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Solving Fractional Riccati Differential equation with Caputo- Fabrizio fractional derivative

Eman Abuteen

Department of Basic Scientific Sciences, Faculty of Engineering Technology, Al-Balqa Applied University, Jordan

Abstract. This article offers an analytical solution for the fractional Riccati differential equation in three distinct cases. These cases are determined by the discriminant and the analytical solution based on the properties of the Caputo-Fabrizio fractional derivative and integral. Several examples were tested using this analytical solution. It is noteworthy that various methods have yielded related results as indicated in the literature.

2020 Mathematics Subject Classifications: 34A08, 26A33, 34A34, 65L05

Key Words and Phrases: Caputo-Fabrizio fractional operator, Riccati differential equation, Fractional differential equation

1. Introduction

Fractional calculus serves as a smooth extension of classical calculus, exploring integrals and derivatives of non-integer order [19],[34],[2]. This extension opens the door to a multitude of applications and real-world phenomena. In various disciplines, including engineering, physics, chemistry, biology, economics, control theory, and other different fields, Fractional calculus has evolved into a pivotal tool. It achieves this by transforming complex problems in these domains into mathematical models using fractional orders. Despite these advancements, challenges persist in solving several models that employ fractional differential operators. Recent progress in the theory and applications of fractional calculus has been observed, introducing analytical methods for resolving fractional differential equations, such as Adomain decomposition method [9], variational iteration method [44], the continuous and discrete symmetry methods [17],[41],[15],[16].
Numerous fractional operators find usage in the literature, some are more widely adopted, such as Riemann-Liouville and Caputo operators . The integral kernel of commonly utilized fractional operators is characterized by singularity. To tackle the singularity challenge and attain efficient and dependable modeling results in recent times, Caputo and Fabrizio

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Email address: dr.eman.abuteen@bau.edu.jo (E. Abuteen)

have introduced an effective fractional order Caputo-Fabrizio derivative featuring a non-singular kernel. A comprehensive presentation of the essential traits of the Caputo-Fabrizio derivative is provided in [31],[14],[29],[13],[12]. Later, many authors turned to the Caputo-Fabrizio derivative to model diverse engineering problems [14],[29],[13],[3],[10],[13],[39].
The Riccati differential equation is employed across diverse disciplines like physics, engineering, biology, control theory, signal processing, and finance [25],[6],[20]. The fractional Riccati equation holds significance in numerous physics and engineering contexts [36],[33],[43],[40],[8],[11],[23],[35],[45]. Many investigators have examined the numerical solution of this problem [24],[22],[21],[30],[42],[5],[7].More convenientreferences for this equation can be found in [18],[37],[1],[30],[38],[27].
This article is organized as follows: Section 2 reviews some concepts and properties of the Caputo-Fabrizio fractional derivative along with its corresponding integral. An analytical solution of any nonhomogeneous fractional differential equation is presented. In detail, solutions of the fractional Riccati differential equation for different cases by using the solution of the Caputo-Fabrizio fractional nonhomogeneous differential equation are presented in the third section. Section 4 is dedicated to conducting various numerical problems.

2. Preliminaries

In this section, we introduce some basic definitions and theorems related to Caputo-Fabrizio fractional derivative and integral.

Definition 1. *[14]*
The Caputo-Fabrizio fractional derivative for a smooth function $f : [a, \infty) \to \mathbb{R}$ is defined by

$$D^{\alpha} f(t) = \frac{1}{1-\alpha} \int_a^t e^{\left(\frac{-\alpha}{1-\alpha}(t-s)\right)} f\prime(s) ds, \tag{1}$$

were $a, \alpha \in \mathbb{R}$ and $\alpha \in (0, 1)$.

Definition 2. *[14]*
The Caputo-Fabrizio fractional integral for a smooth function $f : [a, \infty) \to \mathbb{R}$ is defined by

$$I^{\alpha} f(t) = (1-\alpha)(f(t) - f(a)) + \alpha \int_a^t f(s) ds, \tag{2}$$

were $a, \alpha \in \mathbb{R}$ and $\alpha \in (0, 1)$.

Theorem 1. *[28]*
Let $a, \alpha \in \mathbb{R}$ with $\alpha \in (0, 1)$.Then we have

$$(D^{\alpha} I^{\alpha} f(t)) = f(t) - e^{\left(\frac{-\alpha}{1-\alpha}(t-a)\right)} f(a), \tag{3}$$

Theorem 2. *[28]*
Let $a, \alpha \in \mathbb{R}$ with $\alpha \in (0,1)$. For a smooth function, $f : [a, \infty) \to \mathbb{R}$ the equality

$$(D^\alpha f(t))\prime = \frac{1}{1-\alpha} f\prime(t) - \frac{\alpha}{1-\alpha} D^\alpha f(t) \tag{4}$$

Corollary 1. *Let $a, \alpha \in \mathbb{R}$ with $\alpha \in (0,1)$.For a smooth function $f : [a, \infty) \to \mathbb{R}$ we have*

$$\int_a^t D^\alpha f(s) ds = \frac{1}{\alpha}(f(t) - f(a)) - \frac{1-\alpha}{\alpha} D^\alpha f(t) \tag{5}$$

Proof.
Integrate both sides of the equation (4) in theorem 2, we get

$$D^\alpha f(t) = \frac{1}{1-\alpha}(f(t) - f(a)) - \frac{\alpha}{1-\alpha} \int_a^t D^\alpha f(s) ds$$

Therefore, we have

$$\int_a^t D^\alpha f(s) ds = \frac{1}{\alpha}(f(t) - f(a)) - \frac{1-\alpha}{\alpha} D^\alpha f(t).$$

Theorem 3. *[32]*
Let $a, \alpha \in \mathbb{R}$ with $\alpha \in (0,1)$. Then we have

$$I^\alpha(D^\alpha f(t)) = f(t) - f(a), \tag{6}$$

Proof.
Using the definition of CF integral, we have

$$I^\alpha(D^\alpha f(t)) = (1-\alpha) D^\alpha f(t) + \alpha \int_a^t D^\alpha f(s) ds$$

Applying equation (5), we get

$$I^\alpha(D^\alpha f(t)) = (1-\alpha) D^\alpha f(t) + \alpha \left(\frac{1}{\alpha}(f(t) - f(a)) - \frac{1-\alpha}{\alpha} D^\alpha f(t) \right) = f(t) - f(a).$$

Now we apply the definitions and properties of Caputo-Fabrizio fractional derivative and integral on the nonhomogeneous fractional differential equation

$$D^\alpha f(t) = g(t) \tag{7}$$

Derive both sides and use theorem2, we get

$$\frac{-\alpha}{1-\alpha} D^\alpha f(t) + \frac{1}{1-\alpha} f\prime(t) = g\prime(t) \tag{8}$$

Integrate equation (8) and use the previous properties of Caputo-Fabrizio fractional derivative.
The solution of the nonhomogeneous fractional differential equation (7) is

$$f(t) = (1-\alpha)(g(t) - g(a)) + \alpha \int_a^t g(s) ds + f(a) \tag{9}$$

3. Solution of Fractional order Riccati differential equation

The typical format for the general form fractional Riccati differential equations is as follows:

$$D^{\alpha}y(t) = Ay^2(t) + By(t) + C, 0 < \alpha \leq 1. \tag{10}$$

With the initial condition, $y(0) = y_0$.
Using equation (9), the solution of fractional Riccati differential equation (10) is

$$y(t)-y(0) = (1-\alpha)\left((Ay^2 + By + C) - (A(y(0))^2 + By(0) + C)\right)+\alpha\int_0^t (Ay^2(s)+By(s)+C)ds \tag{11}$$

By deriving both sides, we get

$$y\prime(t) = (1-\alpha)\left(2Ay(t)y\prime(t) + By\prime(t)\right) + \alpha(Ay^2(t) + By(t) + C) \tag{12}$$

which is equivalent to

$$\frac{y\prime(t)}{Ay^2(t) + By(t) + C} - (1-\alpha)\frac{(2Ay(t)y\prime(t) + By\prime(t))}{Ay^2(t) + By(t) + C} = \alpha \tag{13}$$

To find a general solution for Fractional order Riccati differential equation, we analyze the equation $Ay^2(t) + By(t) + C$, under 3 distinct cases

Case 1: Assume that the discriminant $\triangle = B^2 - 4AC > 0$
The fractional Riccati differential equations can be reformulated as

$$D^{\alpha}y(t) = Ay^2 + By + C = (a_1y + b_1)(a_2y + b_2) \tag{14}$$

Using equation (13), we have

$$\frac{y\prime(t)}{(a_1y + b_1)(a_2y + b_2)} - (1-\alpha)\frac{(2Ay(t)y\prime(t) + By\prime(t))}{Ay^2(t) + By(t) + C} = \alpha$$

Using partial fractions, we get

$$y\prime(t)\left(\frac{A_1}{a_1y + b_1} - \frac{A_2}{a_2y + b_2}\right) - (1-\alpha)\frac{(2Ay(t)y\prime(t) + By\prime(t))}{Ay^2(t) + By(t) + C} = \alpha \tag{15}$$

Where $A_1 = \dfrac{a_1}{b_2a_1 - b_1a_2}$ and $A_2 = \dfrac{a_2}{b_1a_2 - b_2a_1}$

The solution of the fractional Riccati differential equation will be

$$ln|a_1y + b_1|^{\frac{A_1}{a_1} - (1-\alpha)} + ln|a_2y + b_2|^{\frac{A_2}{a_2} - (1-\alpha)} = \alpha t + c \tag{16}$$

So, we have the solution is

$$|a_1y + b_1|^{\frac{1}{b_2a_1 - b_1a_2} - (1-\alpha)}|a_2y + b_2|^{\frac{1}{b_1a_2 - b_2a_1} - (1-\alpha)} = ce^{\alpha t} \tag{17}$$

Case 2: If the discriminant $\triangle = B^2 - 4AC = 0$
The fractional Riccati differential equation (10)takes the form

$$D^{\alpha}y(t) = (ay(t) + b)^2, 0 < \alpha \leq 1. \tag{18}$$

With the initial condition, $y(0) = y_0$.
Using equation (13) we have

$$\frac{y\prime(t)}{(ay(t) + b)^2} - (1-\alpha)\frac{(2Ay(t)y\prime(t) + By\prime(t))}{Ay^2(t) + By(t) + C} = \alpha \tag{19}$$

Integrate equation (19), the solution will have the following form

$$|ay + b|^{2(\alpha-1)} = ce^{\overline{ay+b}}\,^{1}e^{\alpha t}, y \neq \frac{-b}{a} \tag{20}$$

Case 3: If the discriminant $\triangle = B^2 - 4AC < 0$
The fractional Riccati differential equation (10)

$$D^{\alpha}y(t) = Ay^2 + By + C, 0 < \alpha \leq 1.$$

The equation $Ay^2 + By + C$ is irreducible, so by integrating equation (13), we get

$$\frac{1}{A}tan^{-1}\frac{2Ay + B}{\sqrt{4AC - B^2}} = (1-\alpha)ln|Ay^2 + By + C| + \alpha t + c \tag{21}$$

4. Numerical examples

We employ the general solution of the fractional Riccati differential equation in all three cases to analyze the following examples and contrast them with alternative methods.

Example 1. *Consider the following fractional logistic differential equation*

$$D^{\alpha}y(t) = y - y^2, y(0) = \frac{1}{2} \tag{22}$$

This differential equation is classified under case 1 and by applying equation (17), we find that the solution to be

$$|y|^{\alpha}|1-y|^{\alpha-2} = \left(\frac{1}{4}\right)^{2\alpha-2}e^{\alpha t} \tag{23}$$

If $\alpha = 1$, then the solution is

$$y = \frac{1}{e^{-t} + 1} \tag{24}$$

This solution agrees with several solutions for fractional logistic differential equations see [24],[5],[7],[32],[4].

Comparisons for different values of α are shown in Figure 1.

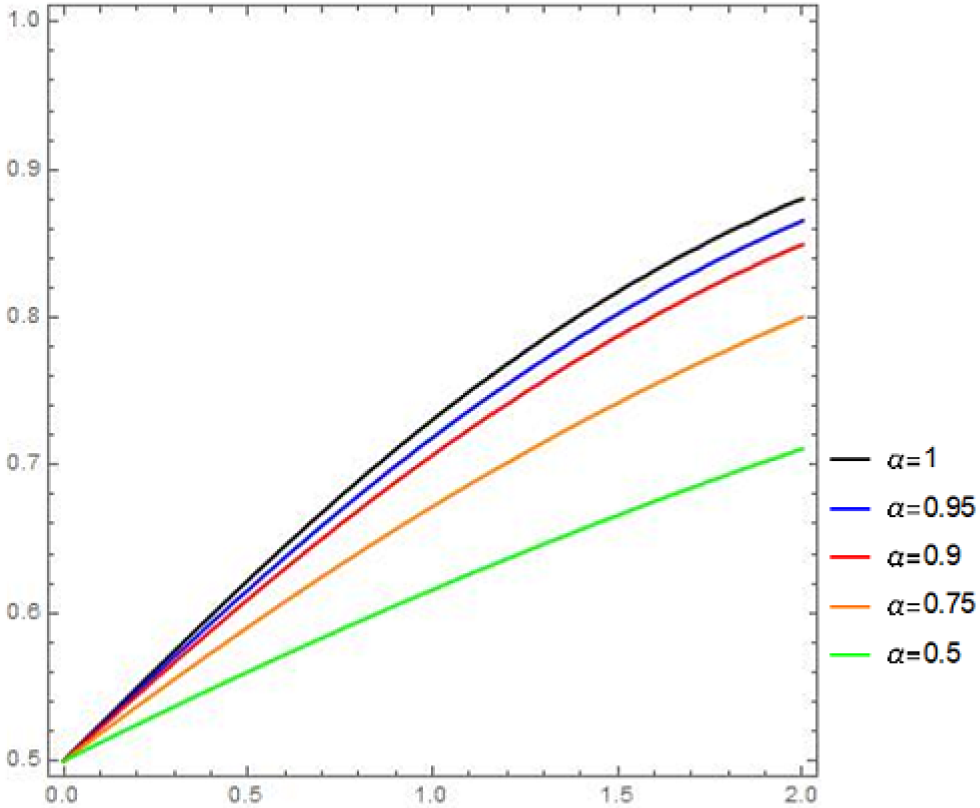

Figure 1: Solutions of the fractional differential equation (22) for different values of α.

Example 2. *Consider the following fractional differential equation*

$$D^{\alpha}y(t) = -y^2 + 2y + 1, y(0) = 0 \tag{25}$$

Based on positive discriminant, this equation is associated with case 1. By applying equation (17), the solution is characterized by

$$|y-(1-\sqrt{2})|^{\frac{1}{2\sqrt{2}}-1+\alpha}|(1+\sqrt{2})-y|^{\alpha-1-\frac{1}{2\sqrt{2}}} = \left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^{\frac{1}{2\sqrt{2}}} e^{\alpha t} \tag{26}$$

Substituting $\alpha = 1$ in equation (26), we obtain the solution as

$$y = \frac{e^{2\sqrt{2}t}-1}{(\sqrt{2}-1)e^{2\sqrt{2}t}+(\sqrt{2}+1)} \tag{27}$$

This solution is equivalent to

$$y = 1 + \sqrt{2}tanh\left(\sqrt{2}t + \frac{1}{2}ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right) \tag{28}$$

Which is compatible with the one found in [36],[33],[43],[40], [35], [26].

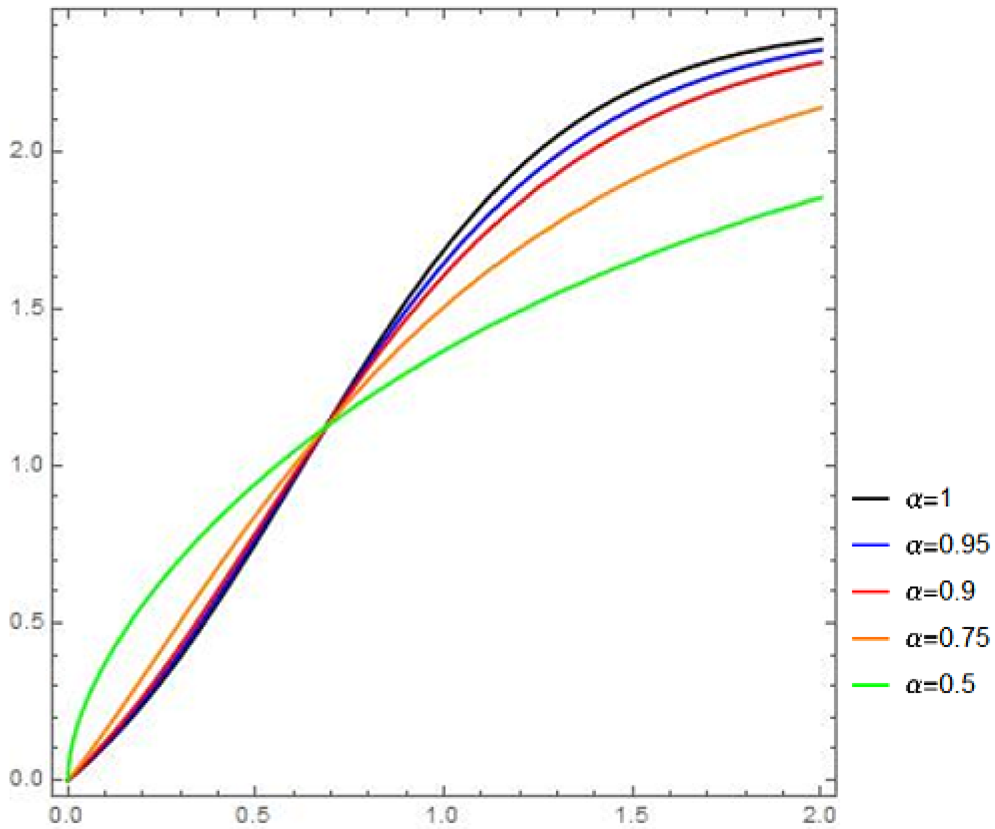

Figure 2: Solutions of the fractional differential equation (25) for different values of α.

Example 3. *Consider the following fractional differential equation*

$$D^{\alpha}y(t) = y^2 + 4y + 4, y(0) = 0 \tag{29}$$

This example is classified as case 2. Therefore, the solution for this case follows the pattern of equation (20), and thereafter, the solution is

$$|y+2|^{2(\alpha-1)} = 2^{2(\alpha-1)}e^{\alpha t + \frac{1}{y+2} - \frac{1}{2}} \tag{30}$$

Where the exact solution for $\alpha = 1$ is

$$y = \frac{4t}{1-2t} \tag{31}$$

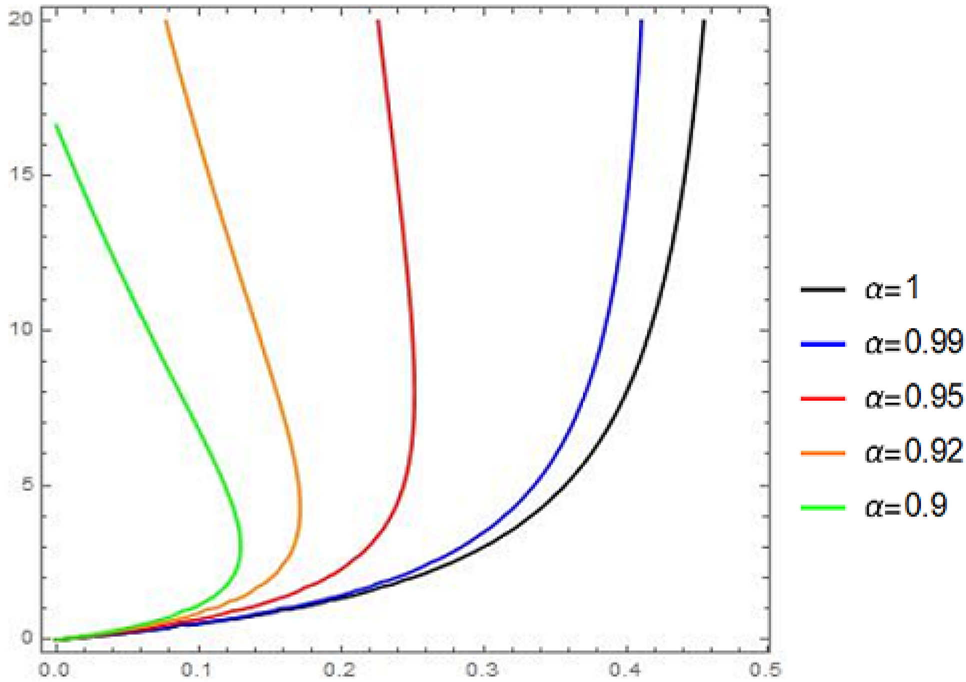

Figure 3: Solutions of the fractional differential equation (29) for different values of α.

Example 4. *Consider the following fractional differential equation*

$$D^{\alpha}y(t) = y^2 + 1, y(0) = 0 \tag{32}$$

We notice that this example is categorized under case 3. Therefore, according to the equation (21), the solution comes out to be

$$tan^{-1}y = (1-\alpha)ln(y^2+1) + \alpha t \tag{33}$$

The exact solution for $\alpha = 1$ is
$y = tan(t)$
We can see that the exact solution agrees with our solution.
For comparisons with this solution and figures see [1] Figure 4 shows a comparison of different values of α.

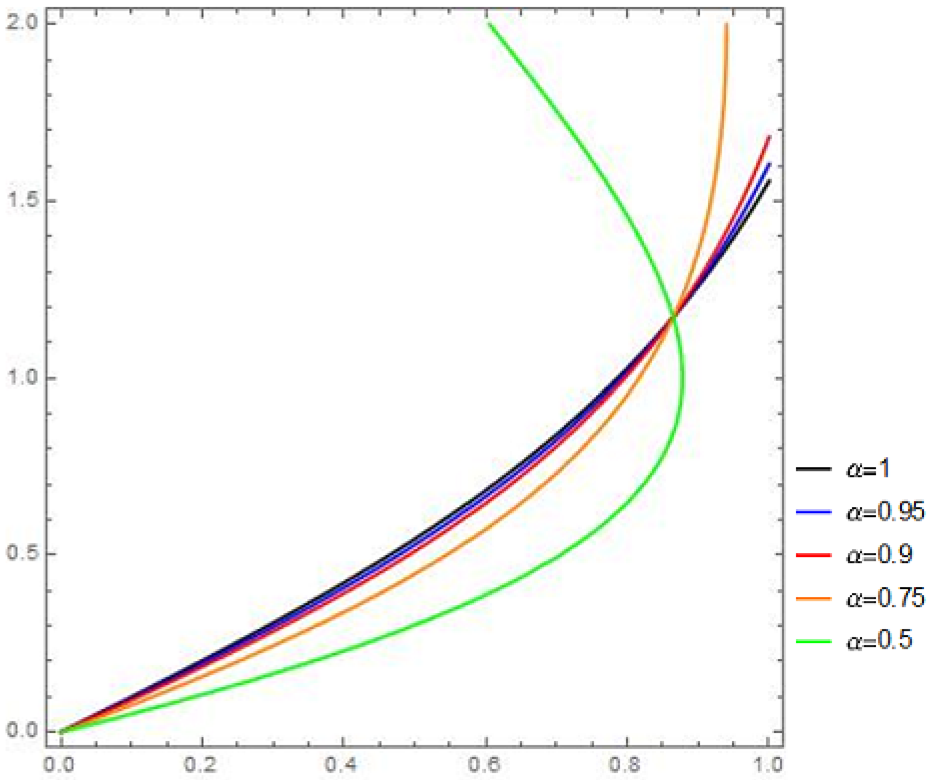

Figure 4: Solutions of the fractional differential equation (32) for different values of α for $0 \leq t \leq 1$

5. Conclusion

In this paper, we categorize the fractional Riccati differential equation under three cases and apply Caputo-Fabrizio fractional derivative and integral properties on these cases to find analytical solutions of these equations. We have applied the analytical solution to various examples and provided figures that demonstrate a strong agreement with the solutions presented in the literature. We plan in a future work to apply Caputo-Fabrizio fractional derivative and integral on other different fractional differential equations.

References

[1] Rami AlAhmad, Qusai AlAhmad, and Ahmad Abdelhadi. Solution of fractional autonomous ordinary differential equations. 2021.

[2] Kilbas Anatoliĭ Aleksandrovich, Srivastava Hari M, and Trujillo Juan J. *Theory and applications of fractional differential equations*, volume 204. elsevier, 2006.

[3] Amal Alshabanat, Mohamed Jleli, Sunil Kumar, and Bessem Samet. Generalization of caputo-fabrizio fractional derivative and applications to electrical circuits. *Frontiers in Physics*, 8:64, 2020.

[4] I. Area and J.J. Nieto. Power series solution of the fractional logistic equation. *Physica A: Statistical Mechanics and its Applications*, 573:125947, 2021.

[5] Iván Area and Juan J Nieto. Fractional-order logistic differential equation with mittag–leffler-type kernel. *Fractal and Fractional*, 5(4):273, 2021.

[6] Ahmed H Arnous, Mohammad Mirzazadeh, Seithuti Moshokoa, Sarang Medhekar, Qin Zhou, Mohammad F Mahmood, Anjan Biswas, and Milivoj Belic. Solitons in optical metamaterials with trial solution approach and bäcklund transform of riccati equation. *Journal of Computational and Theoretical Nanoscience*, 12(12):5940–5948, 2015.

[7] Sadia Arshad, Iram Saleem, Ali Akgül, Jianfei Huang, Yifa Tang, and Sayed M Eldin. A novel numerical method for solving the caputo-fabrizio fractional differential equation. *AIMS Math*, 8:9535–9556, 2023.

[8] Nourhane Attia, Ali Akgül, Djamila Seba, and Abdelkader Nour. On solutions of fractional logistic differential equations.

[9] Amin Samimi Behbahan, As' ad Alizadeh, Meysam Mahmoudi, Mahmoud Shamsborhan, Tariq J Al-Musawi, and Pooya Pasha. A new adomian decomposition technique for a thermal analysis forced non-newtonian magnetic reiner-rivlin viscoelastic fluid flow. *Alexandria Engineering Journal*, 80:48–57, 2023.

[10] Sanjay Bhatter, Amit Mathur, Devendra Kumar, and Jagdev Singh. A new analysis of fractional drinfeld–sokolov–wilson model with exponential memory. *Physica A: Statistical Mechanics and its Applications*, 537:122578, 2020.

[11] Haifa Bin Jebreen and Ioannis Dassios. A biorthogonal hermite cubic spline galerkin method for solving fractional riccati equation. *Mathematics*, 10(9):1461, 2022.

[12] Michele Caputo and Mauro Fabrizio. A new definition of fractional derivative without singular kernel. *Progress in Fractional Differentiation & Applications*, 1(2):73–85, 2015.

[13] Michele Caputo and Mauro Fabrizio. Applications of new time and spatial fractional derivatives with exponential kernels. *Progress in Fractional Differentiation & Applications*, 2(1):1–11, 2016.

[14] Michele Caputo and Mauro Fabrizio. On the singular kernels for fractional derivatives. some applications to partial differential equations. *Progr. Fract. Differ. Appl*, 7(2):1–4, 2021.

[15] Youness Chatibi, El Hassan El Kinani, and Abdelaziz Ouhadan. Lie symmetry analysis and conservation laws for the time fractional black–scholes equation. *International Journal of Geometric Methods in Modern Physics*, 17(01):2050010, 2020.

[16] Youness Chatibi, El Hassan El Kinani, and Abdelaziz Ouhadan. On the discrete symmetry analysis of some classical and fractional differential equations. *Mathematical Methods in the Applied Sciences*, 44(4):2868–2878, 2021.

[17] Youness Chatibi, Abdelaziz Ouhadan, et al. Continuous and discrete symmetry methods for fractional differential equations. In *Fractional-Order Modeling of Dynamic Systems with Applications in Optimization, Signal Processing and Control*, pages 1–35. Elsevier, 2022.

[18] Zhoujin Cui. Solutions of some typical nonlinear differential equations with caputo-fabrizio fractional derivative. *AIMS Math*, 7(8):14139–14153, 2022.

[19] Kai Diethelm and Neville J Ford. Analysis of fractional differential equations. *Journal of Mathematical Analysis and Applications*, 265(2):229–248, 2002.

[20] Aristide Halanay and Vlad Ionescu. *Time-Varying Discrete Linear Systems: Input-Output Operators. Riccati Equations. Disturbance Attenuation*, volume 68. Birkhäuser, 2012.

[21] Ishak Hashim, O Abdulaziz, and S Momani. Homotopy analysis method for fractional ivps. *Communications in Nonlinear Science and Numerical Simulation*, 14(3):674–684, 2009.

[22] Shaheed N Huseen. Series solutions of fractional initial-value problems by q-homotopy analysis method. *International Journal of Innovative Science, Engineering & Technology*, 3(1), 2016.

[23] Rui Jin and Linjun Wang. Generalized bell collocation method to solve fractional riccati differential equations. *IAENG International Journal of Applied Mathematics*, 53(1):1–7, 2023.

[24] Sagar R Khirsariya and Snehal B Rao. On the semi-analytic technique to deal with nonlinear fractional differential equations. *Journal of Applied Mathematics and Computational Mechanics*, 22(1):13–26, 2023.

[25] Irena Lasiecka and Roberto Triggiani. *Differential and algebraic Riccati equations with application to boundary/point control problems: continuous theory and approximation theory.* Springer, 1991.

[26] XY Li, BY Wu, RT Wang, et al. Reproducing kernel method for fractional riccati differential equations. In *Abstract and Applied Analysis*, volume 2014. Hindawi, 2014.

[27] Yuanlu Li, Ning Sun, Bochao Zheng, Qi Wang, and Yingchao Zhang. Wavelet operational matrix method for solving the riccati differential equation. *Communications in Nonlinear Science and Numerical Simulation*, 19(3):483–493, 2014.

[28] Yuanlu Li, Ning Sun, Bochao Zheng, Qi Wang, and Yingchao Zhang. Wavelet operational matrix method for solving the riccati differential equation. *Communications in Nonlinear Science and Numerical Simulation*, 19(3):483–493, 2014.

[29] Jorge Losada and Juan J Nieto. Properties of a new fractional derivative without singular kernel. *Progr. Fract. Differ. Appl*, 1(2):87–92, 2015.

[30] S Katrin Lydia, M Mary Jancirani, and A Alphonse Anitha. Numerical solution of nonlinear fractional differential equations using kharrat-toma iterative method. *NVEO-NATURAL VOLATILES & ESSENTIAL OILS Journal— NVEO*, pages 9878–9890, 2021.

[31] GA Mboro Nchama, LL Alfonso, AL Mecıas, and MR Richard. Construction of caputo-fabrizio fractional differential mask for image enhancement. *Progress in Fractional Differentiation and Application*, 2020.

[32] Juan J. Nieto. Solution of a fractional logistic ordinary differential equation. *Applied Mathematics Letters*, 123:107568, 2022.

[33] Zaid Odibat and Shaher Momani. Modified homotopy perturbation method: application to quadratic riccati differential equation of fractional order. *Chaos, Solitons & Fractals*, 36(1):167–174, 2008.

[34] Igor Podlubny. *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. Elsevier, 1998.

[35] H Porki, M Arabameri, and R Gharechahi. Numerical solution of nonlinear fractional riccati differential equations using compact finite difference method. *Iranian Journal of Numerical Analysis and Optimization*, 12(3 (Special Issue)-On the occasion of the 75th birthday of Professor A. Vahidian and Professor F. Toutounian):585–606, 2022.

[36] Mehmet Giyas Sakar, Ali Akgül, and Dumitru Baleanu. On solutions of fractional riccati differential equations. *Advances in Difference Equations*, 2017:1–10, 2017.

[37] V SHAMEEMA and MC RANJINI. An operational matrix method for fractional differential equations with non-singular kernel. *Journal of Fractional Calculus and Applications*, 14(1):157–170, 2023.

[38] Nabil T Shawagfeh. Analytical approximate solutions for nonlinear fractional differential equations. *Applied Mathematics and Computation*, 131(2-3):517–529, 2002.

[39] HongGuang Sun, Yong Zhang, Dumitru Baleanu, Wen Chen, and YangQuan Chen. A new collection of real world applications of fractional calculus in science and engineering. *Communications in Nonlinear Science and Numerical Simulation*, 64:213–231, 2018.

[40] Muhammed I Syam, Azza Alsuwaidi, Asia Alneyadi, Safa Al Refai, and Sondos Al Khaldi. Implicit hybrid methods for solving fractional riccati equation. *J. Nonlinear Sci. Appl*, 12(2):124–134, 2019.

[41] BA Tayyan and AH Sakka. Lie symmetry analysis of some conformable fractional partial differential equations. *Arabian Journal of Mathematics*, 9:201–212, 2020.

[42] RS Teppawar, RN Ingle, and SN Thorat. Some results and applications on conformable fractional kamal transform. *J. Math. Comput. Sci.*, 11(5):6581–6598, 2021.

[43] LI Yuanlu. Solving a nonlinear fractional differential equation using chebyshev wavelets. *Communications in Nonlinear Science and Numerical Simulation*, 15(9):2284–2292, 2010.

[44] Ri Zhang, Nehad Ali Shah, Essam R El-Zahar, Ali Akgül, and Jae Dong Chung. Numerical analysis of fractional-order emden–fowler equations using modified variational iteration method. *Fractals*, 31(02):2340028, 2023.

[45] Eman AA Ziada. Analytical solution of linear and nonlinear fractional differential equations. *Nile Journal of Basic Science*, 1(1):1–13, 2021.