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A family of Analytic Functions Subordinate to Horadam Polynomials

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Abstract. In this paper, we introduce and investigate a family of analytic functions, denoted by $\mathcal{F}(\Pi, \alpha, \beta, \lambda, \delta, \mu)$, defined by means of Horadam polynomials. For functions in this family, we derive the estimations for the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. Moreover, we obtain the classical Fekete-Szegö inequality of functions belonging to this family.

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1. Introduction

Let $\mathcal{A}$ be the family of all analytic functions f that are defined on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 1 - f'(0) = 0$. Any function $f \in \mathcal{A}$ has the following Taylor-Maclarin series expansion:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad \text{where} \quad z \in \mathbb{D}. \tag{1}$$

Let $\mathcal{S}$ denote the class of all functions $f \in \mathcal{A}$ that are univalent in $\mathbb{D}$. Let the functions f and g be analytic in $\mathbb{D}$, we say the function f is subordinate by the function g in $\mathbb{D}$, denoted by $f(z) \prec g(z)$ for all $z \in \mathbb{D}$, if there exists a Schwarz function w, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{D}$, such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. In particular, if the function g is univalent over $\mathbb{D}$ then $f(z) \prec g(z)$ equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D}$. For more information about the subordination principle and univalent functions we refer the readers to to the monographs [11], [10], [14], [23], [25] and the references therein.

In the year 1965, for $a, b, p, q \in \mathbb{R}$, Horadam [17] introduced the sequence $W_n = W_n(a, b; p, q)$ that is defined by the following recurrence relation

$$W_{n+2} = pW_{n+1} + qW_n, \quad \text{for} \quad n \geq 2,$$

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with the initial values $W_0 = a$ and $W_1 = b$. The characteristic equation of this sequence is given by

$$t^2 - pt - q = 0.$$

In addition, the generating function of Horadam sequence is

$$f(t) = \frac{a + t(b - ap)}{1 - pt - qt^2}.$$

The Horadam sequences generalize many famous sequences such as Fibonacci, Lucas, Pell, Pell-Lucas and Jacobsthal sequences. These sequences have been studied for a long time. For more information about these sequences, we refer the readers to the articles [16], [15], the monograph [21] and the references therein.

In the year 1985, Horadam and Mahon [16] defined the Horadam polynomials $h_n(x) = h_n(a, b; p, q)$ by the following recurrence relation:

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x), \quad \text{for} \quad n \in \mathbb{N} \setminus \{1, 2\}, \tag{2}$$

with initial values,

$$h_1(x) = a, \quad h_2(x) = bx, \quad \text{and} \quad h_3(x) = pbx^2 + qa. \tag{3}$$

Moreover, the generating function of Horadam ploynomials is given by

$$\Pi(x, z) = \sum_{n=1}^{\infty} h_n(x) z^{n-1} = \frac{a + (b - ap)xz}{1 - pxz - qz^2}.$$

In this paper, the argument of $x \in \mathbb{R}$ is independent of the argument $z \in \mathbb{C}$; that is $x \neq \Re(z)$. For particular values of a, b, p and q the Horadam polynomials leads to many known polynomials. Below, we list some particular cases of Horadam Polynomials.

- If $a = b = p = q = 1$, we get Fibonacci polynomials $F_n(x)$ whose recurrence relation is
$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x); \text{ with } F_1(x) = 1, F_2(x) = x.$$
- If $a = 2$ and $b = p = q = 1$, we get Lucas Polynomials $L_n(x)$ whose recurrence relation is
$$L_{n-1}(x) = xL_{n-2}(x) + L_{n-3}(x); \text{ with } L_0(x) = 2, L_1(x) = x.$$
- If $a = q = 1$ and $b = p = 2$, we get Pell Polynomials $P_n(x)$ whose recurrence relation is
$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x); \text{ with } P_1(x) = 1, P_2(x) = 2x.$$

- If $a = b = p = 2$ and $q = 1$, we get Pell-Lucas Polynomials $Q_n(x)$ whose recurrence relation is

$$Q_{n-1}(x) = 2xQ_{n-2}(x) + Q_{n-3}(x); \text{ with } Q_0(x) = 2, Q_1(x) = 2x.$$

- If $a = b = p = x = 1$ and $q = 2y$, we get Jacobsthal Polynomials $J_n(y)$ whose recurrence relation is

$$J_n(y) = J_{n-1}(y) + 2yJ_{n-2}(y); \text{ with } J_1(y) = 1, J_2(y) = 1.$$

- If $a = 2$, $b = p = x = 1$ and $q = 2y$, we get Jacobsthal-Lucas Polynomials $\mathfrak{J}_n(y)$ whose recurrence relation is

$$\mathfrak{J}_{n-1}(y) = \mathfrak{J}_{n-2}(y) + 2y\mathfrak{J}_{n-3}(y); \text{ with } \mathfrak{J}_0(y) = 2, \mathfrak{J}_1(y) = 1.$$

- If $a = 1$ and $b = p = 2$, and $q = -1$, we get Chebyshev Polynomials $H_n(x)$ of the second kind whose recurrence relation is

$$H_{n-1}(x) = 2xH_{n-2}(x) - H_{n-3}(x); \text{ with } H_0(x) = 1, H_1(x) = 2x.$$

- If $a = b = 1$ and $p = 2$, and $q = -1$, we get Chebyshev Polynomials $T_n(x)$ of the first kind whose recurrence relation is

$$T_{n-1}(x) = 2xT_{n-2}(x) - T_{n-3}(x); \text{ with } T_0(x) = 1, T_1(x) = x.$$

For more information about Horadam polynomials and its special interesting cases, we refer the readers to the articles [1], [3], [2], [16], [18], [24], [26], [27], [30], the monograph [21], [29] and the references therein.

Recently, many researchers have been studying the geometric function theory, the typical problem in this field is studying a functional made up of combinations of the initial coefficients of the functions $f \in \mathcal{A}$. For a function in the class $\mathcal{S}$, it is well-known that $|a_n|$ is bounded by n. Moreover, the coefficient bounds give information about the geometric properties of those functions. For instance, the bound for the second coefficients of the class $\mathcal{S}$ gives the growth and distortion bounds for the class. In addition, the Fekete-Szegö functional arises naturally in the investigation of univalency of analytic functions. In the year 1933, Fekete and Szegö [13] found the maximum value of $|a_3 - \lambda a_2^2|$, as a function of the real parameter $0 \leq \lambda \leq 1$ for a univalent function f. Since then, the problem of dealing with the Fekete-Szegö functional for $f \in \mathcal{A}$ with any complex λ is known as the classical Fekete-Szegö problem. There are many researchers investigated the Fekete-Szegö functional and the other coefficient estimates problems, for example see the articles [1], [5], [4], [6], [9], [13], [20], [22], [27], [30] and the references therein.

Motivated by the aforementioned research and the papers [3], [2] and [28] we introduce a novel class of analytic functions defined using Horadam polynomials. For functions belong to this function class, we derive estimations for the Taylor-Maclaurin initial coefficients and Fekete-Szegö functional problem. We also present corollaries for subclasses of our class defined by the means of special cases of Horadam polynomials.

2. Preliminaries

In this section we present some information that are curial for the main results of this paper. First, we define our family of analytic functions subordinated by Horadam polynomials, which we denote by $\mathcal{F}(\Pi, \alpha, \beta, \lambda, \delta, \mu)$.

Definition 1. *We say that a function $f \in \mathcal{A}$ in the class $\mathcal{F}(\Pi, \alpha, \beta, \lambda, \delta, \mu)$ if it fulfills the subordination conditions, associated with the Horadam Polynomials, for all $z \in \mathbb{D}$:*

$$\beta\left(\frac{zG'(z)}{G(z)}\right)^{\alpha} + (1-\beta)\left(\frac{zG'(z)}{G(z)}\right)^{\lambda}\left(1+\frac{zG''(z)}{G'(z)}\right)^{1-\lambda} \prec \Pi(x,z)+1-a, \tag{4}$$

where

$$G(z) = \delta\mu z^2 f''(z) + (\mu-\delta)zf'(z) + (1-\mu+\delta)f(z),$$

and

$$1 \leq \alpha \leq 2, 0 \leq \beta \leq 1, 0 \leq \lambda \leq 1 \text{ and } 0 \leq \delta \leq \mu \leq 1.$$

The following are interesting cases related to our presenting class.

(a) If we replace Horadam polynomials by Gegenbaure polynomials $H_n^{(\gamma)}(z,t)$, we obtain the class $\mathcal{F}(H_n^{(\gamma)}(z,t), \alpha, \beta, \lambda, \delta, \mu)$ which was investigated by Sirvastava et al. [28]. For $z \in \mathbb{D}$, $t \in (\frac{1}{2}, 1]$, and $\gamma \geq 0$ the generating function of Gegenbauer polynomials is given by

$$H_n^{(\gamma)}(z,t) = (z^2 - 2tz + 1)^{-\gamma}.$$

(b) If we replace Horadam polynomials by Chebyshev polynomials $H_n(z,t)$ of the second kind, where $\gamma = 1$, we obtain the class $\mathcal{F}(H_n(z,t), \alpha, \beta, \lambda, \delta, \mu)$ which was introduced and studied by Kamali et al. [19].

Now, we present some particular special subclasses which obtained by taking specific values of the parameters involved in our class $\mathcal{F}(\Pi, \alpha, \beta, \lambda, \delta, \mu)$.

- If $\lambda = \delta = \mu = 0$, $\alpha = 1$, and $\beta = 1 - \eta$ where $0 \leq \eta \leq 1$, then we get the class $\mathcal{F}(\Pi, 1, 1-\eta, 0, 0, 0)$. We say $f \in \mathcal{A}$ belong to this class if it satisfies the following subordination:

$$(1-\eta)\left(\frac{zf'(z)}{f(z)}\right) + \eta\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \Pi(x,z)+1-a. \tag{5}$$

  This class investigated by Abrami et al. [1].

- If $\beta = 0$ and $\delta = \mu = 0$, we get the class $\mathcal{F}(\Pi, \alpha, 0, \lambda, 0, 0)$. We say $f \in \mathcal{A}$ belong to this class if it satisfies the following subordination:

$$\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\lambda} \prec \Pi(x,z)+1-a. \tag{6}$$

  This class investigated by Abrami et al. [1].

- If $\alpha = \beta = 1$, we get the class $\mathcal{F}(\Pi, 1, 1, \lambda, \delta, \mu)$. We say $f \in \mathcal{A}$ belong to this class if it satisfies the following subordination:

$$\frac{\mu\delta z^3 f'''(z) + (2\mu\delta + \mu - \delta)z^2 f''(z) + zf'(z)}{\mu\delta z^2 f''(z) + (\mu - \delta)zf'(z) + (1 - \mu + \delta)f(z)} \prec \Pi(x, z) + 1 - a. \tag{7}$$

  If we replace $\Pi(x, z) + 1 - a$ by the Chebyshev polynomials $H_n(z, t)$ of the second kind, we get the class $\mathcal{F}(H_n(z, t), 1, 1, \lambda, \delta, \mu)$ which introduced and studied by Çağlar et al. [8].

- If $\mu = \delta = 0$, we get the class $\mathcal{F}(\Pi, \alpha, \beta, \lambda, 0, 0)$. We say $f \in \mathcal{A}$ belong to this class if it satisfies the following subordination:

$$\beta \left(\frac{zf'(z)}{f(z)}\right)^{\alpha} + (1 - \beta)\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\lambda} \prec \Pi(x, z) + 1 - a. \tag{8}$$

  If we replace $\Pi(x, z)+1-a$ by the Chebyshev polynomials $H_n(z, t)$ of the second kind, we get the class $\mathcal{F}(H_n(z, t), \alpha, \beta, \lambda, 0, 0)$ which introduced and studied by Szatmari et al. [31].

- If $\delta = \lambda = 0$, $\alpha = 1$ and $\beta = 1 - \eta$, we get the class $\mathcal{F}(\Pi, 1, 1 - \eta, 0, 0, \mu)$.We say $f \in \mathcal{A}$ belong to this class if it satisfies the following subordination:

$$(1 - \eta)\left(\frac{zG'(z)}{G(z)}\right) + \eta\left(1 + \frac{zG''(z)}{G'(z)}\right) \prec \Pi(x, z) + 1 - a, \tag{9}$$

  where $G(z) = \mu zf'(z) + (1 - \mu)f(z)$. If we replace $\Pi(x, z) + 1 - a$ by the Chebyshev polynomials $H_n(z, t)$ of the second kind, we get the class $\mathcal{F}(H_n(z, t), \alpha, 1 - \eta, \lambda, 0, 0)$ which introduced and studied by Bulut et al. [7].

- If $\delta = \mu = \beta = \lambda = 0$, we get the class $\mathcal{F}(\Pi, \alpha, 0, 0, 0, 0)$. We say $f \in \mathcal{A}$ belong to this class if it satisfies the following subordination:

$$1 + \frac{zf''(z)}{f'(z)} \prec \Pi(x, z) + 1 - a. \tag{10}$$

  If we replace $\Pi(x, z) + 1 - a$ by the Chebyshev polynomials $H_n(z, t)$ of the second kind, we get the class $\mathcal{F}(H_n(z, t), \alpha, 0, 0, 0, 0)$ which introduced and studied by Dziok et al. [12].

The following lemma (see, for details [20]) is a well-known fact, but it is crucial for the main results of this paper.

Lemma 1. *Let the Schwarz function $w(z)$ be given by:*

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots \quad \textit{where} \quad z \in \mathbb{D},$$

then $|w_1| \leq 1$ and for $t \in \mathbb{C}$

$$|w_2 - tw_1^2| \leq 1 + (|t| - 1)|w_1|^2 \leq \max\{1, |t|\}.$$

The result is sharp for the functions $w(z) = z$ and $w(z) = z^2$.

In this presenting paper, we investigate a family of analytic functions on the open unit disk $\mathbb{D}$, which we denoted by $\mathcal{F}(\Pi, \alpha, \beta, \lambda, \delta, \mu)$ subordinate to Horadam polynomials. For functions in this family, we derive upper bounds for the initial Taylor-Maclarin coefficients $|a_2|$ and $|a_3|$. Furthermore, we examine the corresponding Fekete-Szegö functional problem for functions belong to this family.

3. Coefficient estimates for the function class $\mathcal{F}(\Pi, \alpha, \beta, \lambda, \delta, \mu)$

In this section, we provide bounds for the initial Taylor-Maclaurin coefficients for the functions belong to the class $\mathcal{F}(\Pi, \alpha, \beta, \lambda, \delta, \mu)$ which are given by equation (1).

Theorem 1. *Let the function $f(z)$ given by (1) be in the class $\mathcal{F}(\Pi, \alpha, \beta, \lambda, \delta, \mu)$. Then*

$$|a_2| \leq \frac{|bx|}{A(1 + 2\mu\delta + \mu - \delta)}, \tag{11}$$

$$|a_3| \leq \frac{|bx|}{2B} \max\left\{1, \left|\frac{bxK}{2A^2} - \frac{pbx^2 + qa}{bx}\right|\right\}, \tag{12}$$

where

$$A = \alpha\beta + (1 - \beta)(2 - \lambda),$$
$$B = (\alpha\beta + (1 - \beta)(3 - 2\lambda))(1 + 2(3\mu\delta + \mu - \delta)),$$

and

$$K = \alpha\beta(\alpha - 3) + (1 - \beta)(\lambda^2 + 5\lambda - 8).$$

Proof. Let f belong to the class $\mathcal{F}(\Pi, \alpha, \beta, \lambda, \delta, \mu)$. Then, using Definition 1, we can find an analytic function u such that

$$\beta\left(\frac{zG'(z)}{G(z)}\right)^{\alpha} + (1 - \beta)\left(\frac{zG'(z)}{G(z)}\right)^{\lambda}\left(1 + \frac{zG''(z)}{G'(z)}\right)^{1-\lambda} \prec \Pi(x, u(z)) + 1 - a \tag{13}$$

where $u : \mathbb{D} \to \mathbb{D}$ is given by

$$u(z) = \sum_{n=1}^{\infty} u_n z^n \quad \text{for} \quad z \in \mathbb{D},$$

such that $u(0) = 0$ and $|u(z)| < 1$ for all $z \in \mathbb{D}$. Moreover, it is well-known that (see, for details [11]), $|u_j| \leq$ for all $j \in \mathbb{N}$.

Now, upon comparing the coefficients in both sides of equation (13), we obtain the following equations, where $\gamma = 2\mu\delta + \mu - \delta$,

$$(\alpha\beta + (1 - \beta)(2 - \lambda))(\gamma + 1)a_2 = h_2(x)u_1. \tag{14}$$

$$[\alpha\beta(\alpha - 3) + (1 - \beta)(\lambda^2 + 5\lambda - 8)](\gamma + 1)^2 a_2^2 + \tag{15}$$

$$+4(\alpha\beta+(1-\beta)(3-2\lambda))(2(\gamma+\mu\delta)+1)a_3 = 2h_2(x)u_2+2h_3(x)u_1^2.$$

Using equation (14), $|u_1|\leq 1$ and $h_2(x)=bx$, we get the desired bound of $|a_2|$:

$$|a_2| \leq \frac{|bx|}{[\alpha\beta+(1-\beta)(2-\lambda)](\gamma+1).}$$

In view of equation (14), we can write equation (15) as:

$$a_3 = \frac{h_2(x)}{2B}\left(u_2 - \frac{1}{h_2(x)}\left(\frac{K[h_2(x)]^2}{2A^2} - h_3(x)\right)u_1^2\right).$$

Using Lemma 1, we obtain

$$|a_3| \leq \frac{|h_2(x)|}{2B}\max\left\{1, \left|\frac{K[h_2(x)]^2-2A^2h_3(x)}{2A^2h_2(x)}\right|\right\}.$$

Using the initial values (3), we get the desired estimate of $|a_3|$. This complete the proof.

The following corollaries are just consequences of Theorem 1.

Corollary 1. *If the function $f\in\mathcal{A}$ satisfies the subordination (5), then*

$$|a_2| \leq \frac{|bx|}{1+\eta},$$

and

$$|a_3| \leq \frac{|bx|}{2(1+2\eta)}\max\left\{1, \left|\frac{bx(1+3\eta)}{(1+\eta)^2} + \frac{pbx^2+qa}{bx}\right|\right\}.$$

Corollary 2. *If the function $f\in\mathcal{A}$ satisfies the subordination (6), then*

$$|a_2| \leq \frac{|bx|}{2-\lambda},$$

and

$$|a_3| \leq \frac{|bx|}{2(3-2\lambda)}\max\left\{1, \left|\frac{bx(\lambda^2+5\lambda-8)}{2(2-\lambda)^2} - \frac{pbx^2+qa}{bx}\right|\right\}.$$

Corollary 3. *If the function $f\in\mathcal{A}$ satisfies the subordination (7), then*

$$|a_2| \leq \frac{|bx|}{1+2\mu\delta+\mu-\delta},$$

and

$$|a_3| \leq \frac{|bx|}{2(1+2(3\mu\delta+\mu-\delta))}\max\left\{1, \left|\frac{b^2x^2+pbx^2+qa}{bx}\right|\right\}.$$

Corollary 4. *If the function $f \in \mathcal{A}$ satisfies the subordination (8), then*

$$|a_2| \leq \frac{|bx|}{\alpha\beta + (1-\beta)(2-\lambda)},$$

and

$$|a_3| \leq \frac{|bx|}{2(\alpha\beta + (1-\beta)(3-2\lambda))} \max\left\{1, \left|\frac{bxK}{2A^2} - \frac{pbx^2 + qa}{bx}\right|\right\},$$

Corollary 5. *If the function $f \in \mathcal{A}$ satisfies the subordination (9), then*

$$|a_2| \leq \frac{|bx|}{(1+\eta)(1+\mu)},$$

and

$$|a_3| \leq \frac{|bx|}{2(1+2\eta)(1+2\mu)} \max\left\{1, \left|\frac{bx(1+3\eta)}{(1+\eta)^2} + \frac{pbx^2 + qa}{bx}\right|\right\}.$$

4. Fekete-Szegö functional of the class $\mathcal{F}(\Pi, \alpha, \beta, \lambda, \delta, \mu)$

In this section, we consider the classical Fekete-Szegö problem for our presenting class $\mathcal{F}(\Pi, \alpha, \beta, \lambda, \delta, \mu)$.

Theorem 2. *Let the function f given by (1) be in the class $\mathcal{F}(\Pi, \alpha, \beta, \lambda, \delta, \mu)$. Then for $bx > 0$ and for some $\zeta \in \mathbb{R}$,*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{bx}{2B}, & \text{if } \zeta \in [\zeta_1, \zeta_2] \\ \frac{|2(pbx^2+qa)BA^2(\gamma+1)^2 - b^2x^2(K(\gamma+1)^2 + 4\zeta B)|}{4BA^2(\gamma+1)^2}, & \text{if } \zeta \notin [\zeta_1, \zeta_2], \end{cases} \tag{16}$$

where

$$\zeta_1 = \frac{(\gamma+1)^2(2A^2(pbx^2 - bx + qa) - b^2x^2K)}{4Bb^2x^2},$$

and

$$\zeta_2 = \frac{(\gamma+1)^2(2A^2(pbx^2 + bx + qa) - b^2x^2K)}{4Bb^2x^2}.$$

Proof. In view of equations (14) and (15), we get the following

$$a_3 = \frac{h_2(x)u_2}{2B} + \frac{h_3(x)u_1^2}{2B} - \frac{K[h_2(x)]^2u_1^2}{4BA^2}.$$

For some real number ζ, using equation (15), we have

$$\begin{aligned} a_3 - \zeta a_2^2 =& \frac{h_2(x)u_2}{2B} + \frac{h_3(x)u_1^2}{2B} - \frac{K[h_2(x)]^2u_1^2}{4BA^2} - \frac{\zeta[h_2(x)]^2u_1^2}{(\gamma+1)^2A^2} \\ =& \frac{h_2(x)}{2B}\left\{u_2 + \left(\frac{h_3(x)}{h_2(x)} - \frac{Kh_2(x)}{2A^2} - \frac{2\zeta B h_2(x)}{(\gamma+1)^2A^2}\right)u_1^2\right\} \end{aligned}$$

Using Lemma 1 and the initial values (3), we obtain

$$|a_3 - \zeta a_2^2| \leq \frac{|bx|}{2B} \max\left\{1, \left|\frac{pbx^2+qa}{bx} - \frac{bxK}{2A^2} - \frac{2\zeta Bbx}{(\gamma+1)^2A^2}\right|\right\} \tag{17}$$

Since $bx > 0$, we have

$$\left|\frac{pbx^2+qa}{bx} - \frac{bxK}{2A^2} - \frac{2\zeta Bbx}{(\gamma+1)^2A^2}\right| \leq 1.$$

Solving for ζ we get:

$$\frac{(\gamma+1)^2(2A^2(pbx^2-bx+qa)-b^2x^2K)}{4Bb^2x^2} \leq \zeta \leq \frac{(\gamma+1)^2(2A^2(pbx^2+bx+qa)-b^2x^2K)}{4Bb^2x^2}$$

Hence, inequality (17) becomes

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{bx}{2B}, & \text{if } \zeta \in [\zeta_1, \zeta_2] \\ \left|\frac{pbx^2+qa}{2B} - \frac{b^2x^2K}{4BA^2} - \zeta\left(\frac{bx}{A(\gamma+1)}\right)^2\right|, & \text{if } \zeta \notin [\zeta_1, \zeta_2]. \end{cases}$$

Simplifying the last inequality, we get the desired inequality (16), this completes the proof of Theorem 2.

The following corollaries are just consequences of Theorem 2.

Corollary 6. *If the function $f \in \mathcal{A}$ satisfies the subordination (5), then for $bx > 0$ and for some $\zeta \in \mathbb{R}$*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{bx}{2(1+2\eta)}, & \textit{if } \zeta \in [\zeta_1, \zeta_2] \\ \left|\frac{pbx^2+qa}{2(1+2\eta)} + \frac{b^2x^2(1+3\eta-2\zeta(1+2\eta))}{2(1+2\eta)(1+\eta)^2}\right|, & \textit{if } \zeta \notin [\zeta_1, \zeta_2], \end{cases}$$

where

$$\zeta_1 = \frac{(1+\eta)^2(pbx^2-bx+qa)+b^2x^2(1+3\eta)}{2(1+2\eta)b^2x^2},$$

and

$$\zeta_2 = \frac{(1+\eta)^2(pbx^2+bx+qa)+b^2x^2(1+3\eta)}{2(1+2\eta)b^2x^2}.$$

Corollary 7. *If the function $f \in \mathcal{A}$ satisfies the subordination (6), then for $bx > 0$ and for some $\zeta \in \mathbb{R}$*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{bx}{2(3-2\lambda)}, & \textit{if } \zeta \in [\zeta_1, \zeta_2] \\ \frac{|2(pbx^2+qa)(3-2\lambda)(2-\lambda)^2 - b^2x^2(\lambda^2+5\lambda+4\zeta(3-2\lambda)-8)|}{4(3-2\lambda)(2-\lambda)^2}, & \textit{if } \zeta \notin [\zeta_1, \zeta_2], \end{cases}$$

where

$$\zeta_1 = \frac{2(2-\lambda)^2(pbx^2 - bx + qa) - b^2x^2(\lambda^2 + 5\lambda - 8))}{4(3-2\lambda)b^2x^2},$$

and

$$\zeta_2 = \frac{2(2-\lambda)^2(pbx^2 + bx + qa) - b^2x^2(\lambda^2 + 5\lambda - 8))}{4(3-2\lambda)b^2x^2}.$$

Corollary 8. *If the function $f \in \mathcal{A}$ satisfies the subordination (7), then for $bx > 0$ and for some $\zeta \in \mathbb{R}$*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{bx}{2(1+2(3\mu\delta+\mu-\delta))}, & \text{if } \zeta \in [\zeta_1, \zeta_2] \\ \left| \frac{pbx^2+qa+b^2x^2}{2(1+2(3\mu\delta+\mu-\delta))} - \zeta \left(\frac{bx}{1+2\mu\delta+\mu-\delta}\right)^2 \right|, & \text{if } \zeta \notin [\zeta_1, \zeta_2], \end{cases}$$

where

$$\zeta_1 = \frac{(1+2\mu\delta+\mu-\delta)^2(pbx^2 - bx + qa + b^2x^2)}{2(1+2(3\mu\delta+\mu-\delta))b^2x^2},$$

and

$$\zeta_2 = \frac{(1+2\mu\delta+\mu-\delta)^2(pbx^2 + bx + qa + b^2x^2)}{2(1+2(3\mu\delta+\mu-\delta))b^2x^2}.$$

Corollary 9. *If the function $f \in \mathcal{A}$ satisfies the subordination (8), then for $bx > 0$ and for some $\zeta \in \mathbb{R}$*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{bx}{2(\alpha\beta+(1-\beta)(3-2\lambda))}, & \text{if } \zeta \in [\zeta_1, \zeta_2] \\ \left| \frac{pbx^2+qa}{2(\alpha\beta+(1-\beta)(3-2\lambda))} - \frac{b^2x^2K}{4(\alpha\beta+(1-\beta)(3-2\lambda))A^2} - \zeta \left(\frac{bx}{A}\right)^2 \right|, & \text{if } \zeta \notin [\zeta_1, \zeta_2]. \end{cases}$$

where

$$\zeta_1 = \frac{2A^2(pbx^2 - bx + qa) - b^2x^2K}{4(\alpha\beta+(1-\beta)(3-2\lambda)b^2x^2},$$

and

$$\zeta_2 = \frac{2A^2(pbx^2 + bx + qa) - b^2x^2K}{4(\alpha\beta+(1-\beta)(3-2\lambda)b^2x^2}.$$

Corollary 10. *If the function $f \in \mathcal{A}$ satisfies the subordination (9), then for $bx > 0$ and for some $\zeta \in \mathbb{R}$*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{bx}{2(1+2\eta)(1+2\mu)}, & \text{if } \zeta \in [\zeta_1, \zeta_2] \\ \left| \frac{(pbx^2+qa)(1+\eta)^2+b^2x^2(1+3\eta)}{2(1+2\eta)(1+2\mu)(1+\eta)^2} - \frac{\zeta b^2x^2}{(1+\eta)^2(1+\mu)^2} \right|, & \text{if } \zeta \notin [\zeta_1, \zeta_2]. \end{cases}$$

where

$$\zeta_1 = \frac{(1+\mu)^2[(1+\eta)^2(pbx^2 - bx + qa) + b^2x^2(1+3\eta)]}{2(1+2\eta)(1+2\mu)b^2x^2},$$

and

$$\zeta_2 = \frac{(1+\mu)^2[(1+\eta)^2(pbx^2 + bx + qa) + b^2x^2(1+3\eta)]}{2(1+2\eta)(1+2\mu)b^2x^2}.$$

References

[1] C. Abirami, N. Magesh, and J. Yamini. Initial bounds for certain classes of biunivalent functions defined by Horadam polynomials. *Abstract and Applied Analysis*, Article ID 7391058:8 pages, 2020.

[2] W. Al-Rawashdeh. Applications of Horadam polynomials to a class of close-to-convex functions. *preprint*.

[3] W. Al-Rawashdeh. Horadam polynomials and a class of binivalent functions defined by Ruscheweyh operator. *International Journal of Mathematics and Mathematical Sciences*, Article ID 2573044:7 pages, 2023.

[4] W. Al-Rawashdeh. Coefficient bounds of a class of bi-univalent functions related to Gegenbauer polynomials. *International Journal of Mathematics and Computer Science*, 19:635–642, 2024.

[5] W. Al-Rawashdeh. Fekete-Szegö Functional of a Subclass of Bi-Univalent Functions Associated with Gegenbauer Polynomials. *European Journal of Pure and Applied Mathematics*, In Press, 2024. https://doi.org/10.29020/nybg.ejpam.v17i1.5022.

[6] S. Altınkaya and S. Yalcin. On the Chebyshev polynomial bounds for classes of univalent functions. *Khayyam Journal Mathematics*, 2(1):1–5, 2016.

[7] S. Bulut, N. Magesh, and V.K. Balaji. Certain subclasses of analytic functions associated with the Chebyshev polynomials. *Honam Mathematical Journal*, 40(4):611–619, 2018.

[8] M. Cağlar, H. Orhan, and M. Kamali. Fekete-Szegö problem for a subclass of analytic functions associated with Chebyshev polynomials. *Bol. Soc. Paran. Mat.*, 40, 2022.

[9] J.H. Choi, Y.C. Kim, and T. Sugawa. A general approach to the Fekete-Szegö problem. *Journal of the Mathematical Society of Japan*, 59(3):707–727, 2007.

[10] P. Duren. Subordination in Complex Analysis, Lecture Notes in Mathematics. *Springer, Berlin, Germany*, 599(3):22–29, 1977.

[11] P. Duren. *Univalent functions.* Grundlehren der Mathematischen Wissenschaften 259, Springer-Verlag, New York, 1983.

[12] J. Dziok, R.K. Raina RK, and J. Sokol J. Application of Chebyshev polynomials to classes of analytic functions. *Comptes Rendus de ĺAcadémie des Sciences Paris*, 353(3):433–438, 2015.

[13] M. Fekete and G. Szegö. Eine Bemerkung Über ungerade Schlichte Funktionen. *Journal of London Mathematical Society*, s1-8(3):85–89, 1933.

[14] A. W. Goodman. *Univalent functions.* Mariner Publishing Co. Inc., Boston, 1983.

[15] A. F. Horadam. Jacobsthal representation polynomials. *The Fibonacci Quarterly*, 35(3):137–148, 1997.

[16] A. F. Horadam and J. M. Mahon. Pell and Pell-Lucas polynomials. *The Fibonacci Quarterly*, 23(3):7–20, 1985.

[17] A.F. Horadam. Basic properties of a certain generalized sequence of numbers. *The Fibonacci Quarterly*, 3(3):161–176, 1965.

[18] T. Horzum and E.G. Kocer. On some properties of Horadam polynomials. *International Mathematics Forum*, 4(3):1243–1252, 2009.

[19] M. Kamali, M. Cağlar, E. Deniz, and M. Turabaev. Fekete Szegö problem for a new subclass of analytic functions satisfying subordinate condition associated with Chebyshev polynomials. *Turkish J. Math.*, 45(3):1195–1208, 2012.

[20] F.R. Keogh and E.P. Merkes. A Coefficient inequality for certain classes of analytic functions. *Proceedings of the American Mathematical Society*, 20(3):8–12, 1969.

[21] T. Koshy. Jacobsthal representation polynomials, Second Edition; Pure and Applied Mathematics. 2018.

[22] M. Lewin. On a coefficient problem for bi-univalent functions. *Proceedings of the American Mathematical Society*, 18(1):63–68, 1967.

[23] S. Miller and P. Mocabu. *Differential Subordination: Theory and Applications.* CRC Press, New York, 2000.

[24] K. Muthunagai, G. Saravanan, and S. Baskaran. A subclass with bi-univalence involving Horadam polynomials and its coefficient bounds. *Proyecciones Journal of Mathematics*, 40(3):721–730, 2021.

[25] Z. Nehari. *Conformal Mappings.* McGraw-Hill, New York, 1952.

[26] F. Qi, C. Kizilateş, and W.S. Du. A cloed formula for the Horadam polynomials in terms of a tridiagonal Determinant. *Symmetry*, 11(782), 2019.

[27] H.M. Srivastava, S. Altınkaya, and S. Yalcin. Certain Subclasses of Bi-Univalent Functions Associated with the Horadam Polynomials. *Iranian Journal of Science and Technology, Transaction A: Science*, 34(2):1873–1879, 2019.

[28] H.M. Srivastava, M. Kamali, and A. Urdaletova. A study of the Fekete-Szegö functional and coefficient estimates forvsubclasses of analytic functions satisfying a certain subordination conditionv and associated with the Gegenbauer polynomials. *AIMS Mathematics*, 7(2):2568–2584, 2021.

[29] H.M. Srivastava and H.L. Manocha. *A treatise on generating functions.* Halsted Press, John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

[30] S.R. Swamy. Bi-univalent function subclasses subordinate to Horadam polynomials. *Earthline Journal of Mathematical Sciences*, 11, 2023.

[31] E. Szatmari and Ş. Altınkaya. Coefficient estimates and Fekete-Szegö inequality for a class of analytic functions satisfying subordinate condition associated with Chebyshev polynomials. *Acta Universitatis Sapientiae Mathematica*, 11, 2019.