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Generalized Compactness in Fuzzy Bitopological Spaces

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Abstract. The main objective of this research is to study some types of generalized closed sets in fuzzy bitopology including $(i,j)-g\alpha-cld$, $(i,j)-gs-cld$, $(i,j)-gp-cld$, and $(i,j)-g\beta-cld$. We then present basic theorems for determining their relationships and explain their properties, such as closure and interior. In addition, there are many interesting counterexamples. The last part of the research focuses on compactness as an application of the types of fuzzy generalized closed sets in fuzzy bitopological spaces and their types and explores the relationships between these concepts, their important theories, and some relevant counterexamples. This approach provides a better characterization of fuzzy compactness and allows for more precise characterization in fuzzy bitopology. The results of this study are new to the domain of fuzzy bitopology.

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Key Words and Phrases: Fuzzy bitopological spaces ($fbts$), fuzzy generalized closed sets ($(i,j)-g-cld$), fuzzy generalized closure operator ($(i,j)-g-cl$), fuzzy generalized interior operator ($(i,j)-g-int$), fuzzy generalized continuous ($(i,j)-g-conts$), fuzzy generalized irresolute ($(i,j)-g-irres$), and fuzzy generalized compact ($(i,j)-g-compact$)

1. Introduction

In this project, we prioritized our study on fuzzy bitopology, which was derived from a fuzzy topology first introduced in 1965 by Zadeh [23]. Following this, many researchers have applied fundamental ideas on fuzzy settings from a general topology and improved the concept of fuzzy topology. Chang (1968) introduced fuzzy concepts into fuzzy topology [9]. Kandil (1989) introduced fuzzy bitopological spaces [11]. In addition, generalized fuzzy closed sets were established in a fuzzy topology by Balasubramanian and Sundaram in 1997 [7]. Some scholars have presented many important papers on the development types of fuzzy sets; for example, Singal and Prakash presented a study of a fuzzy pre-open set [20]. Balasubramanian developed a theory of fuzzy β open set [6]. Ahmad and Athar

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found important results on fuzzy semi open sets [2]. In addition, Hakeem and Latha introduced new results for fuzzy α open set [15].

Furthermore, extensive research has been conducted on the concept of generalized closed sets in fuzzy space [14, 24]. Subsequently, many studies have introduced the use of generalized closed sets in fuzzy topologies, such as El-Shafei [10]. Some studies have applied these to the concept of functions that contribute to enriching this research area too [13, 18, 19]. On the other hand, earlier research on compactness informed our study of this topic [1, 21]. A recent study discussed the properties of compactness, but in another field, as G-metric spaces [12] and fuzzy soft space as [22]. In addition, Jamal et al. studied several properties of compact space using regular open sets [16].

The study explores the concept of generalized closed sets in fuzzy bitopological spaces, a flexible framework for studying topological properties and partial membership. It provides a smooth transition between open and closed sets, offering a more flexible definition of closure than traditional closed sets. Also, delves into their interrelationships and highlights important theories and counterexamples. Moreover, because the generalized closed sets have many applications in a range of topological concepts, such as neighborhoods, which are discussed and explained in detail in reference [4], they were applied to connectedness as in [3], also to functions as in [5], but we aim to apply them to another topological topic, that is compactness. It provides better characterizations of fuzzy openness and fuzzy compactness and allows for more precise characterizations, which are important properties in fuzzy bitopology.

Finally, the research is organized as follows. The first section (Introduction) looks at the subject's background and related studies. In Section 2 (preliminaries), we briefly discuss several important concepts pertinent to our investigation. The concept of generalized closed sets is presented in Section 3 (Types of Fuzzy Generalized Closed Groups in Fuzzy Bitopology Space), important theorems and distinctive properties are discussed, and some interesting counterexamples are introduced. Then, we provide crucial definitions of fuzzy generalized compactness in Section 4 (Types of Fuzzy Generalized Compactness in Fuzzy Bitopological Spaces). In Section 5 (Conclusion), we summarize our results.

2. Preliminaries

In the following part, we go over important antecedent notions that are essential to the development of this paper.

Definition 1. *[17] Suppose the set X is not empty and the I sign represents the unit period $[0, 1]$, then the following defined as:*

(1) *an operator with X domain and I range is known as a fuzzy set E, where $E(x) \in (0, 1]$ when $x \in E$, and $E(x) = 0$ in case $x \notin E$.*

(2) *a set D is including E indicated via $E \subseteq D$ if $E(x) \leq D(x)$, whenever $x \in X$*

(3) *E and D combination indicated by $E \vee D$ if $(E \vee D)(x) = max\{E(x), D(x)\}$ $\forall$ $x \in X$.*

(4) *the intersection of* E, D *indicated by* $E \wedge D$ *if* $(E \wedge D)(x) = min\{E(x), D(x)\}\ \forall\ x \in X$.

(5) *the completeness of* E *denoted via* E^c *such that* $(E(x))^c = 1 - E(x),\ \forall\ x \in X$.

The following definitions explain the meaning of fuzzy topology and fuzzy bitopological spaces.

Definition 2. *[17] A fuzzy topology of* X *is a class of fuzzy groups* $\delta \in I$ *that holds the coming three conditions:*

1. 0 *and* 1 *contained in* δ, *where* $0(x) = 0,\ 1(x) = 1$, *whenever* $x \in X$.

2. *For any* $E, D \in \delta$, $E \wedge D \in \delta$.

3. *For any* $(E_{i \in I}) \in \delta$, $\vee_{i \in I} E_i \in \delta$.

The term "fuzzy topological space," or "fts," refers to the pair (X, δ).
The components of δ *are named fuzzy open sets. If* $F^c \in \delta$, *a fuzzy set* F *is mean as fuzzy closed. The collection including all fuzzy closed sets in fuzzy topology* δ *denote by* $\mathcal{F}_\delta$.

Definition 3. *[11] A fuzzy bitopological spaces, or fbts for short,* (X, δ_1, δ_2) *since* X *is not empty,* δ_1, *and* δ_2 *are fuzzy topological spaces on* X. *Over this dissertation* X *perform fuzzy bitopology* (X, δ_1, δ_2), *and* Y *to* (Y, σ_1, σ_2), *where* $i \neq j$, *and* $i, j \in \{1, 2\}$.

In the section that follows, the definitions of fuzzy set interiors and closings are covered.

Definition 4. *[17] Closing and internal of any fuzzy set* M *of* (X, δ) *are indicated also defined as follows:*

$$cl(M) = \wedge\ \{F : M \leq F, F^c \in \delta\}$$
$$int(M) = \vee\ \{O : O \leq M, O \in \delta\}, \quad respectively.$$

The closing, internal, and complements of M *of* X *are indicated by* $\delta_i - cl(M)$, $\delta_i - int(M)$, *and* M_i^c, *respectively, with regard to fuzzy topology* δ_i. *Additionally, we designate the class of all fuzzy* δ_j*-closed by the mathematical symbol* $\mathcal{F}_{\delta_j}$.

One of the work's core tenets is the definition of the fuzzy generalized closed set, which as following:

Definition 5. *[7] Any fuzzy set* N *of* X *is termed fuzzy generalized closed when closure* N *is subset of* U, *wherever* N *is subset of* U *and* U *is fuzzy open.*
i.e., N *is fuzzy generalized closed if* $cl(N) \leq U$, *wherever* $N \leq U$, U *is fuzzy open.*

One of the fundamental ideas in this research is continuous and irresolute mapping, in addition to compactness, they are defined as follows:

Definition 6. *[17] Let* (X, δ) *and* (Y, σ) *be an fts and* f *a function from* X *to* Y. *Then* f *is fuzzy* $\delta-$*continuous if and only if* $f^{-1}(V) \in \delta, \forall V \in \sigma$.

Definition 7. *[8] A mapping $f : (X, \delta) \longrightarrow (Y, \sigma)$ is said to be fuzzy $\delta - \alpha - irresolute$ if $f^{-1}(V)$ is fuzzy $\alpha-$open set in X for each fuzzy $\alpha-$open set V in Y.*

Definition 8. *[9]*

(1) *Any fuzzy topology (X, τ) is named fuzzy compact when every fuzzy open covering X has a limited subcover.*

(2) *Any fuzzy set B of (X, τ) is named a fuzzy compact subset of X when every fuzzy open covering B has a limited subcover.*

An important property in the study of compactness is the finite intersection property, which was define as:

Definition 9. *[9] A class $\{A_i\}$ of fuzzy groups of X is entitled having finite intersection characteristic $(in\ sum, F.I.P)$ when all finite subclass $\{A_{i_1}, A_{i_2}, ..., A_{i_n}\}$ has a non empty intersection $A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_n} \neq \phi$*

3. Types of Fuzzy Generalized Closed Classes in Fuzzy Bitopology Space

In the following section, we discuss some types of fuzzy generalized closed groups, theorems, and relationships, and examine their closure and interiors in an fbts.

Definition 10. *Any fuzzy set H of fbts (X, τ_1, τ_2), where $i, j \in \{0, 1\}, i \neq j$ is called:*

(1) *fuzzy $(i, j)-$generalized $\alpha-$closed (in sum, $(i, j) - g\alpha - cld$) if $\tau_j - \alpha cl(H) \leq W$, wherever $H \leq W$, $W \in \tau_i$.*

(2) *fuzzy $(i, j)-$generalized semi$-$closed (in sum, $(i, j) - gs - cld$) if $\tau_j - scl(H) \leq W$, wherever $H \leq W$, $W \in \tau_i$.*

(3) *fuzzy $(i, j)-$generalized pre$-$closed (in sum, $(i, j) - gp - cld$) if $\tau_j - pcl(H) \leq W$, wherever $H \leq W$, $W \in \tau_i$*

(4) *fuzzy $(i, j)-$generalized $\beta-$closed (in sum, $(i, j) - g\beta - cld$) if $\tau_j - \beta cl(H) \leq W$, wherever $H \leq W$, $W \in \tau_i$.*

(5) *the complement of the above sets are called fuzzy $(i, j) - g\alpha - open$, $(i, j) - gs - open, (i, j) - gp - open$, and $(i, j) - g\beta - open$.*

Remark 1. (1) *We denote the class for every fuzzy $(i, j) - g\alpha-$open, $(i, j) - gs-$open, $(i, j) - gp-$open and $(i, j) - g\beta-$open (resp, fuzzy $(i, j) - g\alpha-$cld, $(i, j) - gs-$cld, $(i, j) - gp-$cld, and $(i, j) - g\beta-$cld) sets in (X, τ_i, τ_j) by $\mathcal{O}^{fg\varphi}_{(i,j)}$ and $\mathcal{F}^{fg\varphi}_{(i,j)}$ resp. Also, we gave the names $(i, j)-g\varphi-$cld and $(i, j)-g\varphi-$open to all fuzzy types of generalized closed and open groups, respectively.*

(2) *In all sections of this research $i, j \in \{0, 1\}, i \neq j$*

From the above Definition10 we conclude the following:

Proposition 1. *Any fuzzy subset E of (X, τ_1, τ_2) considered fuzzy $(i, j) - g\varphi-$open $\Leftrightarrow$ $F \leq \tau_j - \varphi - int(E)$, wherever $F \in \mathcal{F}_{\tau_i}$, and $F \leq E$, where $i, j \in \{0, 1\}, i \neq j$.*

Proof. Assume E is fuzzy $(i, j) - g\varphi-$open. Then E^c is $(i, j) - g\varphi-$cld, thus the condition relation is hold for E^c. Therefore, by using the complent we find $\tau_j - \varphi - cl(E^c) = (\tau_j - \varphi - int(E))^c \leq F^c$ which implies $F \leq \tau_j - \varphi - int(E)$. Conversely, by using Definition 10 and taking the complement for both sides in condition we find E^c is fuzzy $(i, j) - g\varphi-$cld. For that E is fuzzy $(i, j) - g\varphi-$open.

In the section that follows, we define the terms "closure" and "interior" of fuzzy generalized closed sets in fbts field, as well as the key theories, connections between these notions, and their complement.

Definition 11. *For all fbts (X, τ_1, τ_2), $E \in I^X$, $(i, j) - g\varphi - closure$ and $(i, j) - g\varphi - interior$ in regard to E are indicated and defined as shown:*

(i) $(i, j) - g\varphi - cl(E) = \wedge \{F : E \leq F, F \text{ is } (i, j) - g\varphi - cld \}$

(ii) $(i, j) - g\varphi - int(E) = \vee \{O : O \leq E, O \text{ is } (i, j) - g\varphi - open \}$.

Theorem 1. *If E is a fuzzy subset of (X, τ_1, τ_2). Then the coming conditions are met:*

(1) $(\ (i, j) - g\varphi - int(E)\ )^c = (i, j) - g\varphi - cl(E^c)$

(2) $(\ (i, j) - g\varphi - cl(E)\ )^c = (i, j) - g\varphi - int(E^c)$.

Proof. It is clear from the complement low and De Morgan theorem.

Theorem 2. *If (X, τ_1, τ_2) is fbts. Then the next statements are satisfied:*

(1) *Every fuzzy $(i, j) - g - cld$ is fuzzy $(i, j) - g\alpha - cld$.*

(2) *Every fuzzy $(i, j) - g\alpha - cld$ is fuzzy $(i, j) - gp - cld$ and fuzzy $(i, j) - gs - cld$.*

(3) *Every fuzzy $(i, j) - gp - cld$ or fuzzy $(i, j) - gs - cld$ is fuzzy $(i, j) - g\beta - cld$.*

Proof. It is clear from Definition 10 and the relations between types of fuzzy sets where

$$\begin{array}{ccccc} open & \longrightarrow & (\alpha\ open) & \longrightarrow & (s\ open) \\ & & \downarrow & & \downarrow \\ & & (pre\ open) & \longrightarrow & (\beta\ open) \end{array}$$

Remark 2. *The following diagram explaining the relations between all types generalized closed sets in (X, τ_i, τ_j), $i, j \in \{0, 1\}, i \neq j$:*

$$
\begin{array}{ccc}
\tau_j - cld & & \\
\downarrow & & \\
(i,j) - g - cld & & \\
\downarrow & & \\
(i,j) - g\alpha - cld & \longrightarrow & (i,j) - gp - cld \\
\downarrow & & \downarrow \\
(i,j) - gs - cld & \longrightarrow & (i,j) - g\beta - cld
\end{array}
$$

Figure 1: Explain the relations between $(i,j) - g\varphi-$cld sets.

The examples that follow demonstrate that the above diagram's opposite is not typically true.

Example 1. *Suppose* $E, H, R,$ *and* S *fuzzy subsets of* $X = \{a, b\}$ *as follows:*
$E(a, b) = \{0.7, 0.5\}$, $H(a, b) = \{0.7, 0.6\}$, $R(a, b) = \{0.2, 0.4\}$, $S(a, b) = \{0.6, 0.6\}$.
Assume $\tau_1 = \{0, 1, E\}$, *and* $\tau_2 = \{0, 1, H, R\}$. *Then we can see that* S *is fuzzy* $(1, 2)-g-cld$ *never fuzzy* $\tau_2 - g-cld$, *since* $S \leq H \in \tau_2$ *and* $cl_2(S) \nleq H$.

The coming example show that fuzzy $(1, 2) - g\alpha - cld \nRightarrow$ fuzzy $(1, 2) - g - cld$.

Example 2. *Suppose* $E, H, R,$ *and* S *fuzzy subsets of* $X = \{a, b\}$ *as follows:*
$E(a, b) = \{0.5, 0.4\}$, $H(a, b) = \{0.7, 0.5\}$, $R(a, b) = \{0.4, 0.3\}$, $S(a, b) = \{0.3, 0.4\}$.
Assume $\tau_1 = \{0, 1, E\}$, *and* $\tau_2 = \{0, 1, H, R\}$. *Then we can see that* S *is fuzzy* $(1, 2) - g\alpha-cld$ *never fuzzy* $(1, 2) - g-cld$, *since* $S \leq E \in \tau_1$, *and* $cl_2(S) = H^c \nleq E$.

In the following example we explain that fuzzy $(1, 2)-gs-cld \nRightarrow$ fuzzy $(1, 2)-g\alpha-cld$.

Example 3. *Suppose* $E, H, R,$ *and* S *fuzzy subsets of* $X = \{a, b\}$ *as follows:*
$E(a, b) = \{0.7, 0.5\}$, $H(a, b) = \{0.5, 0.4\}$, $R(a, b) = \{0.4, 0.3\}$, $S(a, b) = \{0.5, 0.5\}$.
Assume $\tau_1 = \{0, 1, E\}$, *and* $\tau_2 = \{0, 1, H, R\}$. *Then we can see that* S *is fuzzy* $(1, 2) - gs-cld$ *never fuzzy* $(1, 2) - g\alpha-cld$, *since* $S \leq E \in \tau_1$, *and* $\alpha - cl_2(S) = H^c \nleq E$.

The next example shows that fuzzy $(1, 2) - gp - cld \nRightarrow$ fuzzy $(1, 2) - g\alpha - cld$.

Example 4. *Suppose* $E, H, R,$ *and* S *fuzzy subsets of* $X = \{a, b\}$ *as follows:*
$E(a, b) = \{0.7, 0.5\}$, $H(a, b) = \{0.6, 0.8\}$, $R(a, b) = \{0.4, 0.3\}$, $S(a, b) = \{0.2, 0.4\}$.
Assume $\tau_1 = \{0, 1, E\}$, *and* $\tau_2 = \{0, 1, H, R\}$. *Then we can see that* S *is fuzzy* $(1, 2) - gp-cld$ *never fuzzy* $(1, 2) - g\alpha-cld$, *since* $S \leq E \in \tau_1$, *and* $\alpha - cl_2(S) = R^c \nleq E$.

The example below indicates that fuzzy $(1, 2) - g\beta - cld \nRightarrow$ fuzzy $(1, 2) - gs - cld$.

Example 5. *Suppose* $E, H, R,$ *and* S *fuzzy subsets of* $X = \{a, b\}$ *as follows:*
$E(a, b) = \{0.5, 0.7\}$, $H(a, b) = \{0.6, 0.5\}$, $R(a, b) = \{0.4, 0.3\}$, $S(a, b) = \{0.5, 0.5\}$.
Assume $\tau_1 = \{0, 1, E\}$, *and* $\tau_2 = \{0, 1, H, R\}$. *Then we can see that* S *is fuzzy* $(1, 2) - g\beta-cld$ *never fuzzy* $(1, 2) - gs-cld$, *since* $S \leq E \in \tau_1$, *and* $s - cl_2(S) = F(a, b) = \{0.6, 0.5\} \nleq E$.

Likewise, the following example shows that fuzzy $(1,2)-g\beta-cld \nRightarrow$ fuzzy $(1,2)-gp-cld$.

Example 6. *Suppose $E, H, R,$ and S fuzzy subsets of $X = \{a, b\}$ as follows:*
$E(a,b) = \{0.5, 0.7\}$, $H(a,b) = \{0.4, 0.6\}$, $R(a,b) = \{0.3, 0.4\}$, $S(a,b) = \{0.5, 0.5\}$.
Assume $\tau_1 = \{0, 1, E\}$, and $\tau_2 = \{0, 1, H, R\}$. Then we can see that S is fuzzy $(1,2)-g\beta-cld$ never fuzzy $(1,2)-gp-cld$, since $S \leq E \in \tau_1$, and $p-cl_2(S) = F(a,b) = \{0.6, 0.5\} \nleq E$.

Theorem 3. *Assume (X, τ_1, τ_2) is fbts and E is fuzzy τ_i-open $(resp, \tau_i - cld)$. Then, the statements below are equal:*

(1) *E is fuzzy $(i,j)-g\varphi-cld$ $(resp, fuzzy\ (i,j)-g\varphi-open)$.*

(2) *E is fuzzy $\tau_j-\varphi-cld$ $(resp, fuzzy\ \tau_j-\varphi-open)$.*

Proof. Suppose $E \in \tau_i$, and fuzzy $(i,j)-g\varphi-$cld. Then $\tau_j-\varphi-cl(E) \leq E$, and hence E is fuzzy $\tau_j-\varphi-$cld. Conversely, it is obvious in Theorem 2, also from Figure (1).

Theorem 4. *Let $E \in \tau_i$ and be fuzzy $(i,j)-g\alpha-cld$. Then $E \wedge F$ is fuzzy $(i,j)-g\varphi-cld$, wherever $F \in \mathcal{F}_{\tau_j}$.*

Proof. As $E \in \tau_i$, and fuzzy $(i,j)-g\alpha-$cld, then by Theorem 3 E is fuzzy $\tau_j-\alpha-$cld. After that, $E \wedge F$ is fuzzy $\tau_j-\alpha-$cld, which implies that it is fuzzy $(i,j)-g\alpha-$cld. Therefore by Figure (1) we conclude $E \wedge F$ is fuzzy $(i,j)-g\varphi-$cld.

Corollary 1. *Suppose $A \in \mathcal{F}_i$, and fuzzy $(i,j)-g\alpha-open$. Thereafter $A \vee F$ is fuzzy $(i,j)-g\varphi-$open, whenever $F \in \tau_j$.*

Theorem 5. *Finite union of fuzzy $(i,j)-g\varphi-cld$ of (X, τ_1, τ_2) is fuzzy $(i,j)-g\varphi-cld$.*

Proof. Assume E, and D are fuzzy $(i,j)-g\varphi-$cld in fbts (X, τ_1, τ_2). Then $E \vee D$ is fuzzy $(i,j)-g\varphi-$cld. It follows from the fact $\tau_j-\varphi-cl(E \vee D) = \tau_j-\varphi-cl(E) \vee \tau_j-\varphi-cl(D)$.

Corollary 2. *If E, and D are fuzzy $(i,j)-g\varphi-$open. Thereafter $E \wedge D$ is fuzzy $(i,j)-g\varphi-$open.*

Remark 3. *The finite intersection of fuzzy $(i,j)-g\varphi-cld$ in fbts (X, τ_1, τ_2) is not fuzzy $(i,j)-g\varphi-cld$ in general.*
We show that by the following example for the specific type that is fuzzy $(i,j)-g\alpha-cld$.
Suppose $E, H, R, D_1,$ and D_2 are fuzzy subsets of $X = \{a, b\}$ as below:
$E(a,b) = \{0.6, 0.6\}$, $H(a,b) = \{0.7, 0.8\}$, $R(a,b) = \{0.6, 0.7\}$, $D_1(a,b) = \{0.5, 0.7\}$,
$D_2(a,b) = \{0.8, 0.5\}$. *Assume $\tau_1 = \{0, 1, E\}$, and $\tau_2 = \{0, 1, H, R\}$. Then D_1 and D_2 are fuzzy $(1,2)-g\alpha-$ cld, but $D_1 \wedge D_2$ is not fuzzy $(1,2)-g\alpha-cld$.*

Corollary 3. **(1)** *The finite itersection of fuzzy $(i,j)-g\varphi-open$ in fbts (X, τ_1, τ_2) is fuzzy $(i,j)-g\varphi-open$.*

(2) *The finite union of fuzzy $(i,j)-g\varphi-open$ in fbts (X, τ_1, τ_2) is not fuzzy $(i,j)-g\varphi-open$ in general.*

4. Types of Fuzzy Generalized Compactness in Fuzzy Bitopological Spaces

This section introduces the idea of generalized compactness in fuzzy bitoplogy and characterize it in terms of important theorems and some properties.

Definition 12. *The space X of fbts (X, δ_1, δ_2) is named fuzzy $(i, j) - g\varphi-$compact when all fuzzy $(i, j) - g\varphi-$open cover for X has a finite subcover.*
In addition, A fuzzy subset A of fbts (X, δ_1, δ_2) is called fuzzy $(i, j) - g\varphi-$compact subset of X when all fuzzy $(i, j) - g\varphi-$open cover for A has a finite subcover.

Example 7. *Suppose $A(a, b) = \{0.5, 0.5\}$ is fuzzy subset of $X = \{a, b\}$, and the fuzzy topologies $\delta_1 = \{0, 1\}$, $\delta_2 = \{0, 1, A\}$. Then X is fuzzy $(1, 2) - g\varphi-$compact space. Furthermore, A is fuzzy $(1, 2) - g\varphi-$compact subset of X.*

Corollary 4. *In any fbts (X, δ_1, δ_2) if δ_i is a fuzzy indiscrete topology, then (X, δ_1, δ_2) is fuzzy $(i, j) - g\varphi-$compact, and any subset of it is fuzzy $(i, j) - g\varphi-$compact.*

Theorem 6. *All fuzzy $(i, j) - g\varphi-$cld subset of fuzzy $(i, j) - g\varphi-$compact space is $(i, j) - g\varphi-$compact.*

Proof. Assume E is fuzzy $(i, j)-g\varphi-$cld, and $\{G_i : i \in I\}$ is fuzzy $(i, j)-g\varphi-$open cover for E. Then, E^c is fuzzy $(i, j) - g\varphi-$open, and hence $\{G_i, E^c : i \in I\}$ is $(i, j) - g\varphi-$open cover for X. Then $\exists$ finite subcover to X, which is $\{G_{i_j}, E^c : j = 1, 2, ..., n\}$, and hence $\exists$ finite subcover of E, which is $\{G_{i_j} : i \in I, j = 1, 2, ..., n\}$. Therefore, E is fuzzy $(i, j) - g\varphi-$compact.

Corollary 5. *All fuzzy δ_j-cld subset of fuzzy $(i, j) - g\varphi-$compact space is fuzzy $(i, j) - g\varphi-$compact too.*

Theorem 7. *If (X, δ_1, δ_2) is fuzzy $(i, j) - g\varphi-$compact space, thus it is fuzzy δ_j-compact space.*

Proof. Suppose $\{G_j : j \in I\}$ is an open cover of (X, δ_j). Then from $Figure(1)$ and Theorm 2, $\{G_i : i \in I\}$ is consider fuzzy $(i, j) - g\varphi-$open cover to X, after that $\{G_i\}$ has finite subcover. Therefore, X is fuzzy δ_j-compact space.

Theorem 8. *If (X, δ_1, δ_2) is fuzzy δ_i-cld and δ_j-compact space. After that, it is fuzzy $(i, j) - g\varphi-$compact.*

Proof. Assume $\{G_i : i \in I\}$ is fuzzy $(i, j) - g\varphi-$open cover for X. As X is δ_i-cld, then by Theorem 3 $\{G_i : i \in I\}$ is fuzzy δ_j-open cover to X, but X is δ_j-compact, after that $\exists$ finite subcover. Therefore X is fuzzy $(i, j) - g\varphi-$compact space.

Theorem 9. *In fbts (X, δ_1, δ_2). The next explanations are true:*

(1) $\forall$ *fuzzy $(i, j)-g\beta-$compact is fuzzy $(i, j)-gp-$compact and fuzzy $(i, j)-gs-$compact.*

(2) $\forall$ *fuzzy* $(i,j)-gp-$*compact or fuzzy* $(i,j)-gs-$*compact is fuzzy* $(i,j)-g\alpha-$*compact.*

(3) $\forall$ *fuzzy* $(i,j)-g\alpha-$ *compact is fuzzy* δ_j-*compact.*

Proof. Obviously from Definition 12 and the relations between types of $(i,j)-g\varphi-cld$ sets in Theorem 2 and *Figure* (1).

The diagram below explains the relationships between all types of fuzzy $(i,j)-g\varphi-$compact:

$$\begin{array}{ccc} (i,j)-g\beta-compact & \longrightarrow & (i,j)-gp-compact \\ \downarrow & & \downarrow \\ (i,j)-gs-compact & \longrightarrow & (i,j)-g\alpha-compact \\ & & \downarrow \\ & & \delta_j-compact \end{array}$$

Figure 2: Explain the relations between $(i,j)-g\varphi-$compact.

Remark 4. *In general, the opposite of the aforementioned graph is not true, and this is clear from Definition 12 and the relations between* $(i,j)-g\varphi-cld$ *sets in Theory 2 and Examples 2 through 1. In addition, we can see that the concepts of the fuzzy* $(i,j)-gp-$*compact space and* $(i,j)-gs-$*compact space are independent.*

Theorem 10. *If* E, D *are fuzzy* $(i,j)-g\varphi-$*compact subsets of* (X,δ_1,δ_2). *Then* $E\wedge D$ *is fuzzy* $(i,j)-g\varphi-$*compact.*

Proof. Suppose $\{G_i : i\in I\}$ is fuzzy $(i,j)-g\varphi-$open cover of $E\wedge D$. Since $E\wedge D\leq E$, and $E\wedge D\leq D$, then $\{G_i : i\in I\}\leq\{U_i : i\in I, such\ that\ E\leq\cup_{i=1}U_i\}\wedge\{V_i : i\in I, such\ that\ D\leq\cup_{i=1}V_i\}$. Then by Corollary 2, and as E,D are fuzzy $(i,j)-g\varphi-$compact, then $\{G_i\}$ has a finite subcover $\{G_{i_j} : j=1,2,...,n\}$. Therefore $E\wedge D$ is fuzzy $(i,j)-g\varphi-$compact.

Definition 13. *A mapping* $f:(X,\delta_1,\delta_2)\rightarrow(Y,\sigma_1,\sigma_2)$ *is named fuzzy* $(i,j)-$*generalized* $\varphi-$*continuous (shortly,* $(i,j)-g\varphi-conts$*) when the opposite image of each fuzzy open of* (Y,σ_j) *is fuzzy* $(i,j)-g\varphi-open$ *of* X.

By using the complement, we find:

Theorem 11. *Suppose* $f:(X,\delta_1,\delta_2)\rightarrow(Y,\sigma_1,\sigma_2)$. *Then* f *is fuzzy* $(i,j)-g\varphi-conts\Leftrightarrow$ $\forall$ *fuzzy closed set* V *at* $(Y,\sigma_j), f^{-1}(V)$ *is fuzzy* $(i,j)-g\varphi-cld$ *set at* X.

Theorem 12. *The portrait* $(i,j)-g\varphi-conts$ *of fuzzy* $(i,j)-g\varphi-$*compact is fuzzy* δ_j-*compact.*

Proof. Suppose $f:(X,\delta_1,\delta_2)\rightarrow(Y,\sigma_1,\sigma_2)$ is fuzzy $(i,j)-g\varphi-conts$, surjective mapping, and (X,δ_1,δ_2) is fuzzy $(i,j)-g\varphi-$compact space. Assume that $\{B_j : j\in I\}$ is

δ_j-open cover for Y, thus $\{f^{-1}(B_j) : j \in I\}$ is fuzzy $(i,j) - g\varphi-$open cover for X, then it has finite subcover for X, and since f is surjective mapping, so $\exists$ $\{B_1, B_2, ..., B_n\}$ finite subcover for Y. Therefore Y is fuzzy δ_j-compact.

Corollary 6. *The δ_j-continuous image of $(i,j) - g\varphi-$compact is δ_j-compact.*

Definition 14. *A mapping $f : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$ is named fuzzy $(i,j)-$generalized $\varphi-$ irresolute mapping $(shortly, (i,j) - g\varphi - irres)$ when the opposite image of all fuzzy $(i,j) - g\varphi - open$ set of X is fuzzy $(i,j) - g\varphi - open$ of Y.*

By using the complement we find:

Theorem 13. *Suppose $f : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$. Then f is fuzzy $(i,j) - g\varphi - irres \Leftrightarrow$ for all fuzzy $(i,j) - g\varphi-cld$ V at Y, $f^{-1}(V)$ is fuzzy $(i,j) - g\varphi - cld$ at X.*

Theorem 14. *If $f : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$ is fuzzy $(i,j) - g\varphi - irres$ mapping, and E is fuzzy $(i,j) - g\varphi-$compact set of X. Thus $f(E)$ is fuzzy $(i,j) - g\varphi-$compact of Y.*

Proof. Suppose $f : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$ is fuzzy $(i,j) - g\varphi - irres$, onto mapping, also
$\{V_i : i \in I\}$ is fuzzy $(i,j) - g\varphi-$open cover of $f(E)$. As f is onto, then
$f(E) \leq f(\cup_{j=1}^{n} f^{-1}(V_{i_j})) \leq \cup_{j=1}^{n} V_{i_j}$. So, $f(E)$ is fuzzy $(i,j) - g\varphi-$compact at Y.

Corollary 7. *When $f : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$ is fuzzy $(i,j) - g\varphi - irres$, onto mapping, and X is fuzzy $(i,j) - g\varphi-$compact. After that, Y is fuzzy $(i,j) - g\varphi-$compact.*

Theorem 15. *If (X, δ_1, δ_2) is fbts. So X is fuzzy $(i,j) - g\varphi-$compact $\Leftrightarrow$ $\forall$ $\{F_i\}$ of fuzzy $(i,j) - g\varphi-$cld sets of X satisfying F.I.P $(Definition\ 9)$ has itself a non empty intersection.*

Proof. Suppose (X, δ_1, δ_2) is fuzzy $(i,j) - g\varphi-$compact, $\{F_i : i \in I\}$ is fuzzy $(i,j) - g\varphi-$cld sets of X satisfying F.I.P, and $\cap\{F_i : i \in I\} = \phi$. Then $X = \cup\{{F_i}^c : i \in I\}$. let $U = ({F_i}^c)$ is fuzzy $(i,j) - g\varphi-$open cover for X. As X is fuzzy $(i,j) - g\varphi-$compact, then U most contain finite subcover of X and $X = (\cap_{j=1}^{n} F_{i_j})^c$, which implies $\cap_{j=1}^{n} F_{i_j} = \phi$. This runs counter to the hypothesis that F_i has F.I.P.
Conversely, assume X is not compact and $\cap$ $\{F_i : i \in I\} \neq \phi$, where $\{F_i\}$ is collection of fuzzy $(i,j) - g\varphi-$cld subsets at X has F.I.P. Then $\exists$ $U = \{G_i : i \in I\}$ is $(i,j) - g\varphi-$open cover for X, that is lacking finite subcover of X, then $\{X - G_{i_1}, X - G_{i_2}, ..., X - G_{i_n}\}$ is a class of $(i,j) - g\varphi-$cld sets has F.I.P, and hence $\cap$ $\{X - G_i\} = X - \cup$ $G_i \neq \phi$, then $X \neq \cup_{i=1} G_i$. The reality that U is a fuzzy $(i,j) - g\varphi-$open cover for X is in conflict with this.

5. Conclusion and Future Studies

In this research, we explore the relationships between different types of generalized closed sets in a new domain, which is a fuzzy bitopological space. In addition, we explored the interconnections between these sets by some counterexamples. After that, we scrutinize the fundamental theorems and distinctive characteristics associated with these concepts. Also, we applied them to fuzzy compactness and studied their theorems, properties, and relationships. Through this in-depth analysis, we contribute to a better comprehension of these key ideas in the context of fuzzy bitopological spaces. This work also opens up new horizons for the future study of these sets in other fields of fuzzy sets, such as regular sets, study them in more than two topologies, or in another domain, such as fuzzy soft spaces.

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