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A Fuzzy Semibipolar Soft Filter and Its Association with Green's Relation $\mathcal{N}$

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Abstract. This paper introduces the concept of fuzzy semibipolar soft filters in ordered groupoids. It is defined in the form of fuzzy semibipolar soft sets over universal sets. Then, a corresponding example is proposed. At this point, a necessary and sufficient condition for fuzzy semibipolar soft filters is provided. Finally, Green's relation $\mathcal{N}$ on ordered groupoids is described in terms of fuzzy semibipolar soft filters.

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Key Words and Phrases: Green's relation, Ordered groupoid, Fuzzy filter, Soft set

1. Introduction and Earlier Works

The fuzzy set theory was introduced by Zadeh [27] in 1965. Zadeh found that the traditional crisp set is not capable of explaining the whole thing realistically. He proposed fuzzy sets to solve this problem. Thus, he presented a classical fuzzy set as a function from a universe to the unit interval. Based on this point, the fuzzy set theory is regarded as an effective mathematical approach to algebraic structures. The concept of fuzzy groups was first introduced by Rosenfeld [25] in 1971. At this point, the fuzzy sets-based ideal theory was also proposed. According to the definition of Zadeh's fuzzy set theory, Rosenfeld was the first who consider the case when a domain of functions is a groupoid. Moreover, he introduced the idea of fuzzy subgroupoids and fuzzy left (resp., right and two-sided) ideals. In 1981, Kuroki [17] first proposed the notion of fuzzy ideals in semigroups. In particular, a systematic exposition of fuzzy semigroups by Mordeson et al. appeared in [22]. In structural development, fuzzy ordered groupoids and ordered semigroups were proposed

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by Kehayopulu and Tsingelis [11, 12]. Khan et al. [14] gave the idea of fuzzy ideals in ordered semigroups. Furthermore, other algebraic structures are considered under the concept of fuzzy sets, continuously. When the focus is on the context of Geen's relations, the combination of fuzzy ideals and Green's relations in semigroups was studied by McLean and Kummer [20] in 1992. Based on this interesting point, Kehayopulu and Tsingelis [12] presented the notion of fuzzy filters in ordered semigroups. Then, some authors used such a concept to describe Green's relations. The concept of fuzzy filters for various algebraic structures has been studied by many authors. In 2007, Green's relation $\mathcal{N}$ was characterized in terms of fuzzy filters of ordered groupoids by Kehayopulu [13]. In 2012, Green's relation $\mathcal{N}$ was characterized in terms of fuzzy filters of ordered gamma-semigroups by Iampan and Siripitukdet [7]. In 2013, Green's relation $\mathcal{N}$ was characterized in terms of fuzzy filters of ordered gamma-groupoids by Iampan and Siripitukdet [8]. As reviewed above, the concept of Green's relations plays an important role in studying the structure of semigroups. Based on this point, it is considered in the sense of computer science. In 2011, the notion of Green's relations and their use in automata theory was introduced by Colcombet [2]. In 2019, the concept of Green's relations in deterministic finite automata was proposed by Fleischer and Kufleitner [5]. In 2022, the notion of Green's relations in L-E-fuzzy skew lattices was presented by Zhi et al. [28].

In 1999, Molodtsov [21] initiated the concept of soft set theory as a tuple that is associated with a set of parameters and a function from a parameter set to the power set of a universal set. The major advantage of soft set theory is that it does need not to bother with any additional information about the data such as probability in statistics or possibility value in fuzzy set theory. In other words, the soft set theory can be considered as an extended notion of fuzzy set theory. The research of the theory for combining the soft set with other mathematical theories has been developed by many authors. This literature is contained in the review on soft set-based parameter reduction and decision-making [3]. In 2001, Maji et al. [19] proposed the notion of fuzzy soft set theory in terms of soft set theory. In this way, a power set of a universe is replaced by a set of all fuzzy subsets of a universe. The application area of this such as the decision support system [15] and the real-life problem: classification of wood materials to prevent fire-related injuries and deaths [16]. As mentioned before, up to the present, there has been much practical application of fuzzy soft set theory, especially the use of the fuzzy soft set to solve the decision-making problem. Then, the context of bipolarity attracts some researchers for continuous development. Shabir and Naz [26] proposed the notion of bipolar soft set theory. This notion is generated by the hybridization of the bipolarity concept of Dubois and Prade [4] and soft set theory. They have studied the notion of the bipolarity of information in terms of soft sets. Bipolar soft set theory is described by two soft sets, one of which provides positive information and the other provides negative information. Therefore, in 2014, Naz and Shabir [23] proposed the notion of fuzzy bipolar soft sets based on fuzzy soft set theory and bipolar soft set theory. Fuzzy bipolar soft set theory has the potential to handle the bipolarity of the information about some objects with the help of two functions. One function handles the positivity of the information, while the other function measures the negativity. To support solving COVID-19, Ali et al. [1]

presented the issue of ranking the effectiveness of COVID-19 tests via fuzzy bipolar soft expert sets. Furthermore, they verified a comparative analysis of fuzzy soft expert sets and fuzzy bipolar soft sets. To solve uncertainty in algebraic structures, Hakim et al. [6] introduced the notion of fuzzy bipolar soft prime ideal theory in ordered semigroups. Recently, the concept of fuzzy semibipolar soft sets was developed as one of the important dimensions of fuzzy bipolar soft sets. Such an idea was proposed by Prasertpong [24]. In this way, a fuzzy semibipolar soft set is generated by two fuzzy soft sets induced by the same parameter set. Then, for each parameter element of a fuzzy semibipolar soft set, there is both positive and negative information. A single parameter related to two-way information is a prominent point of this concept in which the notion of fuzzy bipolar soft sets has no such rule. At this point, Prasertpong proposed Green's relations $\mathcal{L}$ and $\mathcal{R}$ defined by fuzzy semibipolar soft left ideals and fuzzy semibipolar soft right ideals, respectively. As a developing point of filters based on bipolarity contexts, in this paper, we shall introduce the concept of filters in terms of fuzzy semibipolar soft sets, namely, fuzzy semibipolar soft filters. Afterward, Green's relation $\mathcal{N}$ will be considered via fuzzy semibipolar soft filters. In other words, Green's relation $\mathcal{N}$ will be described by the rule of fuzzy semibipolar soft filters. To achieve this, we shall recall the connection between filter classes and Green's relation $\mathcal{N}$ together with mathematical tools for the fuzzy set theory-based descriptions of these.

Throughout this paper, U denotes a non-empty universal set. f is said to be a fuzzy subset of U if it is a function from U to the closed unit interval $[0, 1]$ [27]. Throughout this paper, $\mathcal{F}(U)$ denotes a collection of all fuzzy subsets of U. In this way, 1_U is denoted as a fuzzy subset of U defined by $1_U(u) = 1$ for all $u \in U$, and 0_U is denoted as a fuzzy subset of U defined by $0_U(u) = 0$ for all $u \in U$ [27]. Obviously, 1_U is the greatest element of $\mathcal{F}(U)$, and 0_U is the least element of $\mathcal{F}(U)$ [27]. For $f, g \in \mathcal{F}(U)$, the notation $f\widetilde{\wedge}g$ (resp., $f\widetilde{\vee}g$ and $f\widetilde{+}g$) is denoted as the fuzzy subset of U given by

$$(f\widetilde{\wedge}g)(u) = \min\{f(u), g(u)\}$$

$$\text{(resp., } (f\widetilde{\vee}g)(u) = \max\{f(u), g(u)\} \text{ and } (f\widetilde{+}g)(u) = f(u) + g(u))$$

for all $u \in U$. For $f, g \in \mathcal{F}(U)$, $f\widetilde{\leq}g$ is denoted by meaning $f(u) \leq g(u)$ for all $u \in U$ [27]. At this point, the statement $f\widetilde{\geq}g$ means $g\widetilde{\leq}f$. Let $\{f_i : i \in I\}$ be a non-empty collection of all fuzzy subsets of U. Define

$$\widetilde{\bigwedge}_{i\in I} f_i : U \to [0,1] | u \mapsto (\widetilde{\bigwedge}_{i\in I} f_i)(u) := \inf\{f_i(u) : i \in I\}$$

and

$$\widetilde{\bigvee}_{i\in I} f_i : U \to [0,1] | u \mapsto (\widetilde{\bigvee}_{i\in I} f_i)(u) := \sup\{f_i(u) : i \in I\}.$$

Then $\widetilde{\bigwedge}_{i\in I} f_i, \widetilde{\bigvee}_{i\in I} f_i \in \mathcal{F}(U)$ [13]. In addition, it is true that

$$\widetilde{\bigwedge}_{i\in I} f_i = \inf\{f_i : i \in I\} \text{ and } \widetilde{\bigvee}_{i\in I} f_i = \sup\{f_i : i \in I\}[13].$$

In the following, $\mathcal{P}(U)$ denotes a collection of subsets of U. V denotes a non-empty universal set. Let X be a non-empty subset of V. If F is a function from X to $\mathcal{P}(U)$, then (F, X) is said to be a soft set over U with respect to X. As the understanding of the soft set, U is said to be a universe of all alternative objects of (F, X), and V is said to be a set of all parameters of (F, X), where parameters are attributes, characteristics or statements of alternative objects in U. For any element $x \in X$, $F(x)$ is considered as a set of x-approximate elements (or x-alternative objects) of (F, X) [21].

Throughout this work, X and Y are denoted as two non-empty subsets of V.

Definition 1. *[24] The triple notation $(\mathbb{F}, \neg\mathbb{F}, X)$ is called a fuzzy semibipolar soft set (briefly, FSSS) over U with respect to X if $\mathbb{F} : X \to \mathcal{F}(U)$ and $\neg\mathbb{F} : X \to \mathcal{F}(U)$ are disjoint functions such that*

$$\mathbb{F}(x)\widetilde{+}\neg\mathbb{F}(x) = 1_U$$

for all $x \in X$.

Definition 2. *[24] Let $\mathfrak{F} := (\mathbb{F}, \neg\mathbb{F}, X)$ and $\mathfrak{G} := (\mathbb{G}, \neg\mathbb{G}, Y)$ be FSSSs over U with respect to X and Y, respectively. $\mathfrak{F}$ is a fuzzy semibipolar soft subset of $\mathfrak{G}$, denoted by $\mathfrak{F}\widetilde{\subseteq}\mathfrak{G}$, if $X \subseteq Y$ and $\mathbb{F}(x)\widetilde{\leq}\mathbb{G}(x)$ and $\neg\mathbb{F}(x)\widetilde{\geq}\neg\mathbb{G}(x)$ for all $x \in X$. At this point, we say that $\mathfrak{G}$ is a fuzzy semibipolar soft superset of $\mathfrak{F}$. We write $\mathfrak{G}\widetilde{\supseteq}\mathfrak{F}$. Furthermore, $\mathfrak{F}$ is equal to $\mathfrak{G}$ if $\mathfrak{F}\widetilde{\subseteq}\mathfrak{G}$ and $\mathfrak{F}\widetilde{\supseteq}\mathfrak{G}$.*

Definition 3. *[24] If $(\mathbb{F}_A, \neg\mathbb{F}_A, X)$ is a given FSSS over U with respect to X defined by $\mathbb{F}(x) = 1_U$ and $\neg\mathbb{F}(x) = 0_U$ for all $x \in X$, then it is called a relative whole FSSS over U with respect to X. In the following, we use the notation $\mathfrak{W}_X := (\mathbb{W}_X, \neg\mathbb{W}_X, X)$ instead of a relative whole FSSS over U with respect to X.*

Proposition 1. *[24] Let $\{(\mathbb{F}_i, \neg\mathbb{F}_i, X) : i \in I\}$ be a non-empty set of all FSSSs over U with respect to X. Define*

$$\widetilde{\bigcap}_{i\in I}\mathbb{F}_i : X \to \mathcal{F}(U)|x \mapsto (\widetilde{\bigcap}_{i\in I}\mathbb{F}_i)(x) := \widetilde{\bigwedge}_{i\in I}\mathbb{F}_i(x)$$

and

$$\widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i : X \to \mathcal{F}(U)|x \mapsto (\widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i)(x) := \widetilde{\bigvee}_{i\in I}\neg\mathbb{F}_i(x).$$

Then $(\widetilde{\bigcap}_{i\in I}\mathbb{F}_i, \widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i, X)$ belongs to the collection of all FSSSs over U.

Remark 1. *[24] According to Proposition 1, it is true that*

$$(\widetilde{\bigcap}_{i\in I}\mathbb{F}_i, \widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i, X)\widetilde{\subseteq}(\mathbb{F}_j, \neg\mathbb{F}_j, X)$$

for every $j \in I$.

Proposition 2. *[24] Let $\{(\mathbb{F}_i, \neg\mathbb{F}_i, X) : i \in I\}$ be a non-empty set of all FSSSs over U with respect to X. Then*

$$(\widetilde{\bigcap}_{i\in I}\mathbb{F}_i, \widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i, X) = (\inf_X\{\mathbb{F}_i : i \in I\}, \sup_X\{\neg\mathbb{F}_i : i \in I\}, X).$$

Definition 4. *[24] Let $A \subseteq X$ be given. If $(\mathbb{F}_A, \neg\mathbb{F}_A, X)$ is a FSSS over U with respect to X defined by*

$$\mathbb{F}_A(x) = \begin{cases} 1_U & \textit{if}\, x \in A, \\ 0_U & \textit{if}\, x \notin A, \end{cases} \textit{ and } \neg\mathbb{F}_A(x) = \begin{cases} 0_U & \textit{if}\, x \in A, \\ 1_U & \textit{if}\, x \notin A \end{cases}$$

for all $x \in X$, then it is called a FSSS over U concerning A. In the specificity, if $\mathfrak{F} := (\mathbb{F}, \neg\mathbb{F}, X)$ is any FSSS over U with respect to X and a fixed element $x \in X$ such that $\mathbb{F}(x) = 1_U$ and $\neg\mathbb{F}(x) = 0_U$, then we say that $\mathfrak{F}$ contains x.

Remark 2. *[24] According to Definition 4, it is true that for any subsets A and B of X, $A \subseteq B$ if and only if $(\mathbb{F}_A, \neg\mathbb{F}_A, X)\widetilde{\subseteq}(\mathbb{F}_B, \neg\mathbb{F}_B, X)$. In addition, an equality case is also true.*

Recall that a groupoid is a non-empty set V together with a binary operation $*$ on V, denoted by $(V, *)$. Generally, if $(V, *)$ is a groupoid, then $x * y$ is denoted by xy for all $x, y \in V$. Given two non-empty subsets X and Y of a groupoid $(V, *)$, the product $X * Y$ (simply XY) is defined by $XY = \{xy : x \in X \text{ and } y \in Y\}$. Let $\leq_V$ be a given binary relation on V. An ordered groupoid, denoted by $(V, *, \leq_V)$, is a groupoid $(V, *)$ whose elements of V are partially ordered by $\leq_V$ satisfying with the property that for all $x, y, z \in V, x \leq_V y$ implies $xz \leq_V yz$ and $zx \leq_V zy$. We usually write simply V instead of $(V, *, \leq_V)$. Recall that an ordered semigroup is defined as an ordered associative groupoid.

To benefit for the characterization of Green's relations, we shall review some concepts in an ordered groupoid V as follows. In 1987, Kehayopulu [9] was the first to verify filters. A non-empty subset X of V is called a filter of V if

(i) for all $x, y \in V, xy \in X$ implies $x \in V$ and $y \in V$;

(ii) for all $x, y \in V, x \in X$ and $x \leq_V y \in V$ imply $y \in X$.

For each $x \in V$, $N(x)$ is denoted as a filter of V generated by x. The Green's relation $\mathcal{N}$ on U in [10] is defined as

$$\mathcal{N} := \{(x, y) \in V \times V : N(x) = N(y)\}.$$

As an existential classical relation of Green's relation $\mathcal{N}$ above, this paper contains novel knowledge based on fuzzy semibipolar soft set theory as follows.

- The concept of fuzzy semibipolar soft filters in ordered groupoids is introduced. A corresponding example is proposed. A necessary and sufficient condition for fuzzy semibipolar soft filters is examined.
- Green's relation $\mathcal{N}$ on ordered groupoids is described in terms of fuzzy semibipolar soft filters.

Finally, the work is summarized.

2. Main Results

In this section, we use the previous fundamental notion to study the characterization of a novel filters-based Green's relation $\mathcal{N}$ on ordered groupoids. To achieve the goal, in the starting point, we construct the concept of filters in terms of fuzzy semibipolar soft sets as follows.

Definition 5. *Let* $(X, *, \leq_X)$ *be an ordered groupoid and* $\mathfrak{F} := (\mathbb{F}, \neg\mathbb{F}, X)$ *a FSSS over* U *with respect to* X*.*

(*i*) $\mathfrak{F}$ *is called a fuzzy semibipolar soft subgroupoid if it satisfies*

- $\mathbb{F}(xy)\widetilde{\geq}\mathbb{F}(x)\widetilde{\wedge}\mathbb{F}(y)$ *and* $\neg\mathbb{F}(xy)\widetilde{\leq}\neg\mathbb{F}(x)\widetilde{\vee}\neg\mathbb{F}(y)$ *for all* $x, y \in X$.

(*ii*) $\mathfrak{F}$ *is called a fuzzy semibipolar soft filter (briefly, FSSF) if it satisfies two conditions below.*

- $\mathbb{F}(xy) = \mathbb{F}(x)\widetilde{\wedge}\mathbb{F}(y)$ *and* $\neg\mathbb{F}(xy) = \neg\mathbb{F}(x)\widetilde{\vee}\neg\mathbb{F}(y)$ *for all* $x, y \in X$.
- *For any* $x, y \in X$, $x \leq_X y$ *implies* $\mathbb{F}(x)\widetilde{\leq}\mathbb{F}(y)$ *and* $\neg\mathbb{F}(x)\widetilde{\geq}\neg\mathbb{F}(y)$.

Example 1. *Let* $U := \{u : u \text{ is a natural number and } 2024 \leq u \leq 2033\}$ *be given, and let* $\mathcal{F}$ *be a family of subsets of* U *defined by the set*

$$\{P := \{u \in U : u \text{ is a prime number}\}, C := \{u \in U : u \text{ is a composite number}\}\}.$$

Then, it is clear that $\mathcal{F}$ *is a partition of* U*. Define two fuzzy subsets* α *and* $\neg\alpha$ *of* U *by*

$$\alpha(u) = \begin{cases} 0.8 & \text{if } u \in C, \\ 0.2 & \text{if } u \in P, \end{cases} \text{ and } \neg\alpha(u) = \begin{cases} 0.2 & \text{if } u \in C, \\ 0.8 & \text{if } u \in P \end{cases}$$

for all $u \in U$*. Let* $V := \{x_i : i \text{ is a natural number}\}$*, and let* $X := \{x_1, x_2, x_3, x_4\} \subseteq V$ *be a set whose elements of* X *are partially ordered by* $\leq_X$ *satisfying*

$$\leq_X := \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_4, x_4), (x_1, x_2), (x_3, x_4), (x_4, x_3)\}.$$

Define a binary operation $*$ *on* X *by multiplication rules as Table 1 below.*

Table 1: The table of multiplication rules on X

$*$	x_1	x_2	x_3	x_4
x_1	x_1	x_1	x_3	x_4
x_2	x_1	x_2	x_3	x_4
x_3	x_3	x_4	x_3	x_4
x_4	x_3	x_4	x_4	x_3

Then, it is routine to verify that $(X, *, \leq_X)$ is an ordered groupoid. Define a function $\mathbb{F} : X \to \mathcal{F}(U)$ by

$$\mathbb{F}(x) = \alpha \text{ and } \mathbb{F}(y) = \neg\alpha$$

for all $x \in \{x_1, x_2\}, y \in \{x_3, x_4\}$. Define a function $\neg\mathbb{F} : X \to \mathcal{F}(U)$ by

$$\neg\mathbb{F}(x) = \neg\alpha \text{ and } \neg\mathbb{F}(y) = \alpha$$

for all $x \in \{x_1, x_2\}, y \in \{x_3, x_4\}$. Then, we compute that

$$\mathbb{F}(x)\widetilde{+}\neg\mathbb{F}(x) = 1_U$$

for all $x \in X$. Hence $(\mathbb{F}, \neg\mathbb{F}, X)$ is a FSSS over U. We verify that

$$\mathbb{F}(xy) = \mathbb{F}(x)\widetilde{\wedge}\mathbb{F}(y) \text{ and } \neg\mathbb{F}(xy) = \neg\mathbb{F}(x)\widetilde{\vee}\neg\mathbb{F}(y)$$

for all $x, y \in X$. In addition, for any $x, y \in X$, $x \leq_X y$ implies $\mathbb{F}(x)\widetilde{\leq}\mathbb{F}(y)$ and $\neg\mathbb{F}(x)\widetilde{\geq}\neg\mathbb{F}(y)$. Therefore $(\mathbb{F}, \neg\mathbb{F}, X)$ is a FSSF. This is provided according to Definition 5 above.

Proposition 3. *Let $(X, *, \leq_X)$ be an ordered groupoid, and let $\{(\mathbb{F}_i, \neg\mathbb{F}_i, X) : i \in I\}$ be a non-empty collection of all FSSSs over U with respect to X. If $(\mathbb{F}_i, \neg\mathbb{F}_i, X)$ is a fuzzy semibipolar soft subgroupoid (resp., a FSSF) for all $i \in I$, then $(\widetilde{\bigcap}_{i\in I}\mathbb{F}_i, \widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i, X)$ is a fuzzy semibipolar soft subgroupoid (resp., a FSSF).*

Proof. Assume that $(\mathbb{F}_i, \neg\mathbb{F}_i, X)$ is a fuzzy semibipolar soft subgroupoid for all $i \in I$. Let $x, y \in X$. Then $\mathbb{F}_i(xy)\widetilde{\geq}\mathbb{F}_i(x)\widetilde{\wedge}\mathbb{F}_i(y)$ and $\neg\mathbb{F}_i(xy)\widetilde{\leq}\neg\mathbb{F}_i(x)\widetilde{\vee}\neg\mathbb{F}_i(y)$ for all $i \in I$. Thus

$$\mathbb{F}_i(xy)\widetilde{\geq}\mathbb{F}_i(x)\widetilde{\geq}\inf\{\mathbb{F}_i(x) : i \in I\} =: (\widetilde{\bigcap}_{i\in I}\mathbb{F}_i)(x)$$

or

$$\mathbb{F}_i(xy)\widetilde{\geq}\mathbb{F}_i(y)\widetilde{\geq}\inf\{\mathbb{F}_i(y) : i \in I\} =: (\widetilde{\bigcap}_{i\in I}\mathbb{F}_i)(y)$$

and

$$\neg\mathbb{F}_i(xy)\widetilde{\leq}\neg\mathbb{F}_i(x)\widetilde{\leq}\sup\{\neg\mathbb{F}_i(x) : i \in I\} =: (\widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i)(x)$$

or

$$\neg\mathbb{F}_i(xy)\widetilde{\leq}\neg\mathbb{F}_i(y)\widetilde{\leq}\sup\{\neg\mathbb{F}_i(y) : i \in I\} =: (\widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i)(y)$$

for all $i \in I$. For each $i \in I$,

$$\mathbb{F}_i(xy)\widetilde{\geq}(\widetilde{\bigcap}_{i\in I}\mathbb{F}_i)(x)\widetilde{\wedge}(\widetilde{\bigcap}_{i\in I}\mathbb{F}_i)(y) \text{ and } \neg\mathbb{F}_i(xy)\widetilde{\leq}(\widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i)(x)\widetilde{\vee}(\widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i)(y).$$

Hence

$$(\widetilde{\bigcap}_{i\in I}\mathbb{F}_i)(xy) := \inf\{\mathbb{F}_i(xy) : i \in I\}\widetilde{\geq}(\widetilde{\bigcap}_{i\in I}\mathbb{F}_i)(x)\widetilde{\wedge}(\widetilde{\bigcap}_{i\in I}\mathbb{F}_i)(y)$$

and

$$(\widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i)(xy) := \sup\{\neg\mathbb{F}_i(xy) : i \in I\}\widetilde{\leq}(\widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i)(x)\widetilde{\vee}(\widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i)(y).$$

This implies that $(\widetilde{\bigcap}_{i\in I}\mathbb{F}_i, \widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i, X)$ is a fuzzy semibipolar soft subgroupoid.

Assume $(\mathbb{F}_i, \neg\mathbb{F}_i, X)$ is a FSSF for all $i \in I$. Let $x, y \in X$. Then

$$\mathbb{F}_i(xy) = \mathbb{F}_i(x)\widetilde{\wedge}\mathbb{F}_i(y) \text{ and } \neg\mathbb{F}_i(xy) = \neg\mathbb{F}_i(x)\widetilde{\vee}\neg\mathbb{F}_i(y)$$

for all $i \in I$. Hence

$$\mathbb{F}_i(xy)\widetilde{\leq}\mathbb{F}_i(x), \mathbb{F}_i(xy)\widetilde{\leq}\mathbb{F}_i(y), \neg\mathbb{F}_i(xy)\widetilde{\geq}\neg\mathbb{F}_i(x), \text{and } \neg\mathbb{F}_i(xy)\widetilde{\geq}\neg\mathbb{F}_i(y)$$

for all $i \in I$. Whence

$$(\widetilde{\bigcap}_{i\in I}\mathbb{F}_i)(xy) := \widetilde{\bigcap}\{\mathbb{F}_i(xy)|i \in I\}\widetilde{\leq}\widetilde{\bigcap}\{\mathbb{F}_i(x)|i \in I\} =: (\widetilde{\bigcap}_{i\in I}\mathbb{F}_i)(x),$$

$$(\widetilde{\bigcap}_{i\in I}\mathbb{F}_i)(xy) := \widetilde{\bigcap}\{\mathbb{F}_i(xy)|i \in I\}\widetilde{\leq}\widetilde{\bigcap}\{\mathbb{F}_i(y)|i \in I\} =: (\widetilde{\bigcap}_{i\in I}\mathbb{F}_i)(y),$$

$$(\widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i)(xy) := \widetilde{\bigcup}\{\neg\mathbb{F}_i(xy)|i \in I\}\widetilde{\geq}\widetilde{\bigcup}\{\neg\mathbb{F}_i(x)|i \in I\} =: (\widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i)(x),$$

$$(\widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i)(xy) := \widetilde{\bigcup}\{\neg\mathbb{F}_i(xy)|i \in I\}\widetilde{\geq}\widetilde{\bigcup}\{\neg\mathbb{F}_i(y)|i \in I\} =: (\widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i)(y).$$

Therefore

$$(\widetilde{\bigcap}_{i\in I}\mathbb{F}_i)(xy)\widetilde{\leq}(\widetilde{\bigcap}_{i\in I}\mathbb{F}_i)(x)\widetilde{\wedge}(\widetilde{\bigcap}_{i\in I}\mathbb{F}_i)(y)$$

and

$$(\widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i)(xy)\widetilde{\geq}(\widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i)(x)\widetilde{\vee}(\widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i)(y).$$

It is easy to verify that $(\widetilde{\bigcap}_{i\in I}\mathbb{F}_i, \widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i, X)$ is a fuzzy semibipolar soft subgroupoid. Thus

$$(\widetilde{\bigcap}_{i\in I}\mathbb{F}_i)(xy)\widetilde{\geq}(\widetilde{\bigcap}_{i\in I}\mathbb{F}_i)(x)\widetilde{\wedge}(\widetilde{\bigcap}_{i\in I}\mathbb{F}_i)(y)$$

and

$$(\widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i)(xy)\widetilde{\leq}(\widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i)(x)\widetilde{\vee}(\widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i)(y).$$

It follows that

$$(\widetilde{\bigcap}_{i\in I}\mathbb{F}_i)(xy) = (\widetilde{\bigcap}_{i\in I}\mathbb{F}_i)(x)\widetilde{\wedge}(\widetilde{\bigcap}_{i\in I}\mathbb{F}_i)(y)$$

and

$$(\widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i)(xy) = (\widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i)(x)\widetilde{\vee}(\widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i)(y).$$

Assume that $x \leq_X y$. Then

$$\mathbb{F}_i(x)\widetilde{\leq}\mathbb{F}_i(y) \text{ and } \neg\mathbb{F}_i(x)\widetilde{\geq}\neg\mathbb{F}_i(y)$$

for all $i \in I$. Thus

$$(\widetilde{\bigcap}_{i\in I}\mathbb{F}_i)(x) := \widetilde{\bigcap}\{\mathbb{F}_i(x)|i \in I\}\widetilde{\leq}\widetilde{\bigcap}\{\mathbb{F}_i(y)|i \in I\} =: (\widetilde{\bigcap}_{i\in I}\mathbb{F}_i)(y)$$

and

$$(\widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i)(x) := \widetilde{\bigcup}\{\neg\mathbb{F}_i(x)|i \in I\}\widetilde{\geq}\widetilde{\bigcup}\{\neg\mathbb{F}_i(y)|i \in I\} =: (\widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i)(y).$$

As a consequence, $(\widetilde{\bigcap}_{i\in I}\mathbb{F}_i, \widetilde{\bigcup}_{i\in I}\neg\mathbb{F}_i, X)$ is a FSSF.

Notation 1. *For an ordered groupoid $(X, *, \leq_X)$ and a FSSS $\mathfrak{F} := (\mathbb{F}, \neg\mathbb{F}, X)$ over U, we denote the notation*

$$N_{\mathfrak{F}} := \{\mathfrak{G} := (\mathbb{G}, \neg\mathbb{G}, X) : \mathfrak{G} \textit{ is a FSSF over } U \textit{ and } \mathfrak{F}\widetilde{\subseteq}\mathfrak{G}\}.$$

Remark 3. *As Notation 1 above, we observe that $(\mathbb{W}_X, \neg\mathbb{W}_X, X)$ belongs to $N_{\mathfrak{F}}$. Then $N_{\mathfrak{F}}$ is a non-empty subset of a non-empty collection of all FSSSs over U. By Proposition 1, we see that a FSSS $(\widetilde{\bigcap}\mathbb{G}, \widetilde{\bigcup}\neg\mathbb{G}, X)$ over U in which $(\mathbb{G}, \neg\mathbb{G}, X)$ belongs to $N_{\mathfrak{F}}$ exists in such non-empty collection of all FSSSs over U. Using Proposition 3, we obtain that the FSSS $(\widetilde{\bigcap}\mathbb{G}, \widetilde{\bigcup}\neg\mathbb{G}, X)$ over U in which $(\mathbb{G}, \neg\mathbb{G}, X)$ belongs to $N_{\mathfrak{F}}$ is a FSSF.*

Proposition 4. *Let $(X, *, \leq_X)$ be a given ordered groupoid. Let $\mathfrak{F} := (\mathbb{F}, \neg\mathbb{F}, X)$ be a given FSSS over U. Then, a FSSS $(\widetilde{\bigcap}\mathbb{G}, \widetilde{\bigcup}\neg\mathbb{G}, X)$ over U in which $(\mathbb{G}, \neg\mathbb{G}, X)$ belongs to $N_{\mathfrak{F}}$ is an element of $N_{\mathfrak{F}}$.*

Proof. By Remark 3, we have a FSSS $(\widetilde{\bigcap}\mathbb{G}, \widetilde{\bigcup}\neg\mathbb{G}, X)$ over U in which $(\mathbb{G}, \neg\mathbb{G}, X)$ belongs to $N_{\mathfrak{F}}$ is a FSSF. Suppose $(\mathbb{G}, \neg\mathbb{G}, X) \in N_{\mathfrak{F}}$. Then $\mathfrak{F}\widetilde{\subseteq}(\mathbb{G}, \neg\mathbb{G}, X)$. Let $x \in X$. Then $\mathbb{F}(x)\widetilde{\leq}G(x)$ and $\neg\mathbb{F}(x)\widetilde{\geq}\neg G(x)$. Hence

$$\mathbb{F}(x)\widetilde{\leq}\inf\{G(x)\} =: (\widetilde{\bigcap}G)(x) \text{ and } \neg\mathbb{F}(x)\widetilde{\geq}\sup\{\neg G(x)\} =: (\widetilde{\bigcup}\neg G)(x).$$

It follows that $\mathfrak{F}\widetilde{\subseteq}(\widetilde{\bigcap}\mathbb{G}, \widetilde{\bigcup}\neg\mathbb{G}, X)$. This means that $(\widetilde{\bigcap}\mathbb{G}, \widetilde{\bigcup}\neg\mathbb{G}, X) \in N_{\mathfrak{F}}$.

Proposition 5. *Let $(X, *, \leq_X)$ be a given ordered groupoid. Let $\mathfrak{F} := (\mathbb{F}, \neg\mathbb{F}, X)$ be a given FSSS over U. Then, a FSSS $(\widetilde{\bigcap}\mathbb{G}, \widetilde{\bigcup}\neg\mathbb{G}, X)$ over U in which $(\mathbb{G}, \neg\mathbb{G}, X)$ belongs to $N_{\mathfrak{F}}$ is a fuzzy semibipolar soft subset of $(\mathbb{H}, \neg\mathbb{H}, X)$ for every $(\mathbb{H}, \neg\mathbb{H}, X) \in N_{\mathfrak{F}}$.*

Proof. From Remark 1, it follows that the statement holds.

Definition 6. *Let $(X, *, \leq_X)$ be an ordered groupoid and $\mathfrak{F} := (\mathbb{F}, \neg\mathbb{F}, X)$ a FSSS over U. A FSSS $(\mathbb{G}, \neg\mathbb{G}, X)$ over U is called a FSSF generated by $\mathfrak{F}$ if $(\mathbb{G}, \neg\mathbb{G}, X) \in N_{\mathfrak{F}}$ and $(\mathbb{G}, \neg\mathbb{G}, X)\widetilde{\subseteq}(\mathbb{H}, \neg\mathbb{H}, X)$ for all $(\mathbb{H}, \neg\mathbb{H}, X) \in N_{\mathfrak{F}}$.*

Remark 4. *According to Propositions 4 and 5, it is easy to see that a FSSS $(\widetilde{\bigcap}\mathbb{G}, \widetilde{\bigcup}\neg\mathbb{G}, X)$ over U in which $(\mathbb{G}, \neg\mathbb{G}, X)$ belongs to $N_{\mathfrak{F}}$ is a FSSF generated by $\mathfrak{F}$.*

Proposition 6. *Let $(X, *, \leq_X)$ be an ordered groupoid and $\mathfrak{F} := (\mathbb{F}, \neg\mathbb{F}, X)$ a FSSS over U. If $(\mathbb{G}, \neg\mathbb{G}, X)$ is a FSSF over U generated by $\mathfrak{F}$, then a FSSS $(\widetilde{\bigcap}\mathbb{H}, \widetilde{\bigcup}\neg\mathbb{H}, X)$ over U in which $(\mathbb{H}, \neg\mathbb{H}, X)$ belongs to $N_{\mathfrak{F}}$ is equal to $(\mathbb{G}, \neg\mathbb{G}, X)$.*

Proof. Suppose $(\mathbb{G}, \neg\mathbb{G}, X)$ is a FSSF over U generated by $\mathfrak{F}$. Then $(\mathbb{G}, \neg\mathbb{G}, X) \in N_{\mathfrak{F}}$. By Remark 1, we get that a FSSS $(\widetilde{\bigcap}\mathbb{H}, \widetilde{\bigcup}\neg\mathbb{H}, X)$ over U in which $(\mathbb{H}, \neg\mathbb{H}, X)$ belongs to $N_{\mathfrak{F}}$ is a fuzzy semibipolar soft subset of $(\mathbb{G}, \neg\mathbb{G}, X)$. From Proposition 4, it follows that the FSSS $(\widetilde{\bigcap}\mathbb{H}, \widetilde{\bigcup}\neg\mathbb{H}, X)$ over U in which $(\mathbb{H}, \neg\mathbb{H}, X)$ belongs to $N_{\mathfrak{F}}$ is an element of $N_{\mathfrak{F}}$. By the assumption, we obtain that the FSSS $(\widetilde{\bigcap}\mathbb{H}, \widetilde{\bigcup}\neg\mathbb{H}, X)$ over U in which $(\mathbb{H}, \neg\mathbb{H}, X)$ belongs to $N_{\mathfrak{F}}$ is a fuzzy semibipolar soft superset of $(\mathbb{G}, \neg\mathbb{G}, X)$. The proof is complete.

Notation 2. *For an ordered groupoid $(X, *, \leq_X)$ and a FSSS $\mathfrak{F} := (\mathbb{F}, \neg\mathbb{F}, X)$ over U, we denote by $N(\mathfrak{F})$ a FSSF over U generated by $\mathfrak{F}$. That is, a FSSS $(\widetilde{\bigcap}\mathbb{G}, \widetilde{\bigcup}\neg\mathbb{G}, X)$ over U in which $(\mathbb{G}, \neg\mathbb{G}, X)$ belongs to $N_{\mathfrak{F}}$ is denoted as $N(\mathfrak{F})$.*

Notation 3. *For an ordered groupoid $(X, *, \leq_X)$ and $x \in X$, we denote the notation*

$$N_x := \{\mathfrak{F} := (\mathbb{F}, \neg\mathbb{F}, X) : \mathfrak{F} \textit{ is a FSSF over } U, \mathbb{F}(x) = 1_U, \textit{ and } \neg\mathbb{F}(x) = 0_U\}.$$

Remark 5. *In Notation 3, we observe that $(\mathbb{W}_X, \neg\mathbb{W}_X, X)$ belongs to N_x. Then N_x is a non-empty subset of a non-empty collection of all FSSSs over U. By Proposition 1, we see that a FSSS $(\widetilde{\bigcap}\mathbb{G}, \widetilde{\bigcup}\neg\mathbb{G}, X)$ over U in which $(\mathbb{G}, \neg\mathbb{G}, X)$ belongs to N_x exists in the non-empty collection of all FSSSs over U. By Proposition 3, it follows that the FSSS $(\widetilde{\bigcap}\mathbb{G}, \widetilde{\bigcup}\neg\mathbb{G}, X)$ over U in which $(\mathbb{G}, \neg\mathbb{G}, X)$ belongs to N_x is a FSSF.*

Theorem 1. *If $(X, *, \leq_X)$ is an ordered groupoid and $x \in X$, then*

$$N_x = N_{\mathfrak{F}_{\{x\}} := (\mathbb{F}_{\{x\}}, \neg\mathbb{F}_{\{x\}}, X)}.$$

Proof. Let $\mathfrak{F} := (\mathbb{F}, \neg\mathbb{F}, X) \in N_x$. Then, we get that $\mathfrak{F}$ is a FSSF over U. Moreover, $\mathbb{F}(x) = 1_U$ and $\neg\mathbb{F}(x) = 0_U$. Thus $\mathfrak{F} \in N_{\mathfrak{F}_{\{x\}}}$. Indeed, let $y \in X$ be given.

Case 1. Assume $x = y$. Then

$$\mathbb{F}(y) = \mathbb{F}(x) = 1_U \text{ and } \neg\mathbb{F}(y) = \neg\mathbb{F}(x) = 0_U.$$

Furthermore, observe that $\mathbb{F}_{\{x\}}(y) = 1_U$ and $\neg\mathbb{F}_{\{x\}}(y) = 0_U$. It follows that $\mathbb{F}_{\{x\}}(y) = \mathbb{F}(y)$ and $\neg\mathbb{F}_{\{x\}}(y) = \neg\mathbb{F}(y)$. Thus $\mathfrak{F}\widetilde{\supseteq}\mathfrak{F}_{\{x\}}$.

Case 2. Assume $x \neq y$. Then

$$\mathbb{F}_{\{x\}}(y) = 0_U\widetilde{\leq}\mathbb{F}(y) \text{ and } \neg\mathbb{F}_{\{x\}}(y) = 1_U\widetilde{\geq}\mathbb{F}(y).$$

Whence $\mathfrak{F}\widetilde{\supseteq}\mathfrak{F}_{\{x\}}$.

Conversely, let $\mathfrak{G} := (\mathbb{G}, \neg\mathbb{G}, X) \in N_{\mathfrak{F}_{\{x\}}}$. Then $\mathfrak{G}$ is a FSSF over U and $\mathfrak{G}\widetilde{\supseteq}\mathfrak{F}_{\{x\}}$. It is true that

$$1_U = \mathbb{F}_{\{x\}}(x)\widetilde{\leq}\mathbb{G}(x) \text{ and } 0_U = \neg\mathbb{F}_{\{x\}}(x)\widetilde{\geq}\neg\mathbb{G}(x).$$

Observe that

$$\mathbb{G}(x) = 1_U \text{ and } \neg\mathbb{G}(x) = 0_U,$$

which yields $\mathfrak{G} \in N_x$. Therefore $N_x = N_{\mathfrak{F}_{\{x\}}}$.

In Propositions 7 and 8 below, it is not hard to verify that two arguments are true.

Proposition 7. *Let $(X, *, \leq_X)$ be an ordered groupoid and $\mathfrak{F} := (\mathbb{F}, \neg\mathbb{F}, X)$ a FSSS over U. If $\mathfrak{F}$ is a FSSF, then $\mathbb{F}^{-1}(1_U) = \emptyset$ (resp.,$\neg\mathbb{F}^{-1}(0_U) = \emptyset$) or $\mathbb{F}^{-1}(1_U)$ (resp.,$\neg\mathbb{F}^{-1}(0_U)$) is a filter of X.*

Proposition 8. *Let $(X, *, \leq_X)$ be an ordered groupoid and $x \in X$. Then*

$$(\mathbb{F}_{\mathbb{G}^{-1}(1_U)}, \neg\mathbb{F}_{\neg\mathbb{G}^{-1}(0_U)}, X)\widetilde{\subseteq}(\mathbb{G}, \neg\mathbb{G}, X)$$

for all $(\mathbb{G}, \neg\mathbb{G}, X) \in N_x$.

In the following, a necessary and sufficient condition for FSSFs is introduced.

Lemma 1. *Let $(X, *, \leq_X)$ be an ordered groupoid, and let A be a non-empty subset of X. Then A is a filter of X if and only if the FSSS $(\mathbb{F}_A, \neg\mathbb{F}_A, X)$ over U concerning A is a FSSF.*

Proof. Suppose that A is a filter of X and let $x, y \in X$. Then, we consider the following two cases.

Case 1. Assume $xy \in A$. Then

$$\mathbb{F}_A(xy) = 1_U \text{ and } \neg\mathbb{F}_A(xy) = 0_U.$$

Furthermore, we obtain that $x \in A$ and $y \in A$. Hence

$$\mathbb{F}_A(x) = 1_U = \mathbb{F}(y) \text{ and } \neg\mathbb{F}_A(x) = 0_U = \neg\mathbb{F}(y).$$

Thus

$$\mathbb{F}_A(xy) = 1_U = \mathbb{F}_A(x)\widetilde{\wedge}\mathbb{F}(y) \text{ and } \neg\mathbb{F}_A(xy) = 0_U = \mathbb{F}_A(x)\widetilde{\vee}\mathbb{F}_A(y).$$

Case 2. Assume $xy \notin A$. Then

$$\mathbb{F}_A(xy) = \emptyset \text{ and } \neg\mathbb{F}_A(xy) = 1_U.$$

Moreover, we obtain that $x \notin A$ or $y \notin A$. Thus

$$(\mathbb{F}_A(x) = 0_U \text{ and } \neg\mathbb{F}_A(x) = 1_U) \text{ or } (\mathbb{F}_A(y) = 0_U \text{ and } \neg\mathbb{F}_A(y) = 1_U).$$

Hence

$$\mathbb{F}_A(xy) = 0_U = \mathbb{F}_A(x)\widetilde{\wedge}\mathbb{F}_A(y) \text{ and } \neg\mathbb{F}_A(xy) = 1_U = \neg\mathbb{F}_A(x)\widetilde{\vee}\neg\mathbb{F}_A(y).$$

Next, assume that $x \leq_X y$. Then, we consider the following two cases.

Case 1. Assume $x \in A$. Then

$$\mathbb{F}_A(x) = 1_U \text{ and } \neg\mathbb{F}_A(x) = 0_U.$$

By the hypothesis, we have $y \in A$. Hence $\mathbb{F}_A(y) = 1_U$ and $\neg\mathbb{F}_A(y) = 0_U$. Whence

$$\mathbb{F}_A(x) = 1_U = \mathbb{F}_A(y) \text{ and } \neg\mathbb{F}_A(x) = 0_U = \neg\mathbb{F}_A(y).$$

Case 2. Suppose $x \notin A$. Then

$$\mathbb{F}_A(x) = 0_U\widetilde{\leq}\mathbb{F}_A(y) \text{ and } \neg\mathbb{F}_A(x) = 1_U\widetilde{\geq}\neg\mathbb{F}_A(y).$$

This implies that $(\mathbb{F}_A, \neg\mathbb{F}_A, X)$ is a FSSF over U.

Conversely, suppose $(\mathbb{F}_A, \neg\mathbb{F}_A, X)$ is a FSSF over U. Then, it is easy to verify that A is a subgroupoid of X. Let $x, y \in X$ and $xy \in A$. Then

$$1_U = \mathbb{F}_A(xy) = \mathbb{F}_A(x)\widetilde{\wedge}\mathbb{F}_A(y) \text{ and } 0_U = \neg\mathbb{F}_A(xy) = \neg\mathbb{F}_A(x)\widetilde{\vee}\neg\mathbb{F}_A(y).$$

We observe that

$$\mathbb{F}_A(x) = \mathbb{F}_A(y) = 1_U \text{ and } \neg\mathbb{F}_A(x) = \neg\mathbb{F}_A(y) = 0_U.$$

It follows that $x, y \in A$. Next, assume that $x \leq_X y$ and $x \in A$. Then

$$1_U = \mathbb{F}_A(x)\widetilde{\leq}\mathbb{F}_A(y) \text{ and } 0_U = \neg\mathbb{F}_A(x)\widetilde{\geq}\neg\mathbb{F}_A(y).$$

We obtain that

$$\mathbb{F}_A(y) = 1_U \text{ and } \neg\mathbb{F}_A(y) = 0_U,$$

which yields $y \in A$. This implies that A is a filter of X.

Theorem 2. *If* $(X, *, \leq_X)$ *is an ordered groupoid and* $x \in X$, *then*

$$N(\mathbb{F}_{\{x\}}, \neg\mathbb{F}_{\{x\}}, X) = (\mathbb{F}_{N(x)}, \neg\mathbb{F}_{N(x)}, X).$$

Proof. By Notation 2 and Theorem 1, we observe that a FSSS $(\widetilde{\bigcap}\mathbb{G}, \widetilde{\bigcup}\neg\mathbb{G}, X)$ over U in which $(\mathbb{G}, \neg\mathbb{G}, X)$ belongs to N_x is equal to the FSSS $N(\mathbb{F}_{\{x\}}, \neg\mathbb{F}_{\{x\}}, X)$ over U. By Remark 1, we have the FSSS $(\widetilde{\bigcap}\mathbb{G}, \widetilde{\bigcup}\neg\mathbb{G}, X)$ over U in which $(\mathbb{G}, \neg\mathbb{G}, X)$ belongs to N_x is a fuzzy semibipolar soft subset of $(\mathbb{G}, \neg\mathbb{G}, X)$ for all $(\mathbb{G}, \neg\mathbb{G}, X) \in N_x$. Thus $N(\mathbb{F}_{\{x\}}, \neg\mathbb{F}_{\{x\}}, X)\widetilde{\subseteq}(\mathbb{G}, \neg\mathbb{G}, X)$ for all $(\mathbb{G}, \neg\mathbb{G}, X) \in N_x$. Since $N(x)$ is a filter of X, we have $(\mathbb{F}_{N(x)}, \neg\mathbb{F}_{N(x)}, X)$ is a FSSF over U due to Lemma 1. Note that $\mathbb{F}_{N(x)}(x) = 1_U$ and $(\neg\mathbb{F}_{N(x)})(x) = 0_U$. Then $(\mathbb{F}_{N(x)}, \neg\mathbb{F}_{N(x)}, X) \in N_x$. This means that

$$N(\mathbb{F}_{\{x\}}, \neg\mathbb{F}_{\{x\}}, X)\widetilde{\subseteq}(\mathbb{F}_{N(x)}, \neg\mathbb{F}_{N(x)}, X).$$

On the other hand, we shall prove that $(\mathbb{F}_{N(x)}, \neg\mathbb{F}_{N(x)}, X)$ is a fuzzy semibipolar soft subset of $(\mathbb{H}, \neg\mathbb{H}, X)$ for all $(\mathbb{H}, \neg\mathbb{H}, X) \in N_x$. Suppose $(\mathbb{H}, \neg\mathbb{H}, X) \in N_x$. Then $\mathbb{H}(x) = 1_U$ and $\neg\mathbb{H}(x) = 0_U$. Therefore $x \in \mathbb{H}^{-1}(1_U) \cap \neg\mathbb{H}^{-1}(0_U)$. From Proposition 7, we have $\mathbb{H}^{-1}(1_U)$ and $\neg\mathbb{H}^{-1}(0_U)$ are filters of X containing x. It follows that $N(x) \subseteq \mathbb{H}^{-1}(1_U)$ and $N(x) \subseteq \neg\mathbb{H}^{-1}(0_U)$. From Remark 2 and Proposition 8, it follows that

$$(\mathbb{F}_{N(x)}, \neg\mathbb{F}_{N(x)}, X)\widetilde{\subseteq}(\mathbb{F}_{\mathbb{H}^{-1}(1_U)}, \neg\mathbb{F}_{\neg\mathbb{H}^{-1}(0_U)}, X)\widetilde{\subseteq}(\mathbb{H}, \neg\mathbb{H}, X).$$

Observe that a FSSS $(\inf_X\{\mathbb{H}\}, \sup_X\{\neg\mathbb{H}\}, X)$ over U in which $(\mathbb{H}, \neg\mathbb{H}, X)$ belongs to N_x is a fuzzy semibipolar soft superset of $(\mathbb{F}_{N(x)}, \neg\mathbb{F}_{N(x)}, X)$. By Proposition 2, we see that

$$(\widetilde{\bigcap}\mathbb{H}, \widetilde{\bigcup}\neg\mathbb{H}, X) = (\inf_X\{\mathbb{H}\}, \sup_X\{\neg\mathbb{H}\}, X).$$

That is, we get that the FSSS $(\widetilde{\bigcap}\mathbb{H}, \widetilde{\bigcup}\neg\mathbb{H}, X)$ over U in which $(\mathbb{H}, \neg\mathbb{H}, X)$ belongs to N_x is a fuzzy semibipolar soft superset of $(\mathbb{F}_{N(x)}, \neg\mathbb{F}_{N(x)}, X)$. By Notation 2 and Theorem 1, we obtain that

$$(\mathbb{F}_{N(x)}, \neg\mathbb{F}_{N(x)}, X)\widetilde{\subseteq}N(\mathbb{F}_{\{x\}}, \neg\mathbb{F}_{\{x\}}, X).$$

This implies that

$$N(\mathbb{F}_{\{x\}}, \neg\mathbb{F}_{\{x\}}, X) = (\mathbb{F}_{N(x)}, \neg\mathbb{F}_{N(x)}, X).$$

Notation 4. *For an ordered groupoid* $(X, *, \leq_X)$ *and* $x \in X$, *we denote the notation*

$$N^B(x) := N(\mathbb{F}_{\{x\}}, \neg\mathbb{F}_{\{x\}}, X).$$

Remark 6. *By Notation 2, Notation 4 and Theorem 1, observe that a FSSS* $(\widetilde{\bigcap}\mathbb{F}, \widetilde{\bigcup}\neg\mathbb{F}, X)$ *over* U *in which* $(\mathbb{F}, \neg\mathbb{F}, X)$ *belongs to* N_x. *Then, by Proposition 3, we see that* $N^B(x) \in N_x$ *and* $N^B(x)\widetilde{\subseteq}(\mathbb{G}, \neg\mathbb{G}, X)$ *for all* $(\mathbb{G}, \neg\mathbb{G}, X) \in N_x$. *From Definition 4, it follows that* $N^B(x)$ *is a FSSF over* U *containing* x. *This remark leads to the following definition.*

Definition 7. *Let $(X, *, \leq_X)$ be an ordered groupoid and $x \in X$. We call $N^B(x)$ the FSSF over U generated by x.*

We are now ready for the presentation of a binary relation induced by a FSSF over U.

Definition 8. *Let $(X, *, \leq_X)$ be an ordered groupoid. We define relations $\mathcal{N}^B$ on X as follows:*

$$\mathcal{N}^B := \{(x, y) \in X \times X : N^B(x) = N^B(y)\}.$$

As mentioned above, we shall verify the relationship between two binary relations $\mathcal{N}$ and $\mathcal{N}^B$ as the following theorem.

Theorem 3. *If $(X, *, \leq_X)$ is an ordered groupoid, then $\mathcal{N}$ and $\mathcal{N}^B$ are identical.*

Proof. Let $x, y \in X$ be given. Then, we obtain that

$$\begin{aligned}
(x, y) \in \mathcal{N}^B &\iff N^B(x) = N^B(y) \\
&\iff N(\mathbb{F}_{\{x\}}, \neg\mathbb{F}_{\{x\}}, X) = N(\mathbb{F}_{\{y\}}, \neg\mathbb{F}_{\{y\}}, X) \\
&\iff (\mathbb{F}_{N(x)}, \neg\mathbb{F}_{N(x)}, X) = (\mathbb{F}_{N(y)}, \neg\mathbb{F}_{N(y)}, X) \\
&\iff N(x) = N(y) \\
&\iff (x, y) \in \mathcal{N}
\end{aligned}$$

due to Remark 2, Theorem 2, Notation 4, and Definition 8. Whence $\mathcal{N} = \mathcal{N}^B$.

3. Conclusions

As studied before, there has been much technical association of fuzzy semibipolar soft set theory, especially the use of the FSSFs to describe Green's relation $\mathcal{N}$. Observe that Green's relation $\mathcal{N}$ on groupoids in this work is induced by the two-way function and that extends the concept of the one-way function in [13]. In addition, we obtained that if a FSSFs-based binary relation exists, then two parameters of a fuzzy semibipolar soft set are related under Green's relation $\mathcal{N}$. For the context of filters, Mahmood et al.[18] proposed the fundamentals of fuzzy filters in terms of rough set theory, in which rough set theory is an important concept for dealing with vagueness problems. In the future, to extend such a concept via a two-way function, the notion of FSSFs will be considered under rough set theory in the next step.

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