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Closeness Energy of Non-Commuting Graph for Dihedral Groups

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Abstract. This paper focuses on the non-commuting graph for dihedral groups of order $2n$, D_{2n}, where $n \geq 3$. We show the spectrum and energy of the graph corresponding to the closeness matrix. The result is that the obtained energy is always twice its spectral radius and is never an odd integer. Moreover, it is classified as hypoenergetic.

2020 Mathematics Subject Classifications: 05C25, 05C50, 15A18, 20D99

Key Words and Phrases: Closeness matrix, Energy of a graph, Non-commuting graph, Dihedral Groups

1. Introduction

Let G be a group and $Z(G)$ be a center of G. The non-commuting graph of G, denoted as Γ_G, has vertex set $G \backslash Z(G)$ and two distinct vertices v_p, v_q in Γ_G are connected by an edge whenever $v_p v_q \neq v_q v_p$ [1]. Many authors have studied non-commuting graphs for various kinds of groups. According to Abdollahi [1], Γ_G is always connected and its diameter is always 2. Accordingly, (d_{pq}), which is the shortest path between v_p and v_q, is well defined in Γ_G. This discussion continues by examining the isomorphic properties of two non-commuting graphs related to the isomorphic properties of the corresponding groups.

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The energy of Γ_G, $E(\Gamma_G)$, is calculated by adding all the absolute values of its eigenvalues. This definition was pioneered by Gutman [6]. There is a classification of graphs based on energy value [11]. Also, Sun et al. have shown that the clique path has the maximum distance of eigenvalues and energy [23]. As has been shown in the literature, the adjacency energy of a graph is never an odd integer and it is never its square root either [4, 12].

A graph matrix based on the distance between two vertices was introduced by Indulal and Gutman [7]. Readers may refer to [8] for information regarding degree product distance energy. In addition, Jog and Gurjar [9] discusses the degree sum exponent distance of graphs. Accordingly, Romdhini et al. [16] investigated signless Laplacian energies of interval-valued fuzzy graphs. In addition, Zheng and Zhou [24] presented the closeness eigenvalues of graphs.

Throughout this work, the vertex set for Γ_G is the non-abelian dihedral group of order $2n$, where $n \geq 3$, denoted by $D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle$ [3]. The center of D_{2n} is either $Z(D_{2n}) = \{e\}$ for n is odd, or $\left\{e, a^{\frac{n}{2}}\right\}$ for n is even. The centralizer of the element a^i in D_{2n} is $C_{D_{2n}}(a^i) = \{a^j : 1 \leq j \leq n\}$ and for the element $a^i b$ is either $C_{D_{2n}}(a^i b) = \{e, a^i b\}$, if n is odd or $C_{D_{2n}}(a^i b) = \left\{e, a^{\frac{n}{2}}, a^i b, a^{\frac{n}{2}+i} b\right\}$, if n is even.

Several authors have examined the energy of commuting and non-commuting graphs involving D_{2n} as the set of vertex. By considering the eigenvalues of the degree sum and degree subtraction matrices, Romdhini and Nawawi [17, 19] and Romdhini et al. [22] formulated the energy. In [18, 21], the sum of the degree exponent and the maximum and minimum degree energies were presented for D_{2n}. Therefore, the purpose of this paper is to formulate the energy based on the closeness matrix for Γ_G on D_{2n}.

2. Preliminaries

In this part, we begin with the definition of the closeness matrix of a graph.

Definition 1. *[24] Let d_{pq} be the distance between vertex v_p and v_q. The closeness matrix of order $n \times n$ associated with Γ_G is given by $C(\Gamma_G) = [c_{pq}]$ whose (p, q)-th entry is*

$$c_{pq} = \begin{cases} 2^{-d_{pq}}, & \text{if } v_p \neq v_q \\ 0, & \text{if } v_p = v_q. \end{cases}$$

The closeness energy of Γ_G can be written by

$$E_C(\Gamma_G) = \sum_{i=1}^{n} |\lambda_i|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of $C(\Gamma_G)$.

The spectral radius of Γ_G corresponding with closeness matrix is

$$\rho_C(\Gamma_G) = max\{|\lambda| : \lambda \in Spec_C(\Gamma_G)\}.$$

We know that Γ_G has $2n-1$ and $2n-2$ vertices for odd and even n, respectively, then Γ_G corresponding to the $C-$matrix can be classified as hypoenergetic graph if the $C-$energy complies with the statement below:

$$[11]E_C(\Gamma_G) < \begin{cases} 2n-1, & \text{for odd } n \\ 2n-2, & \text{for even } n, \end{cases}$$

The following theorem is useful to construct the closeness matrix of Γ_G. We define $G_1 = \{a^i : 1 \le i \le n\} \backslash Z(D_{2n})$ and $G_2 = \{a^i b : 1 \le i \le n\}$.

Theorem 1. *[10] For a non-commuting graph for G, Γ_G,*

(i) if $G = G_1$, then $\Gamma_G \cong \bar{K}_m$, where $m = |G_1|$,

(ii) if $G = G_2$, then $\Gamma_G \cong \begin{cases} K_n, & \text{if } n \text{ is odd} \\ K_n - \frac{n}{2}K_2, & \text{if } n \text{ is even.} \end{cases}$

where $\frac{n}{2}K_2$ denotes $\frac{n}{2}$ copies of K_2.

Lemma 1. *[5] The adjacency spectrum of K_n is $\{(n-1)^{(1)}, (-1)^{(n-1)}\}$.*

In order to simplify the determinant in the characteristic polynomial of Γ_G, we need the following three results.

Lemma 2. *[13] If a, b, c and d are real numbers, then the determinant of the $(n_1+n_2)\times(n_1+n_2)$ matrix of the form*

$$\begin{vmatrix} (\lambda+a)I_{n_1} - aJ_{n_1} & -cJ_{n_1\times n_2} \\ -dJ_{n_2\times n_1} & (\lambda+b)I_{n_2} - bJ_{n_2} \end{vmatrix}$$

can be simplified in an expression as

$$(\lambda+a)^{n_1-1}(\lambda+b)^{n_2-1}\left(\left(\lambda-(n_1-1)\,a\right)\left(\lambda-(n_2-1)\,b\right)-n_1n_2cd\right),$$

where $1 \le n_1, n_2 \le n$ and $n_1+n_2 = n$.

Theorem 2. *[20] If s, t are real numbers, then the characteristic polynomial of an $n\times n$ matrix*

$$M = \begin{bmatrix} t(J-I)_{\frac{n}{2}} & t(J-I)_{\frac{n}{2}} + sI_{\frac{n}{2}} \\ t(J-I)_{\frac{n}{2}} + sI_{\frac{n}{2}} & t(J-I)_{\frac{n}{2}} \end{bmatrix}$$

is

$$P_M(\lambda) = (\lambda-s+2t)^{\frac{n}{2}-1}\left(\lambda-s-(n-2)\,t\right)(\lambda+s)^{\frac{n}{2}}.$$

Theorem 3. *[20] If r, s, t, u are real numbers, then the characteristic polynomial of an $(2n-2)\times(2n-2)$ matrix*

$$M = \begin{bmatrix} r(J-I)_{n-2} & tJ_{(n-2)\times\frac{n}{2}} & tJ_{(n-2)\times\frac{n}{2}} \\ tJ_{\frac{n}{2}\times(n-2)} & u(J-I)_{\frac{n}{2}} & u(J-I)_{\frac{n}{2}} + sI_{\frac{n}{2}} \\ tJ_{\frac{n}{2}\times(n-2)} & u(J-I)_{\frac{n}{2}} + sI_{\frac{n}{2}} & u(J-I)_{\frac{n}{2}} \end{bmatrix}$$

is

$$\begin{aligned} P_M(\lambda) = {} & (\lambda+r)^{n-3}(\lambda-s+2u)^{\frac{n}{2}-1}(\lambda+s)^{\frac{n}{2}} \\ & \left(\lambda^2-(s+(n-2)u+r(n-3))\,\lambda+r(n-3)\,(s+(n-2)u)-n(n-2)t^2\right). \end{aligned}$$

3. Main Results

In this section, we begin with the distance between two distinct vertices in Γ_G.

Theorem 4. *Let Γ_G be the non-commuting graph on $G = G_1 \cup G_2$. For two distinct vertices $v_p, v_q \in V(\Gamma_G)$, then the distance between v_p and v_q*

(i) for the odd n, $d_{pq} = \begin{cases} 2, & \text{if } v_p, v_q \in G_1 \\ 1, & \text{otherwise,} \end{cases}$ *, and*

(ii) for the even n, $d_{pq} = \begin{cases} 2, & \text{if } v_p, v_q \in G_1 \\ 2, & v_p \in G_2, v_q \in \left\{a^{\frac{n}{2}+i}b\right\} \\ 1, & \text{otherwise.} \end{cases}$

Proof. For odd n case, since $C_{D_{2n}}(a^i) = \{a^j : 1 \le j \le n\}$, then the vertex a^i, for $1, 2, \ldots, n-1$, is not adjacent to all vertices of G_1, however, it always has an edge with all members of G_2. Thus, it is proven that $d_{pq} = 1$, where v_p belongs to G_1 and $v_q \in G_2$, or vice versa. Suppose now two distinct vertices $a^p, a^q \in G_1$ with $p \neq q$, meaning from a^p there are two vertices that must be passed to arrive at the terminal vertex v^q, they are one of a^ib and v^q itself. From this fact, we then get $d_{pq} = 2$.

While for the even n case, the centralizer of a^ib in D_{2n} is $\{e, a^ib\}$ implies that for $1 \le i \le n$, vertex a^ib is connected with all other elements of $G_1 \cup G_2$. Therefore, for $v_p, v_q \in G_2$, it is shown that $d_{pq} = 1$. Now when n is even, as a result of $C_{D_{2n}}(a^ib) = \{e, a^{\frac{n}{2}}, a^ib, a^{\frac{n}{2}+i}b\}$ for all $1 \le i \le n$, then vertices a^ib and $a^{\frac{n}{2}+1}b$ are always disconnected in Γ_G. Hence, for $v_p \in G_2$ and $v_q \in \left\{a^{\frac{n}{2}+i}b\right\}$, $d_{pq} = 2$. This also applies vice versa when $v_q \in G_2$ and $v_p \in \left\{a^{\frac{n}{2}+i}b\right\}$. However, when one of v_p and v_q is not in $\left\{a^{\frac{n}{2}+i}b\right\}$, then $d_{pq} = 1$.

The closeness energy of the non-commuting graph on G, for $G = G_1$ or $G = G_2$ is presented in the Theorem below:

Theorem 5. *Let Γ_G be the non-commuting graph on G.*

(i) If $G = G_1$, then $E_C(\Gamma_G)$ is undefined, and

(ii) If $G = G_2$, then $E_C(\Gamma_G) = \begin{cases} n-1, & \text{if } n \text{ is odd} \\ n-\frac{3}{2}, & \text{if } n \text{ is even.} \end{cases}$.

Proof.

(i) For $G = G_1$ case, by Theorem 1 (1), $\Gamma_G \cong \bar{K}_m$, where $m = |G_1|$. Then Γ_G consists of m isolated vertices which implies the distance of every pair vertices of G_1 is undefined.

(ii) For the second case when $G = G_2$, we first proceed if n is odd. Again, by Theorem 1 (2), $\Gamma_G \cong K_n$. Then every pair of vertices are at distance 1. Now the closeness matrix of Γ_G is $C(\Gamma_G) = c_{pq}$, with $(p,q)-$entry if $v_p \neq v_q$ is 2^{-1}, and zero if $v_p = v_q$. Hence,

$$C(\Gamma_G) = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & \frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & 0 \end{bmatrix} = \frac{1}{2}A(K_n).$$

In other words, $C(\Gamma_G)$ is the product of $\frac{1}{2}$ and the adjacency matrix of K_n. Therefore, from Lemma 1, the closeness energy of Γ_G will be $n-1$.

Meanwhile for the even n, by Theorem 1, $\Gamma_G \cong K_n - \frac{n}{2}K_2$, then the distance between every pair a^ib and $a^{\frac{n}{2}+i}$ for all $1 \leq i \leq n$ is 2, and 1, otherwise. Thus, $C(\Gamma_G) = c_{pq}$ and for $v_p \neq v_q$,

$$c_{ij} = \begin{cases} \frac{1}{4}, & \text{if } v_p = a^ib,\ v_q = a^{\frac{n}{2}+i}b,\ 1 \leq i \leq n \\ \frac{1}{2}, & \text{if } v_p = a^ib,\ v_q \neq a^{\frac{n}{2}+i}b,\ 1 \leq i \leq n \\ 0, & otherwise. \end{cases}$$

Now we can construct $C(\Gamma_G)$ as follows:

$$C(\Gamma_G) = \begin{bmatrix} 0 & \cdots & \frac{1}{2} & \frac{1}{4} & \cdots & \frac{1}{2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \cdots & 0 & \frac{1}{2} & \cdots & \frac{1}{4} \\ \frac{1}{4} & \cdots & \frac{1}{2} & 0 & \cdots & \frac{1}{2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \cdots & \frac{1}{4} & \frac{1}{2} & \cdots & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(J-I)_{\frac{n}{2}} & \frac{1}{4}(J-I)_{\frac{n}{2}} + \frac{1}{4}I_{\frac{n}{2}} \\ \frac{1}{4}(J-I)_{\frac{n}{2}} + \frac{1}{4}I_{\frac{n}{2}} & \frac{1}{2}(J-I)_{\frac{n}{2}} \end{bmatrix}.$$

In this case, we have four block matrices of $C(\Gamma_G)$:

$$C(\Gamma_G) = \begin{bmatrix} A & B \\ B & A \end{bmatrix},$$

where A is a matrix of order $\frac{n}{2}$ with zero diagonal entries and all of the non-diagonal entries as $\frac{1}{2}$ and B is the matrix of order $\frac{n}{2}$ with diagonal entries are $\frac{1}{4}$ and the non-diagonal entries are $\frac{1}{2}$. By Theorem 2 with $s = \frac{1}{4}$ and $t = \frac{1}{2}$, Equation 2 is

$$P_{C(\Gamma_G)}(\lambda) = \left(\lambda + \frac{3}{4}\right)^{\frac{n}{2}-1}\left(\lambda + \frac{3}{4} - \frac{1}{2}n\right)\left(\lambda + \frac{1}{4}\right)^{\frac{n}{2}}.$$

Therefore, using the roots of Equation 2, the closeness energy of Γ_G is

$$E_C(\Gamma_G) = \left(\frac{n}{2}\right)\left|-\frac{1}{4}\right| + \left(\frac{n}{2}-1\right)\left|-\frac{3}{4}\right| + \left|\frac{1}{2}n - \frac{3}{4}\right| = n - \frac{3}{2}.$$

Theorem 6. *The characteristic polynomial of* Γ_G*, where* $G = G_1 \cup G_2$*, is*

(i) for n is odd: $P_{C(\Gamma_G)}(\lambda) = (\lambda+2)^{n-2}(\lambda+1)^{n-1}\left(\lambda^2 - (3n-5)\lambda + (n-1)(n-4)\right)$,

(ii) for n is even: $P_{D(\Gamma_G)}(\lambda) = \lambda^{\frac{n}{2}-1}(\lambda+2)^{n-3+\frac{n}{2}}\left(\lambda^2 - 3(n-2)\lambda + n(n-4)\right)$.

Proof.

(i) When n is odd and $G = G_1 \cup G_2$, by Theorem 4, we have the distance of every pair of vertices. Since $Z(D_{2n}) = \{e\}$, consequently, Γ_G has $2n-1$ vertices. They are $n-1$ vertices of a^i, for $1 \le i \le n-1$, and n vertices of a^ib, $1 \le i \le n$. Hence, from Definition 1, $C(\Gamma_G)$ is a $(2n-1)\times(2n-1)$ matrix as the following:

$$C(\Gamma_G) = \begin{bmatrix} 0 & \dots & \frac{1}{4} & \frac{1}{2} & \dots & \frac{1}{2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{4} & \dots & 0 & \frac{1}{2} & \dots & \frac{1}{2} \\ \frac{1}{2} & \dots & \frac{1}{2} & 0 & \dots & \frac{1}{2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{2} & \dots & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}(J-I)_{n-1} & \frac{1}{2}J_{(n-1)\times n} \\ \frac{1}{2}J_{(n-1)\times n} & \frac{1}{2}(J-I)_n \end{bmatrix}.$$

Now the characteristic polynomial of Equation 1 is

$$P_{C(\Gamma_G)}(\lambda) = |\lambda I_{2n-1} - C(\Gamma_G)| = \begin{vmatrix} (\lambda+2)I_{n-1} - 2J_{n-1} & -J_{(n-1)\times n} \\ -J_{n\times(n-1)} & (\lambda+1)I_n - J_n) \end{vmatrix}.$$

Using Lemma 2, with $a = \frac{1}{4}$, $b = \frac{1}{2}$, $c = d = \frac{1}{2}$, and $n_1 = n-1$, $n_2 = n$, then we obtain the formula of $P_{C(\Gamma_G)}(\lambda)$,

$$P_{C(\Gamma_G)}(\lambda) = \left(\lambda + \frac{1}{4}\right)^{n-2}\left(\lambda + \frac{1}{2}\right)^{n-1}\left(\lambda^2 + 1 - \left(\frac{3}{4}n\right)\lambda - \frac{1}{8}(n+2)(n-1)\right)$$

(ii) Now for the even n case and $G = G_1 \cup G_2$, we know that $Z(D_{2n}) = \{e, a^{\frac{n}{2}}\}$. Then, the cardinality of the vertex set of Γ_G is $2n-2$ with detail $n-2$ vertices of a^i, for $1 \le i < \frac{n}{2}, \frac{n}{2} < i < n$, and n vertices of a^ib, for $1 \le i \le n$. Following the result of Theorem 4 and by Definition 1, then $C(\Gamma_G)$ is a $(2n-2)\times(2n-2)$ matrix as the following:

$$C(\Gamma_G) = \begin{bmatrix} 0 & \dots & \frac{1}{4} & \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{4} & \dots & 0 & \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} \\ \frac{1}{2} & \dots & \frac{1}{2} & 0 & \dots & \frac{1}{2} & \frac{1}{4} & \dots & \frac{1}{2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{2} & \dots & 0 & \frac{1}{2} & \dots & \frac{1}{4} \\ \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{4} & \dots & \frac{1}{2} & 0 & \dots & \frac{1}{2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{4} & \frac{1}{2} & \dots & 0 \end{bmatrix}$$

Now we provide nine block matrices of $C(\Gamma_G)$ as follows:

$$C(\Gamma_G) = \begin{bmatrix} \frac{1}{4}(J-I)_{n-2} & \frac{1}{2}J_{(n-2)\times\frac{n}{2}} & \frac{1}{2}J_{(n-2)\times\frac{n}{2}} \\ \frac{1}{2}J_{\frac{n}{2}\times(n-2)} & \frac{1}{2}(J-I)_{\frac{n}{2}} & \frac{1}{2}(J-I)_{\frac{n}{2}}+\frac{1}{4}I_{\frac{n}{2}} \\ \frac{1}{2}J_{\frac{n}{2}\times(n-2)} & \frac{1}{2}(J-I)_{\frac{n}{2}}+\frac{1}{4}I_{\frac{n}{2}} & \frac{1}{2}(J-I)_{\frac{n}{2}} \end{bmatrix}.$$

By Theorem 3 with $r = s = \frac{1}{4}$ and $t = u = \frac{1}{2}$, we then obtain

$$P_{C(\Gamma_G)}(\lambda) = \left(\lambda + \frac{1}{4}\right)^{\frac{3n-6}{2}} \left(\lambda + \frac{3}{4}\right)^{\frac{n}{2}-1} \left(\lambda^2 - \frac{3}{4}(n-2)\lambda - \frac{1}{16}\left(2n^2+n-9\right)\right).$$

Theorem 7. *The $C-$spectral radius for Γ_G, where $G = G_1 \cup G_2$, is*

(i) for n is odd: $\rho_C(\Gamma_G) = \frac{1}{8}\left(3n - 4 + \sqrt{n(17n-16)}\right)$,

(ii) for n is even: $\rho_C(\Gamma_G) = \frac{1}{8}\left(3n - 6 + \sqrt{n(17n-32)}\right)$.

Proof.

(i) According to Theorem 6 (1), for the odd n case gives four eigenvalues. They are $\lambda_1 = -\frac{1}{4}$ of multiplicity $(n-2)$, $\lambda_2 = -\frac{1}{2}$ of multiplicity $(n-1)$, and $\lambda_{3,4} = \frac{1}{8}\left(3n - 4 \pm \sqrt{n(17n-16)}\right)$. Hence, the spectrum of Γ_G as the following:

$$Spec_C(\Gamma_G) = \left\{ \left(\frac{1}{8}\left(3n-4+\sqrt{n(17n-16)}\right)\right)^1, \left(-\frac{1}{4}\right)^{n-2}, \left(-\frac{1}{2}\right)^{n-1}, \left(\frac{1}{8}\left(3n-4-\sqrt{n(17n-16)}\right)\right)^1 \right\}.$$

We take the maximum absolute eigenvalues and get the spectral radius of Γ_G as the desired result.

(ii) For n is even and following Theorem 6 (2) implies that Γ_G has four eigenvalues. They are $\lambda_1 = -\frac{1}{4}$ of multiplicity $n - 3 + \frac{n}{2}$, $\lambda_2 = -\frac{3}{4}$ of multiplicity $\frac{n}{2} - 1$ and $\lambda_{3,4} = \frac{1}{8}\left(3n - 6 \pm \sqrt{n(17n-32)}\right)$. Hence, the spectrum of Γ_G as the following:

$$Spec_C(\Gamma_G) = \left\{ \left(\frac{1}{8}\left(3n-6+\sqrt{n(17n-32)}\right)\right)^1, \left(-\frac{1}{4}\right)^{n-3+\frac{n}{2}}, \left(-\frac{3}{4}\right)^{\frac{n}{2}-1}, \left(\frac{1}{8}\left(3n-6-\sqrt{n(17n-32)}\right)\right)^1 \right\}.$$

The maximum of $|\lambda_i|$, $i = 1, 2, 3, 4$ is the $C-$spectral radius of Γ_G, and we complete the proof.

Theorem 8. *The $C-$energy for Γ_G, where $G = G_1 \cup G_2$, is*

(i) for n is odd: $E_C(\Gamma_G) = \frac{1}{4}\left(3n - 4 + \sqrt{n(17n-16)}\right)$

(ii) for n is even: $E_C(\Gamma_G) = \frac{1}{4}\left(3n - 6 + \sqrt{n(17n-32)}\right)$.

Proof.

(i) By Theorem 7 (1), for the odd n, the $C-$energy of Γ_G can be calculated as follows:

$$\begin{aligned} E_C(\Gamma_G) &= (n-2)\left|-\frac{1}{4}\right| + (n-1)\left|-\frac{1}{2}\right| + \left|\frac{1}{8}\left(3n-4 \pm \sqrt{n(17n-16)}\right)\right| \\ &= \frac{1}{4}\left(3n-4+\sqrt{n(17n-16)}\right). \end{aligned}$$

(ii) For even n, by Theorem 7 (2), then the $C-$energy of Γ_G is

$$\begin{aligned} E_C(\Gamma_G) &= \left(\frac{3n-6}{2}\right)\left|-\frac{1}{4}\right| + \left(\frac{n}{2}-1\right)\left|-\frac{3}{4}\right| + \left|\frac{1}{8}\left(3n-6 \pm \sqrt{n(17n-32)}\right)\right| \\ &= \frac{1}{4}\left(3n-6+\sqrt{n(17n-32)}\right). \end{aligned}$$

As a result of Theorem 8, in the following, we obtain the classification of the closeness energy of Γ_G for D_{2n}, where $G = G_1 \cup G_2$.

Corollary 1. Γ_G *associated with the closeness matrix is hypoenergetic.*

Moreover, based on the energies in Theorem 8, we can conclude the following fact:

Corollary 2. *$C-$energy for Γ_G is never an odd integer.*

The statements in Corollary 2 comply with the well-known facts from [4] and [12]. Furthermore, the comparison between energy in Theorem 8 and its spectral radius in Theorem 7 can be determined in the following statement:

Corollary 3. *$C-$energy for Γ_G is always twice its spectral radius.*

As a future view of this research, we recommend combining them with [2], which is essentially an extension of the graph matrix based on Q-NSS matrix. In addition, this work can be extended to the neutrosophic soft rings and neutrosophic soft field [14, 15].

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References

[1] A Abdollahi, S Akbari, and H R Maimani. Non-commuting graph of a group. *Journal of Algebra*, 298(2):468–492, 2006.

[2] F Al-Sharqi, M U Romdhini, and A Alquran. Group decision-making based on aggregation operator and score function of Q-neutrosophic soft matrix. *Journal of Intelligent and Fuzzy Systems*, 45:305–321, 2023.

[3] M Aschbacher. *Finite Group Theory.* Cambridge University Press, Cambridge, 2000.

[4] R B Bapat and S Pati. Energy of a Graph Is Never an Odd Integer. *Bulletin of Kerala Mathematics Association*, 1:129–132, 2004.

[5] A E Brouwer and W H Haemers. *Spectra of graphs.* Springer, New York, 2011.

[6] I Gutman. The Energy of Graph. *Ber. Math.-Stat. Sekt. Forschungsz. Graz*, 103:1–2, 1978.

[7] G Indulal, I Gutman, and A Vijayakumar. On Distance Energy of Graphs. *MATCH Communications in Mathematical and in Computer Chemistry*, 60:461–472, 2008.

[8] S R Jog and J R Gurjar. Degree Product Distance Energy of Some Graphs. *Asian Journal of Mathematics*, 24(1):42–49, 2018.

[9] S R Jog and J R Gurjar. Degree Sum Exponent Distance Energy of Some Graphs. *Journal of The Indonesian Mathematical Society*, 27(1):67–74, 2021.

[10] S M S Khasraw, I D Ali, and R R Haji. On the non-commuting graph of dihedral group. *Electronic Journal of Graph Theory and Application*, 8(2):233–239, 2020.

[11] X Li, Y Shi, and I Gutman. *Graph Energy.* Springer, New York, 2012.

[12] S Pirzada and I Gutman. Energy of a Graph Is Never the Square Root of an Odd Integer. *Applicable Analysis and Discrete Mathematics*, 2:118–121, 2008.

[13] H S Ramane and S S Shinde. Degree exponent polynomial of graphs obtained by some graph operations. *Electronic Journal of Graph Theory and Application*, 63:161–168, 2017.

[14] M U Romdhini, A Al-Quran, F Al-Sharqi, H Hashim, and A Lutfi. Unveiling an Innovative Approach to Q-Complex Neutrosophic Soft Rings. *International Journal of Neutrosophic Science*, 22(04):20–35, 2023.

[15] M U Romdhini, A Al-Quran, F Al-Sharqi, M K Tahat, and A Lutfi. Exploring the Algebraic Structures of Q-Complex Neutrosophic SoftFields. *International Journal of Neutrosophic Science*, 22(04):93–105, 2023.

[16] M U Romdhini, F Al-Sharqi, A Nawawi, A Al-Quran, and H Rashmanlou. Signless Laplacian energy of interval-valued fuzzy graph and its applications. *Sains Malaysiana*, 52(7):2127–2137, 2023.

[17] M U Romdhini and A Nawawi. Degree Sum Energy of Non-Commuting Graph for Dihedral Groups. *Malaysian Journal of Science*, 41(sp1):34–39, 2022.

[18] M U Romdhini and A Nawawi. Maximum and minimum degree energy of commuting graph for dihedral groups. *Sains Malaysiana*, 51(12):4145–4151, 2022.

[19] M U Romdhini and A Nawawi. Degree subtraction energy of commuting and non-commuting graphs for dihedral groups. *Journal of Mathematical and Computational Science*, 18(3):497–508, 2023.

[20] M U Romdhini and A Nawawi. Distance energy of non-commuting graphs for dihedral groups. *Jordan Journal of Mathematics and Statestics*, page submitted, 2023.

[21] M U Romdhini, A Nawawi, and C Y Chen. Degree Exponent Sum Energy of Commuting Graph for Dihedral Groups. *Malaysian Journal of Science*, 41(sp1):40–46, 2022.

[22] M U Romdhini, A Nawawi, and C Y Chen. Neighbors degree sum energy of commuting and non-commuting graphs for dihedral groups. *Malaysian Journal of Mathematical Sciences*, 17(1):53–65, 2023.

[23] S Sun, K C Das, and Y Shang. On Maximal Energy Distance. *Mathematics*, 9(360):1–7, 2021.

[24] L Zheng and B Zhou. The closeness eigenvalues of graphs. *Journal of Algebraic Combinatorics*, 58:741–760, 2023.