a hyper BN-ideal of H, $a \circledast x \ll I$ and $x \in I$ imply that $a \in I$. Thus, $A \subseteq I$. By Theorem 1(xi), $A \ll I$. □

Theorem 9. *Let I be a reflexive normal hyper BN-ideal of a hyper BN-algebra H. Then $(x \circledast y) \cap I \neq \varnothing$ implies $x \circledast y \ll I$ for all $x, y \in H$.*

Proof. Let $x, y \in H$ such that $(x \circledast y) \cap I \neq \varnothing$ where I is a reflexive normal hyper BN-ideal of H. Since I is reflexive, $x \circledast x \subseteq I$ and $y \circledast y \subseteq I$. By normality of I, $(x \circledast y) \circledast (x \circledast y) \subseteq I$. Since $(x \circledast y) \cap I \neq \varnothing$, there exists $a \in (x \circledast y) \cap I$. Now, $(x \circledast y) \circledast a \subseteq (x \circledast y) \circledast (x \circledast y) \subseteq I$. By Theorem 1$(xi)$, $(x \circledast y) \circledast a \ll I$. Note that $a \in I$ and so, by Lemma 2(ii), $x \circledast y \ll I$. □

Theorem 10. *Let I be a reflexive normal hyper BN-ideal of a hyper BN-algebra H and let A be a subset of H. If $A \ll I$, then $A \subseteq I$.*

Proof. Assume that $A \ll I$ and let $a \in A$. Then there exists $x \in I$ such that $a \ll x$, that is, $0 \in a \circledast x$. Hence, $0 \in (a \circledast x) \cap I$, and so, $(a \circledast x) \cap I \neq \varnothing$. By Theorem 9, $a \circledast x \ll I$. Since I is a hyper BN-ideal of H, we have $a \in I$ so that $A \subseteq I$. □

The next result follows from Theorem 9 and Theorem 10.

Corollary 3. *Let I be a reflexive normal hyper BN-ideal of a hyper BN-algebra H. Then $(x \circledast y) \cap I \neq \varnothing$ implies $x \circledast y \subseteq I$ for all $x, y \in H$.*

Theorem 11. *Every reflexive normal hyper BN-ideal of a hyper BN-algebra H is a strong hyper BN-ideal of H.*

Proof. Let I be a reflexive normal hyper BN-ideal of a hyper BN-algebra H and let $x, y \in H$ such that $(x \circledast y) \cap I \neq \varnothing$ and $y \in I$. Then $x \circledast y \ll I$ by Theorem 9. I being a hyper BN-ideal means that $x \in I$. Hence, I is a strong hyper BN-ideal of H. □

The converse of Theorem 11 is not true as shown in the next example.

Example 19. Consider the hyper BN-algebra $H = \{0, a, b\}$ in Example 1. The sets $I_1 = \{0, a\}$ and $I_2 = \{0, b\}$ are strong hyper BN-ideals in Example 12. I_1 is normal as shown in Example 7 but not reflexive since $b \circledast b = \{0, b\} \nsubseteq I_1$. On the other hand, I_2 is not normal as shown in Example 7 and is not reflexive because $a \circledast a = \{0, a\} \nsubseteq I_2$. Hence, both I_1 and I_2 are not reflexive normal hyper BN-ideals of H.

Since reflexivity and normality are innate in a set, we can conclude that the strong hyper BN-ideal in Theorem 11 is not a reflexive normal hyper BN-ideal. And so, together with Proposition 1(ii), we have the following result.

Corollary 4. *Let I be a reflexive normal set of a hyper BN-algebra H. I is a strong hyper BN-ideal of H if and only if I is a hyper BN-ideal of H.*

4. Quotient Structure of Hyper BN-algebras

In this section, we will be looking at two ways of constructing the quotient structures of hyper BN-algebras.

4.1. Quotient Hyper BN-algebra via Reflexive Normal Hyper SubBN-algebra

We now begin constructing the quotient structure of hyper BN-algebra via reflexive normal hyper subBN-algebra.

Lemma 3. *Let I be a normal subset of a hyper BN-algebra H and $x, y \in H$. Then*

(i) $x \in I \Rightarrow 0 \circledast x \subseteq I$, *and*

(ii) $x \circledast y \subseteq I \Rightarrow y \circledast x \subseteq I$.

Proof. Let I be a normal subset of a hyper BN-algebra H and let $x, y \in H$.

(i) Let $x \in I$. Then $x \circledast 0 = \{x\} \subseteq I$ by Theorem 1(ii). Since I is normal, we have $0 \in I$ and so $0 \circledast 0 = \{0\} \subseteq I$. By normality of I, we have $0 \circledast x = (0 \circledast x) \circledast 0 = (0 \circledast x) \circledast (0 \circledast 0) \subseteq I$. Thus, $0 \circledast x \subseteq I$.

(ii) Let $x \circledast y \subseteq I$. Then for all $a \in x \circledast y$, $a \in I$. By (i), we have $0 \circledast a \subseteq I$ for all $a \in x \circledast y$. Thus, $0 \circledast (x \circledast y) \subseteq I$. But $0 \circledast (x \circledast y) = y \circledast x$ by Theorem 1(iv). Therefore, $y \circledast x \subseteq I$. □

Definition 13. Let $(H, \circledast, 0)$ be a hyper BN-algebra and S be a reflexive normal hyper subBN-algebra of H. We define a relation $\sim_S$ on H by $x \sim_S y$ if and only if $x \circledast y \subseteq S$, where $x, y \in H$.

Lemma 4. *$\sim_S$ is an equivalence relation.*

Proof. Let H be a hyper BN-algebra and S be a reflexive normal hyper subBN-algebra of H. Since S is reflexive, for all $x \in H$, $x \circledast x \subseteq S$. Thus, $x \sim_S x$. Whence, $\sim_S$ is reflexive. Let $x \sim_S y$. Then $x \circledast y \subseteq S$. By Lemma 3$(ii)$, $y \circledast x \subseteq S$. Thus, $y \sim_S x$. Hence, $\sim_S$ is symmetric. Let $x, y, z \in H$ such that $x \sim_S y$ and $y \sim_S z$. Then $x \circledast y \subseteq S$ and $y \circledast z \subseteq S$. By applying Lemma 3$(ii)$ again, we have $z \circledast y \subseteq S$. Now, by normality of S using $x \circledast y \subseteq S$ and $z \circledast y \subseteq S$, we have $x \circledast z = (x \circledast z) \circledast 0 \subseteq (x \circledast z) \circledast (y \circledast y) \subseteq S$. Thus, $x \circledast z \subseteq S$ implying that $x \sim_S z$. Thus, $\sim_S$ is transitive. Therefore, $\sim_S$ is an equivalence relation. □

Definition 14. Let $(H, \circledast, 0)$ be a hyper BN-algebra, S be a reflexive normal hyper subBN-algebra of H, and $\varnothing \neq A, B \subseteq H$. Then $A \sim_S B$ if for all $a \in A$ and $b \in B$, $a \sim_S b$.

Lemma 5. *Let $(H, \circledast, 0)$ be a hyper BN-algebra, S be a reflexive normal hyper subBN-algebra of H, and $\varnothing \neq A, B \subseteq H$. $A \sim_S B$ if and only if $A \circledast B \subseteq S$.*

Proof. Let $A \sim_S B$. Then for all $a \in A$ and $b \in B$, $a \circledast b \subseteq S$. Thus, $A \circledast B = \bigcup_{a \in A, b \in B} (a \circledast b) \subseteq S$. For the converse, suppose $A \circledast B \subseteq S$. Let $a \in A$ and $b \in B$. Then $a \circledast b \subseteq A \circledast B \subseteq S$. Hence, $A \sim_S B$. $\square$

Lemma 6. *Let $(H, \circledast, 0)$ be a hyper BN-algebra, S be a reflexive normal hyper subBN-algebra of H, and $\varnothing \neq A, B \subseteq H$. Then the equivalence class containing 0 is S. In other words, $[0]_{\sim_S} = S$.*

Proof. Let $x \in [0]_{\sim_S}$. Then $\{x\} = x \circledast 0 \subseteq S$. Thus, $x \in S$ and hence $[0]_{\sim_S} \subseteq S$. For the other inclusion, let $x \in S$. Note that $0 \in S$. Since S is a hyper subBN-algebra, $x \circledast 0 \subseteq S$. Thus, $x \sim_S 0$ and $x \in [0]_{\sim_S}$. Hence, $S \subseteq [0]_{\sim_S}$. Therefore, $[0]_{\sim_S} = S$. $\square$

In the following result, we define $H/S = \{[x]_{\sim_S} | x \in H\}$.

Theorem 12. *Let S be a reflexive normal hyper subBN-algebra of a hyper BN-algebra $(H, \circledast, 0)$. Then H/S is a hyper BN-algebra with hyperoperation $\odot$ defined by $[x]_{\sim_S} \odot [y]_{\sim_S} = \{[z]_{\sim_S} | z \in x \circledast y\}$ and $[x]_{\sim_S} \ll_{\sim_S} [y]_{\sim_S}$ if and only if $[0]_{\sim_S} \in [x]_{\sim_S} \odot [y]_{\sim_S}$.*

Proof. If $x \sim_S p$ and $y \sim_S q$, then we have $x \circledast p \subseteq S$ and $y \circledast q \subseteq S$. By normality of S, $(x \circledast y) \circledast (p \circledast q) \subseteq S$. Thus, by Lemma 5, $x \circledast y \sim_S p \circledast q$ and so $[x \circledast y]_{\sim_S} = [p \circledast q]_{\sim_S}$. Hence, the hyperoperation $\odot$ is well-defined.

Now, let $x \in H$. By (HBN_1), $x \ll x$, that is, $0 \in x \circledast x$. Thus, $[0]_{\sim_S} \in \{[z]_{\sim_S} | z \in x \circledast x\} = [x]_{\sim_S} \odot [x]_{\sim_S}$ which implies $[x]_{\sim_S} \ll_{\sim_S} [x]_{\sim_S}$. Hence, (HBN_1) holds for H/S. For (HBN_2), let $[x]_{\sim_S} \in H/S$. Then,

$$[x]_{\sim_S} \odot [0]_{\sim_S} = \{[z]_{\sim_S} | z \in x \circledast 0\} = \{[z]_{\sim_S} | z = x\} = \{[x]_{\sim_S}\}.$$

Finally, let $[w]_{\sim_S} \in ([x]_{\sim_S} \odot [y]_{\sim_S}) \odot [z]_{\sim_S}$ where $[x]_{\sim_S}, [y]_{\sim_S}, [z]_{\sim_S} \in H/S$. Then there exists $u \in x \circledast y$ such that $[w]_{\sim_S} \in [u]_{\sim_S} \odot [z]_{\sim_S}$. Consequently, there exists $w' \in u \circledast z$ such that $[w]_{\sim_S} = [w']_{\sim_S}$. Now, observe that by (HBN_3) for H, we have $w' \in u \circledast z \subseteq (x \circledast y) \circledast z = (0 \circledast z) \circledast (y \circledast x)$. Now, $w' \in a \circledast b$ where $a \in 0 \circledast z$ and $b \in y \circledast x$. This means that $[w']_{\sim_S} \in \{[k]_{\sim_S} | k \in a \circledast b\} = [a]_{\sim_S} \odot [b]_{\sim_S}$. Notice that $a \in 0 \circledast z$ and $b \in y \circledast x$ mean $[a]_{\sim_S} \in \{[l]_{\sim_S} | l \in 0 \circledast z\} = [0]_{\sim_S} \odot [z]_{\sim_S}$ and $[b]_{\sim_S} \in \{[m]_{\sim_S} | m \in y \circledast x\} = [y]_{\sim_S} \odot [x]_{\sim_S}$, respectively. Thus, $[w]_{\sim_S} = [w']_{\sim_S} \in [a]_{\sim_S} \odot [b]_{\sim_S} \subseteq ([0]_{\sim_S} \odot [z]_{\sim_S}) \odot ([y]_{\sim_S} \odot [x]_{\sim_S})$. Since $[w]_{\sim_S}$ is arbitrary, it follows that $([x]_{\sim_S} \odot [y]_{\sim_S}) \odot [z]_{\sim_S} \subseteq ([0]_{\sim_S} \odot [z]_{\sim_S}) \odot ([y]_{\sim_S} \odot [x]_{\sim_S})$. Next, let $[v]_{\sim_S} \in ([0]_{\sim_S} \odot [z]_{\sim_S}) \odot ([y]_{\sim_S} \odot [x]_{\sim_S})$ where $[x]_{\sim_S}, [y]_{\sim_S}, [z]_{\sim_S} \in H/S$. Then there exist $a \in 0 \circledast z$ and $b \in y \circledast x$ such that $[v]_{\sim_S} \in [a]_{\sim_S} \odot [b]_{\sim_S}$. This implies that there exists $v' \in a \circledast b$ such that $[v]_{\sim_S} = [v']_{\sim_S}$. Now, using (HBN_3) for H, we have $v' \in a \circledast b \subseteq (0 \circledast z) \circledast (y \circledast x) = (x \circledast y) \circledast z$. This means that $v' \in d \circledast z$ where $d \in x \circledast y$. Thus, $[v']_{\sim_S} \in \{[n]_{\sim_S} | n \in d \circledast z\} = [d]_{\sim_S} \odot [z]_{\sim_S}$. But $d \in x \circledast y$ means that $[d]_{\sim_s} \in \{[t]_{\sim_S} | t \in x \circledast y\} = [x]_{\sim_S} \odot [y]_{\sim_S}$. Hence, $[v]_{\sim_S} = [v']_{\sim_S} \in [d]_{\sim_S} \odot [z]_{\sim_S} \subseteq ([x]_{\sim_S} \odot [y]_{\sim_S}) \odot [z]_{\sim_S}$. Thus, $([0]_{\sim_S} \odot [z]_{\sim_S}) \odot ([y]_{\sim_S} \odot [x]_{\sim_S}) \subseteq$

$([x]_{\sim_S} \odot [y]_{\sim_S}) \odot [z]_{\sim_S}$. Hence, $([x]_{\sim_S} \odot [y]_{\sim_S}) \odot [z]_{\sim_S} = ([0]_{\sim_S} \odot [z]_{\sim_S}) \odot ([y]_{\sim_S} \odot [x]_{\sim_S})$ for all $[x]_{\sim_S}, [y]_{\sim_S}, [z]_{\sim_S} \in H/S$ holding (HBN_3). Therefore, $(H/S, \odot, [0]_{\sim_S})$ is a hyper BN-algebra. $\square$

Let us illustrate the construction of the quotient structure of a hyper BN-algebra via reflexive normal hyper subBN-algebra.

Example 20. Let $H = \{0, 1, 2, 3, 4\}$ be the hyper BN-algebra in Example 9. Let $S = \{0, 3\}$. Then it has been shown that S is a reflexive normal hyper subBN-algebra of H.

Now, $[0]_{\sim_S} = \{0, 3\} = [3]_{\sim_S}$, $[1]_{\sim_S} = \{1, 2\} = [2]_{\sim_S}$, and $[4]_{\sim_S} = \{4\}$. Hence, $H/S = \{[0]_{\sim_S}, [1]_{\sim_S}, [4]_{\sim_S}\}$ and the hyperoperation $\odot$ is defined by the following Cayley table:

$\odot$	$[0]_{\sim_S}$	$[1]_{\sim_S}$	$[4]_{\sim_S}$
$[0]_{\sim_S}$	$\{[0]_{\sim_S}\}$	$\{[1]_{\sim_S}\}$	$\{[4]_{\sim_S}\}$
$[1]_{\sim_S}$	$\{[1]_{\sim_S}\}$	$\{[0]_{\sim_S}\}$	$\{[4]_{\sim_S}\}$
$[4]_{\sim_S}$	$\{[4]_{\sim_S}\}$	$\{[4]_{\sim_S}\}$	$\{[0]_{\sim_S}\}$

By routine calculations, $(H/S, \odot, [0]_{\sim_S})$ is a hyper BN-algebra.

4.2. Quotient Hyper BN-algebra via Congruence Relation

Now, we will construct the quotient hyper BN-algebra via congruence relation. Also, we will show the relationship between the construction of quotient hyper BN-algebra in **4.1** and the construction here.

Definition 15. Let θ be an equivalence relation on a hyper BN-algebra H and $\varnothing \neq A, B \subseteq H$. Then

(i) $A\theta B$ if there exist $a \in A$ and $b \in B$ such that $a\theta b$;

(ii) $A\overline{\theta}B$ if for every $a \in A$, there exists $b \in B$ such that $a\theta b$ and for every $b \in B$, there exists $a \in A$ such that $a\theta b$;

(iii) θ is called a *congruence relation* on H, if whenever $x\theta y$ and $x'\theta y'$, then $(x \circledast x')\overline{\theta}(y \circledast y')$ for all $x, y, x', y' \in H$.

Not all equivalence relations are congruence relations as shown in the following example.

Example 21. Consider $H = \{0, 1, 2, 3, 4\}$ and its hyperoperation given by the following Cayley table:

$\circledast$	0	1	2	3	4
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$
1	$\{1\}$	$\{0, 1\}$	$\{2\}$	$\{3\}$	$\{4\}$
2	$\{2\}$	$\{2\}$	$\{0, 2\}$	$\{0, 2, 3\}$	$\{2, 4\}$
3	$\{3\}$	$\{3\}$	$\{0, 2, 3\}$	$\{0, 3\}$	$\{3, 4\}$
4	$\{4\}$	$\{4\}$	$\{2, 4\}$	$\{3, 4\}$	$\{0, 4\}$

By routine calculations, H is a hyper BN-algebra. Define a relation θ on H by $\theta = \{(0,0),(0,2),(1,1),(1,3),(2,0),(2,2),(3,1),(3,3),(4,4)\}$. By inspection, θ is an equivalence relation on H. Now, observe that $1\theta 3$ and $2\theta 0$ but $1 \circledast 2 = \{2\}\overline{\not\theta}\{3\} = 3 \circledast 0$ because $(2,3) \notin \theta$. Thus, θ is not a congruence relation on H. Hence, if θ is an equivalence relation on H, then θ need not be a congruence relation on H.

The following lemma shows that $\overline{\theta}$ is transitive on $\mathcal{P}^*(H)$.

Lemma 7. *Let θ be an equivalence relation on H and $\varnothing \neq A, B, C \subseteq H$. If $A\overline{\theta}B$ and $B\overline{\theta}C$, then $A\overline{\theta}C$.*

Proof. Let θ be an equivalence relation on H and $\varnothing \neq A, B, C \subseteq H$. Assume that $A\overline{\theta}B$ and $B\overline{\theta}C$. By Definition 15(ii), for each $a \in A$ (resp. $b \in B$), there exists $b \in B$ (resp. $a \in A$) such that $a\theta b$ (resp. $b\theta a$) and for all $b \in B$ (resp. $c \in C$), there exists $c \in C$ (resp. $b \in B$) such that $b\theta c$ (resp. $c\theta b$). Since θ is an equivalence relation, $a\theta c$ (resp. $c\theta a$). Therefore, $A\overline{\theta}C$. □

Lemma 8. *Let θ be an equivalence relation on H. The following statements are equivalent:*

(i) *θ is a congruence relation on H.*

(ii) *If $x, y \in H$ such that $x\theta y$, then $(x \circledast a)\overline{\theta}(y \circledast a)$ and $(a \circledast x)\overline{\theta}(a \circledast y)$ for all $a \in H$.*

Proof. Let θ be an equivalence relation on H.

$(i) \Rightarrow (ii)$ Let θ be a congruence relation on H and $a, x, y \in H$ such that $x\theta y$. Since $a\theta a$, $(x \circledast a)\overline{\theta}(y \circledast a)$ and $(a \circledast x)\overline{\theta}(a \circledast y)$, by Definition 15$(iii)$.

$(ii) \Rightarrow (i)$ Assume that if $x\theta y$, then $(x \circledast a)\overline{\theta}(y \circledast a)$ and $(a \circledast x)\overline{\theta}(a \circledast y)$ for all $a, x, y \in H$. Let $x, y, x', y' \in H$ such that $x\theta y$ and $x'\theta y'$. Then, $(x \circledast x')\overline{\theta}(y \circledast x')$ and $(y \circledast x')\overline{\theta}(y \circledast y')$. By Lemma 7, $(x \circledast x')\overline{\theta}(y \circledast y')$. By Definition 15$(iii)$, θ is a congruence relation. □

The following proposition tells us that $\sim_S$ in Definition 13 is a congruence relation.

Proposition 3. *Let S be a reflexive normal hyper subBN-algebra of a hyper BN-algebra H. Then $\sim_S$ is a congruence relation on H.*

Proof. By Lemma 4, $\sim_S$ is an equivalence relation. We will show that $\sim_S$ is a congruence relation using Lemma 8. Let $x, y \in H$ with $x \sim_S y$ and let $a \in H$. Then $x \circledast y \subseteq S$. Also, since S is reflexive, $a \circledast a \subseteq S$. By normality of S, $(x \circledast a) \circledast (y \circledast a) \subseteq S$ and $(a \circledast x) \circledast (a \circledast y) \subseteq S$. Also, by Lemma 3$(ii)$, we have $y \circledast x \subseteq S$. Now, $y \circledast x \subseteq S$ and $a \circledast a \subseteq S$ imply $(y \circledast a) \circledast (x \circledast a) \subseteq S$ and $(a \circledast y) \circledast (a \circledast x) \subseteq S$. Now, $(x \circledast a) \circledast (y \circledast a) \subseteq S$ implies that $u \circledast v \subseteq S$ for all $u \in x \circledast a$ and $v \in y \circledast a$. Similarly, $(y \circledast a) \circledast (x \circledast a) \subseteq S$ implies that $v \circledast u \subseteq S$ for all $u \in x \circledast a$ and $v \in y \circledast a$. Thus, for all $u \in x \circledast a$ and $v \in y \circledast a$, $u \sim_S v$ and this means that $(x \circledast a)\overline{\sim_S}(y \circledast a)$ for all $a \in H$ since a is arbitrary. In similar fashion, using $(a \circledast x) \circledast (a \circledast y) \subseteq S$ and $(a \circledast y) \circledast (a \circledast x) \subseteq S$, we will obtain $(a \circledast x)\overline{\sim_S}(a \circledast y)$ for all $a \in H$. Therefore, $\sim_S$ is a congruence relation on H. □

The converse of Proposition 3 is not true in general as shown in the following example:

Example 22. Let $H = \{0, 1, 2, 3, 4, 5\}$ be a set. Define a hyperoperation $\circledast$ on H by the following Cayley table:

$\circledast$	0	1	2	3	4	5
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$
1	$\{1\}$	$\{0,1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$	$\{0,1\}$	$\{0,4\}$	$\{0,5\}$
3	$\{3\}$	$\{3\}$	$\{0,1\}$	$\{0,1\}$	$\{0,4\}$	$\{0,5\}$
4	$\{4\}$	$\{4\}$	$\{0,4\}$	$\{0,4\}$	$\{0,1\}$	$\{0,1\}$
5	$\{5\}$	$\{5\}$	$\{0,5\}$	$\{0,5\}$	$\{0,1\}$	$\{0,1\}$

By routine calculations, H is a hyper BN-algebra. Let $S = \{0, 1\}$ and $\sim_S$ be a relation on H defined by $x \sim_S y$ if and only if $x \circledast y \subseteq S$ for all $x, y \in H$. Then $\sim_S = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}$. By inspection, $\sim_S$ is an equivalence relation. Now, we can verify that $\sim_S$ is a congruence relation using Lemma 8. Note that S is a hyper subBN-algebra of H. Also, it is reflexive because for all $a \in H$ $a \circledast a \subseteq S$. However, it is not normal because $2 \circledast 3 \subseteq S$ and $4 \circledast 4 \subseteq S$ but $(2 \circledast 4) \circledast (3 \circledast 4) = \{0, 1, 4\} \nsubseteq S$. Therefore, S is not a reflexive normal hyper subBN-algebra of H.

The following example will illustrate that the congruence class containing 0 is not necessarily reflexive. This will mean that the construction of the quotient structure via reflexive normal hyper subBN-algebra is just a special case of the construction of the quotient structure via congruence relation.

Example 23. Let $H = \{0, 1, 2, 3, 4\}$ be a set. Define a hyperoperation $\circledast$ on H by the following Cayley table:

$\circledast$	0	1	2	3	4
0	$\{0\}$	$\{1\}$	$\{3\}$	$\{2\}$	$\{4\}$
1	$\{1\}$	$\{0,1\}$	$\{2\}$	$\{3\}$	$\{4\}$
2	$\{2\}$	$\{3\}$	$\{0,2,3\}$	$\{1,2,3\}$	$\{0,1\}$
3	$\{3\}$	$\{2\}$	$\{1,2,3\}$	$\{0,2,3\}$	$\{0,1\}$
4	$\{4\}$	$\{4\}$	$\{0,1\}$	$\{0,1\}$	$\{0,4\}$

By routine calculations, H is a hyper BN-algebra. Let $\theta = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 4)\}$. By inspection, θ is an equivalence relation. Verify that θ is a congruence relation using Lemma 8. Note that $[0]_\theta = \{0, 1\}$ is a hyper subBN-algebra of H. However, it is not reflexive because $2 \circledast 2 = \{0, 2, 3\} \nsubseteq [0]_\theta$. Also, it is not normal because $4 \circledast 3 \subseteq [0]_\theta$ but $(4 \circledast 4) \circledast (3 \circledast 3) = \{0, 1, 2, 3, 4\} \nsubseteq [0]_\theta$. Therefore, $[0]_\theta$ is not a reflexive normal hyper subBN-algebra of H.

In Lemma 6, $[0]_{\sim_S} = S$ where S is a hyper subBN-algebra. The next result will tell us that in general, the congruence class containing 0 is a hyper subBN-algebra.

Theorem 13. *Let θ be a congruence relation on a hyper BN-algebra H. Then $[0]_\theta$ is a hyper subBN-algebra of H.*

Proof. Clearly, $0 \in [0]_\theta$. Now, let $x, y \in [0]_\theta$. Then $x\theta 0$ and $y\theta 0$ which imply that $y\theta x$ by symmetric and transitive properties of θ. Since $y\theta x$ and $0\theta y$, and θ is a congruence relation, we have by Lemma 8, $\{y\} = (y \circledast 0)\overline{\theta}(x \circledast y)$. This means that for all $a \in x \circledast y$, $a\theta y$. By transitive property of θ, $a\theta y$ and $y\theta 0$ imply $a\theta 0$ for all $a \in x \circledast y$. Hence, $x \circledast y \subseteq [0]_\theta$. By Theorem 3, $[0]_\theta$ is a hyper subBN-algebra of H. □

Theorem 14. *Let θ be a congruence relation on a hyper BN-algebra H. Then $[0]_\theta$ is a strong hyper BN-ideal of H. Consequently, it is a hyper BN-ideal and a weak hyper BN-ideal of H.*

Proof. Clearly, $0 \in [0]_\theta$. Now, let $x, y \in H$ such that $(x \circledast y) \cap [0]_\theta$ and $y \in [0]_\theta$. Then there exists $a \in x \circledast y$ such that $a \in [0]_\theta$ and so $a\theta 0$. Hence, $(x \circledast y)\theta 0$. Moreover, $y\theta 0$ implies $0\theta y$ because θ is symmetric. Since $0\theta y$ and θ is a congruence relation on H, by Lemma 8, $\{x\} = (x \circledast 0)\overline{\theta}(x \circledast y)$. This means that for all $b \in x \circledast y$, $x\theta b$. Also, $(x \circledast y)\theta\{0\}$ means that there is an element $b' \in x \circledast y$ such that $b'\theta 0$. Since $b' \in x \circledast y$, we have $x\theta b'$. By transitive property of θ, $x\theta 0$ and $x \in [0]_\theta$. Therefore, $[0]_\theta$ is a strong hyper BN-ideal of H. By Propositions 1(ii) and (i), $[0]_\theta$ is a also a hyper BN-ideal and a weak hyper BN-ideal of H. □

The following example serves as our motivation in the construction of quotient structure via congruence relation.

Example 24. Consider the hyper BN-algebra $H = \{0, 1, 2, 3, 4\}$ in Example 23. The relation θ defined on H is a congruence relation as shown. We have $I = [0]_\theta = \{0, 1\}$, $I_2 = \{2, 3\} = I_3$, and $I_4 = \{4\}$. Let $H/I = \{I_x : x \in H\} = \{I, I_2, I_4\}$. Define a hyperoperation $\otimes$ on H/I by $I_x \otimes I_y = \{I_z : z \in x \circledast y\}$ and the hyperoder $\ll_I$ by $I_x \ll_I I_y$ if and only if $I \in I_x \otimes I_y$. Thus, our Cayley table is as follows:

$\otimes$	I	I_2	I_4
I	$\{I\}$	$\{I_2\}$	$\{I_4\}$
I_2	$\{I_2\}$	$\{I, I_2\}$	$\{I\}$
I_4	$\{I_4\}$	$\{I\}$	$\{I, I_4\}$

Using routine calculations, we can show that $(H/I, \otimes, I)$ is a hyper BN-algebra.

We will now show that using congruence relation, the quotient structure obtained is a hyper BN-algebra.

Theorem 15. *Let θ be a congruence relation on a hyper BN-algebra H such that $I = [0]_\theta$ and $H/I = \{I_x : x \in H\}$, where $I_x = [x]_\theta$ for all $x \in H$. Then H/I with the hyperoperation $\otimes$ and hyperorder $\ll_I$ which are defined as follows:*

$$I_x \otimes I_y = \{I_z : z \in x \circledast y\} \text{ and } I_x \ll_I I_y \text{ if and only if } I \in I_x \otimes I_y$$

is a hyper BN-algebra which we call the quotient hyper BN-algebra.

Proof. Let us show first that the hyperoperation $\otimes$ is well-defined on H/I. Assume $x, y, x', y' \in H$ with $I_x = I_{x'}$ and $I_y = I_{y'}$. Let $J \in I_x \otimes I_y$. Then there exists $u \in x \circledast y$ such that $J = I_u$. Note that $x\theta x'$ and $y\theta y'$. Since θ is a congruence relation on H, it follows that $(x \circledast y)\overline{\theta}(x' \circledast y')$. Hence, there is an element $z' \in x' \circledast y'$ such that $u\theta z'$, and so $J = I_u = I_{z'}$. Thus, $J \in I_{x'} \otimes I_{y'}$. So, $I_x \otimes I_y \subseteq I_{x'} \otimes I_{y'}$. Conversely, let $L \in I_{x'} \otimes I_{y'}$. Then there is an element $v' \in x' \circledast y'$ such that $L = I_{v'}$. Note that $x'\theta x$ and $y'\theta y$. Hence, $(x' \circledast y')\overline{\theta}(x \circledast y)$. Thus, there exists $z \in x \circledast y$ such that $v'\theta z$, so that $L = I_{v'} = I_z$. It means that $L \in I_x \otimes I_y$ and $I_{x'} \otimes I_{y'} \subseteq I_x \otimes I_y$. Therefore, $I_x \otimes I_y = I_{x'} \otimes I_{y'}$ and $\otimes$ is a well-defined hyperoperation on H/I.

Now, we will show that (i) of Definition 5 holds for H/I, that is, $I_x \ll_I I_x$ for all $I_x \in H/I$. Since H is a hyper BN-algebra, $x \ll x$, that is, $0 \in x \circledast x$. Thus, we have $I \in I_x \otimes I_x$.

For Definition 5(ii), we will show that $I_x \otimes I = \{I_x\}$ for all $I_x \in H/I$. H being a hyper BN-algebra means that $x \circledast 0 = \{x\}$. Thus, Definition 5(ii) follows for H/I.

Let $I_w \in (I_x \otimes I_y) \otimes I_z$ where $I_x, I_y, I_z \in H/I$. Then there exists $u \in x \circledast y$ such that $I_w \in I_u \otimes I_z$. Since H is a hyper BN-algebra, we have $w' \in u \circledast z \subseteq (x \circledast y) \circledast z = (0 \circledast z) \circledast (y \circledast x)$ which implies that $I_w = I_{w'} \in (I \otimes I_z) \otimes (I_y \otimes I_x)$. Since I_w is arbitrary, we have $(I_x \otimes I_y) \otimes I_z \subseteq (I \otimes I_z) \otimes (I_y \otimes I_x)$. Conversely, pick an arbitrary element $I_v \in (I \otimes I_z) \otimes (I_y \otimes I_x)$. Then there exist $s \in 0 \circledast z$ and $t \in y \circledast x$ such that $I_s \in I \otimes I_z$ and $I_t \in I_y \otimes I_x$. And so, $I_v \in I_s \otimes I_t$. This means that there is an element $v' \in s \circledast t$ such that $I_v = I_{v'}$. Since H is a hyper BN-algebra, $v' \in s \circledast t \subseteq (0 \circledast z) \circledast (y \circledast x) = (x \circledast y) \circledast z$. Thus, $I_v = I_{v'} \in (I_x \otimes I_y) \otimes I_z$. Since I_v is arbitrary, we have $(I \otimes I_z) \circledast (I_y \otimes I_x) \subseteq (I_x \otimes I_y) \otimes I_z$. Hence, $(I_x \otimes I_y) \otimes I_z = (I \otimes I_z) \otimes (I_y \otimes I_x)$ and Definition 5(iii) holds for H/I.

Therefore, $(H/I, \otimes, I)$ is a hyper BN-algebra. □

Theorem 16. *Let θ be a congruence relation on a hyper BN-algebra H such that $I = [0]_\theta$ and $H/I = \{I_x : x \in H\}$, where $I_x = [x]_\theta$ for all $x \in H$. If H is commutative, then so is H/I.*

Proof. Suppose H is commutative. Then for all $x, y \in H$, $x \circledast y = y \circledast x$. Let $I_x, I_y \in H/I$. Then $I_x \otimes I_y = \{I_z : z \in x \circledast y = y \circledast x\} = I_y \otimes I_x$. Hence, H/I is commutative. □

The converse of Theorem 16 is not necessarily true. H/I in Example 24 is commutative but H is not because $0 \circledast 3 = \{2\} \neq \{3\} = 3 \circledast 0$.

Lemma 9. *Let H be a hyper BN-algebra, θ be a congruence relation on H and $x, y \in H$. If $(x \circledast y)\theta\{0\}$, then $(y \circledast x)\theta\{0\}$.*

Proof. Let H be a hyper BN-algebra and θ be a congruence relation on H. Let $x, y \in H$ such that $(x \circledast y)\theta\{0\}$. Then there exists $a \in x \circledast y$ such that $a\theta 0$. Since $0 \in H$ and θ is a congruence relation, we have $(0 \circledast a)\overline{\theta}(0 \circledast 0) = \{0\}$. This means that for all $s \in 0 \circledast a$, $s\theta 0$. But $s \in 0 \circledast a \subseteq 0 \circledast (x \circledast y) = y \circledast x$ by Theorem 1(iii). Thus, $s \in y \circledast x$ with $s\theta 0$. Therefore, $(y \circledast x)\theta\{0\}$. □

Lemma 9 serves as our motivation in defining regularity of an equivalence relation on a hyper BN-algebra.

Definition 16. Let H be a hyper BN-algebra and θ be an equivalence relation on H. Then θ is called a *regular congruence relation* on H, if θ is a congruence relation on H and whenever $(x \circledast y)\theta\{0\}$, then $x\theta y$ for all $x, y \in H$.

Theorem 17. *Let θ and θ' be regular congruence relations on H with $[0]_\theta = [0]_{\theta'}$. Then $\theta = \theta'$.*

Proof. Let θ and θ' be regular congruence relations on H with $[0]_\theta = [0]_{\theta'}$. Since θ and θ' are both relations on H, we just need to show that $x\theta y$ if and only if $x\theta' y$ for all $x, y \in H$. Let $x\theta y$. Since θ is a congruence relation on H, by Lemma 8, $(x \circledast x)\overline{\theta}(x \circledast y)$. Since $0 \in x \circledast x$, there exists an element $s \in x \circledast y$ such that $0\theta s$. It follows that $s \in [0]_\theta = [0]_{\theta'}$. Thus, $(x \circledast y)\theta'\{0\}$. Now, since θ' is a regular congruence relation on H, we have $x\theta' y$. Conversely, let $x\theta' y$. Then $(x \circledast x)\overline{\theta'}(x \circledast y)$. Since $0 \in x \circledast x$, there exists an element $s \in x \circledast y$ such that $0\theta' s$. It follows that $s \in [0]_{\theta'} = [0]_\theta$. Thus, $(x \circledast y)\theta\{0\}$. Since θ is a regular congruence relation on H, we have $x\theta y$. $\square$

Definition 17. A hyper BN-algebra H that satisfies the condition: if $x \ll y$, then $x = y$ for all $x, y \in H$, is called a *hyper BN_1-algebra*.

Example 25. Consider the hyper BN-algebra $H = \{0, a, b\}$ in Example 1. Then H is a hyper BN_1-algebra. Also, the hyper BN-algebra $H = \{0, 1, 2, 3\}$ in Example 15 is a hyper BN_1-algebra.

Example 26. The hyper BN-algebra $H' = \{0, 1, 2\}$ in Example 2 is not a hyper BN_1-algebra because $1 \ll 2$ but $1 \neq 2$. Also, the hyper BN-algebra $H' = \{0, 1, 2, 3\}$ in Example 3 is not a hyper BN_1-algebra because $2 \ll 3$ but $2 \neq 3$.

Notice that θ in Example 23 is not regular since $(2 \circledast 4)\theta\{0\}$ but $(2, 4) \notin \theta$. The resulting quotient structure which is given in Example 24 is not a hyper BN_1-algebra. To support it further, $I_4 \ll_I I_2$ but $I_4 \neq I_2$.

Example 27. If we consider $\theta = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)\}$ in Example 23. We can show that θ is a regular congruence relation. Now, $I = [0]_\theta = \{0, 1\} = I_1$ and $I_2 = [2]_\theta = \{2, 3, 4\} = I_3 = I_4$. Thus, $H/I = \{I, I_2\}$ and the hyperoperation $\otimes$ is defined by the following Cayley table:

$\otimes$	I	I_2
I	$\{I\}$	$\{I_2\}$
I_2	$\{I_2\}$	$\{I, I_2\}$

Using routine calculations, we can show that $(H/I, \otimes, I)$ is a hyper BN_1-algebra.

We can deduce from Example 27 that if θ is a regular congruence relation on a hyper BN-algebra H, then the resulting structure would be a hyper BN_1-algebra. The following result generalizes this observation.

Theorem 18. *Let H be a hyper BN-algebra, θ be a regular congruence relation on H and $I = [0]_\theta$. Then H/I is a hyper BN_1-algebra.*

Proof. By Theorem 15, H/I is a hyper BN-algebra. Now, let $I_x \ll I_y$ where $I_x, I_y \in H/I$. Then $I \in I_x \otimes I_y$. Hence, there exists $u \in x \circledast y$ such that $I_u = I$ and so, $u\theta 0$. Hence, $(x \circledast y)\theta\{0\}$. Since θ is regular, $x\theta y$. Thus, $I_x = I_y$. Therefore, H/I is a hyper BN_1-algebra. □

5. Conclusion

We have defined various types of ideals for hyper BN-algebras. We also obtained some properties. We showed the general relationship among various types of ideals and hyper subBN-algebras. We established the equivalency of weak hyper BN-ideals and hyper sub BN-algebras. We also found a condition such that a strong hyper BN-ideal become a hyper BN-ideal. Finally, we were able to construct quotient hyper BN-algebras via reflexive normal hyper subBN-algebra and via congruence relation. We likewise showed that the construction via reflexive normal hyper subBN-algebra is just a special case of the construction via congruence relation. Furthermore, we have introduced the notion of hyper BN_1-algebra by giving additional axiom on the definition of hyper BN-algebra. Construction of quotient hyper BN-algebras will result to hyper BN_1-algebras if the congruence relation is regular. For future work, we have currently looked at homomorphisms and isomorphisms on hyper BN-algebras.

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