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A Novel Numerical Scheme for Fractional Bernoulli Equations and the Rössler Model: A Comparative Analysis using Atangana-Baleanu Caputo Fractional Derivative

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Abstract. This study aims to use a novel scheme for the Atangana-Baleanu Caputo fractional derivative (ABC-FD) to solve the fractional Bernoulli equation and the fractional Rössler model. Furthermore, the suggested technique is compared to Runge-Kutta Fourth Order (RK4). The proposed method is efficacious and generates solutions that are indistinguishable from the approximate solutions generated by the RK4 method. Therefore, we can adapt the approach to various systems and develop results that are more accurate. On top of that, the new technique (ABC-FD) can identify chaotic situations. Consequently, this approach can be used to enhance the performance of other systems. In the future, this technique can be employed to determine the numerical solution for a multitude of models applicable in the fields of science and engineering.

2020 Mathematics Subject Classifications: 97M40

Key Words and Phrases: Numerical solution, the Atangana-Baleanu fractional derivative, Initial value problems, Chaos

1. Introduction

Fractional calculus, a science that deals with non-integer order derivatives and integrals has attracted significant attention from a variety of scientific disciplines [42]. Fundamental concepts in fractional calculus were first explained by Oldham and Spanier [42], Gorenflo and Mainardi [22] and Samko et al. [29]. These contributions have had a significant influence on the comprehension and simulation of intricate dynamic systems.

Numerous studies have focused on chaotic behavior within the framework of fractional calculus, as it is a feature shared by many nonlinear systems. Dudkowski et al. [11] brought attention to the fact that dynamical systems might have hidden attractors, highlighting the significance of correctly identifying chaotic regimes. The range of fractional

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calculus applications in diverse domains has been expanded by recent studies by Salah et al. [12], Abdoon et al. [13] and Abu-Ghuwaleh et al. [2], which highlighted the efficacy of fractional methods in solving multi-dimensional equations, influenza modeling, and integrating master theorems, respectively.

There has been a lot of study into the use of fractional calculus approaches to try to understand and regulate engineering systems that exhibit chaotic behavior. Guma et al. [23], Almutairi et al. [4], Xu et al. [20], and Lai et al. [38] show how fractional approaches can be used to study healthcare risk factors, validate efficiency models, and create chaotic systems with multiple attractors, all of which add to our knowledge of chaotic dynamics. Analysing complex circuits [36], synchronizing reaction-diffusion systems [24], and modelling predator-prey dynamics [40] are only a few of the many engineering applications that have connected with fractional calculus.

The methods outlined in the literature have demonstrated broad applicability across diverse domains, showcasing their effectiveness in advancing technologies, these methodologies have found resonance in robotics [9], differential equations [39], image processing [21], control systems [35], solving partial differential equations [25, 31] and various other interdisciplinary arenas. They have facilitated significant strides in Controlling robotic manipulators with fractional-order modeling [9], offering innovative schemes in order to solve fractional differential equations and chaotic systems [39]. Moreover, these techniques have contributed to the development of cutting-edge edge-detection methodologies using fractional derivatives with with kernels that are both nonlocal and nonsingular [21]. In the realm of control systems, these methods have enabled the control of fractional-order systems such as Chua's system [35]. Additionally, they have been instrumental in deriving solutions for intricate mathematical models like the generalized Zakharov equation and initial value problems involving generalized fractional derivatives [25, 31]. Furthermore, recent studies [5, 14, 15, 44, 46] have expanded upon these methodologies, presenting numerical solutions, comparative studies, and applications encompassing symmetric attractors, electric circuits, image encryption, and stability analyses in control systems and image processing. The adaptability and versatility of these methods continue to pave the way for advancements in multidisciplinary fields, illustrating their profound impact on diverse technological landscapes.

The significance of fractional differential equations lies in their ability to accurately capture non-integer order derivatives, making them essential for understanding diverse phenomena in science and engineering. The ABC-FD, as a versatile tool within this framework, is introduced and compared to the Runge-Kutta Fourth Order (RK4) method. The research aims to demonstrate the efficacy of the ABC-FD by generating solutions indistinguishable from RK4, highlighting its adaptability to various systems, and showcasing its ability to identify chaotic situations. The anticipated contributions include enhanced accuracy in solving fractional differential equations and the potential to improve numerical solutions for a multitude of models applicable in scientific and engineering fields, thereby advancing the understanding and application of fractional calculus in diverse domains.

In this study, our proposed formulation distinguishes itself from the work presented in Atangana and Qureshi see [8] in both the introduction and numerical scheme sections. In the introduction, we acknowledge the contributions of Atangana and Qureshi, particularly in their application of fractal-fractional operators to chaotic dynamical systems. Our study builds upon this foundation by introducing a novel numerical scheme that extends and refines their approach. Specifically, in the numerical scheme section, we detail the key distinctions, highlighting modifications in the application of fractal-fractional operators and innovative techniques that contribute to the effectiveness of our proposed method. This comparative discussion aims to underscore the uniqueness and advancements of our formulation within the broader context of fractional calculus and chaotic dynamical systems research.

This study presents a pioneering advancement in the application of fractional calculus, specifically focusing on refining the Atangana-Baleanu Caputo fractional derivative (ABC-FD) approach. The novelty lies in the fine-tuning of ABC-FD parameters, enabling enhanced accuracy in solving the fractional Bernoulli equation and simulating the Rössler model. Moreover, our research uniquely integrates multidisciplinary concepts, demonstrating the applicability of fractional calculus in diverse engineering domains. Validation through real-world case studies showcases the effectiveness of our approach in accurately predicting chaotic behaviour in complex engineering systems. Comparative analyses against existing methods underscore the superior performance and robustness of the proposed ABC-FD scheme. This introduction provides a comprehensive overview of the foundational works in fractional calculus, emphasizing their relevance in studying chaotic behaviour in engineering systems. It sets the stage for the specific focus on the ABC-FD approach and its potential applications in engineering contexts.

This has led to the description of many chaotic systems in published literature, all of which use circuits as their implementation. One of the most significant areas of research is to construct robust chaotic oscillators with simple architecture, either mathematically or circuit. This is one of the most crucial areas for research. Chaotic systems behave in unpredictable and complex ways over time because they are sensitive to initial conditions.

One of the numerous novel methods for studying chaotic systems that have emerged in recent years is asymptotic stability, which pinpoints the precise nature of chaos. The areas of mathematics and science covered by fractional calculus [30, 47] are incredibly diverse; cutting-edge applications are being developed in mathematics, biology, and other subjects [16, 17].

An innovative technique (ABC-FD) was first introduced and in [18]. Developing a fresh method for this derivative can increase its application in a few industries and give significant advantages. Such an innovative plan could have the following possible advantages: enhanced flexibility and precision, enhanced management of intricate systems, and more relevance to actual occurrences. Making use of a new method (ABC-FD) is also new. Choosing the right number method for fractional derivatives is very important for getting a good picture of how the system works. When you use fractional derivatives instead of integer derivatives, you can get different kinds of attractors, bifurcation patterns, and temporary behaviors. It's new to point out and analyze these new processes. Fractional order systems have been used in many areas, as example cryptography, safe communications, and sending data securely.

Investigating The Rössler model, renowned for its ability to depict chaotic dynamics, finds extensive application across scientific, engineering, and interdisciplinary realms. In scientific exploration, it serves as a foundational illustration elucidating chaos theory's fundamental principles, shedding light on nonlinear behaviors, bifurcations, and sensitivity to initial conditions. Engineering domains leverage its chaotic nature for testing control algorithms in systems susceptible to erratic oscillations, such as electronic circuits or chemical reactors. Additionally, the model's chaotic signals contribute to encryption methods and signal processing techniques, ensuring secure communication. Its utility extends to simulating complex natural phenomena like fluid dynamics or biological rhythms while also offering insights into economic and social systems' nonlinear behaviors. This versatility positions the Rössler model as an invaluable tool for educational purposes and as a framework for understanding chaotic dynamics in diverse real-world scenarios.

The implications of the Atangana-Baleanu Caputo fractional derivative approach extend to various engineering domains. The capacity to solve fractional differential equations and represent chaotic systems with high accuracy could have applications in many fields, including control theory, signal processing, and communication networks. Engineers may be able to construct more resilient systems that can deal with nonlinear dynamics and improve the performance of current systems by using the ABC-FD technique. The application of the Atangana-Baleanu Caputo fractional derivative (ABC-FD) method in engineering presents a promising avenue for accurately modeling intricate systems and identifying chaotic tendencies. Its engineering applications are manifold: in communication systems, it could refine protocols for better data transmission in unpredictable settings; in control systems, it might aid in devising strategies to manage nonlinear dynamics, crucial in robotics and industrial automation. Moreover, its potential in signal processing could revolutionize data extraction and analysis in diverse fields, while optimizing system design and enabling predictive maintenance across industries. Biomedical engineering stands to benefit from its insights into biological systems, potentially advancing medical diagnostics and treatment planning. Similarly, in renewable energy, its predictive capabilities could optimize the integration of intermittent sources, contributing to the efficiency of power grids. Overall, the ABC-FD method's ability to model complex systems and discern chaotic behavior heralds' innovation across varied engineering domains, offering solutions to multifaceted challenges.

A modified version of the classical Bernoulli equation namely the fractional Bernoulli equation is a modification that applies notions from fractional calculus to the original Bernoulli equation. In the discipline of fluid mechanics and other related fields, it is utilized to describe the behavior of fluids in non-Newtonian and complicated systems where fractional derivatives play an important role.

There are many different situations in which the fractional Bernoulli equation can be applied, including the following: Bernoulli's equation, in its traditional form, assumes that fluids behave in a Newtonian manner. However, many fluids, such as viscoelastic or power-law fluids, behave in a manner that is not Newtonian. In order to account for these difficulties, the fractional Bernoulli equation was developed.

When you use fractional calculus, you can describe how fluid flows through porous media that have fractal properties. This is called fractal flow media. The use of this equation allows for a more realistic modeling of flow dynamics in materials of this kind.

The study of biofluid dynamics is important because biological systems frequently incorporate fluids that behave in a non-Newtonian manner. Taking into consideration the non-Newtonian properties of blood, fractional calculus can be used to provide a more precise description of blood flow in vessels, for example.

Anomalous Transport Phenomena: Fractional calculus can be used to describe systems that show anomalous diffusion or transport. This is when particles move in a way that doesn't follow the usual rules of diffusion. Using this equation, one can better analyze and comprehend the occurrence of such phenomena.

Additionally, fractional derivatives and integrals are incorporated into the fractional Bernoulli equation, which is an extension of the traditional Bernoulli equation that allows it to handle more complicated situations. There is a wide range of applications for it, particularly in areas that deal with complex fluid dynamics, porous media, and systems that exhibit non-Newtonian behavior.

This study is driven by the need for innovative solutions in numerically handling complex fractional differential equations, specifically the fractional Bernoulli equation and the fractional Rössler model. To address this, we introduce the Atangana-Baleanu Caputo fractional derivative (ABC-FD) method and aim to provide a numerical solution surpassing conventional methods, as demonstrated through a comparative analysis with the Runge-Kutta Fourth Order (RK4) method. Leveraging MATLAB, our focus extends beyond solution generation to emphasize the crucial aspect of maintaining numerical stability in fractional procedures. The study's objectives include showcasing the ABC-FD method's accuracy and stability while highlighting its potential applications in signal processing, control theory, communication systems, and other engineering fields. By doing so, we contribute valuable tools and insights to advance the understanding and application of fractional calculus.

The rest of this paper is organized as follows. In Section 2, we start by presenting some preliminaries and basic definitions which will be needed in the sequel. In section 3, we shall present a numerical scheme for the ABC fractional derivative. In section 4, the applications of the ABC-FD scheme are illustrated. In section 5 we shall discuss the numerical result. Finally, in section 6, we present our study's conclusions.

2. Preliminaries and basic definitions

Firstly, we present briefly the fractional operators that are needed in the sequel.

Definition 1. *Let $q \in [1, \infty)$ and Ω be open subset of $\mathbb{R}$, the Sobolev space $H^q(\Omega)$ is defined by [3]:*

$$H^q(\Omega) = \{f \in\ L^2(\Omega) :\ D^\beta f \in L^2(\Omega),\ forall\ |\beta| \leq q\}. \tag{1}$$

Definition 2. *The Atangana-Baleanu Caputo (ABC) fractional derivative of a function*

$y(\tau) \in H^1(0,c),\ c > 0$ with $\alpha \in [0,1]$ is defined as [7]:

$$\begin{matrix} ABC \\ 0 \end{matrix} D_t^{\alpha} y(t) = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_0^t y(\tau) E_{\alpha} \left(-\frac{\alpha}{1-\alpha} (t-\tau)^{\alpha} \right) d\tau,\ 0 \le \alpha \le 1, \tag{2}$$

where $B(\alpha)$ denotes a normalization function obeying $B(0) = B(1)$.

Definition 3. *The Mittag-Leffler function can be expressed as follows [37]:*

$$E_{\alpha}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)} \tag{3}$$

Definition 4. *The Atangana-Baleanu fractional integral of a function $y(\tau) \in H^1(0,c)$, $c > 0$ is as follows [7]:*

$${}_0^{AB} I_t^{\alpha} y(t) = \frac{1-\alpha}{M(\alpha)} y(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t y(\tau)(t-\tau)^{\alpha-1} d\tau,\ 0 \le \alpha \le 1. \tag{4}$$

3. 3. Numerical scheme for the ABC-FD

This section investigates a novel scheme for the (ABC-FD), of the form that was first presented in [43]:

$$\begin{cases} \begin{matrix} ABC \\ 0 \end{matrix} D_t^{\alpha} x(t) = E(t,\ x(t)), \\ \begin{matrix} ABC \\ 0 \end{matrix} D_t^{\alpha} y(t) = H(t,\ y(t)), \\ \begin{matrix} ABC \\ 0 \end{matrix} D_t^{\alpha} z(t) = P(t,\ z(t)). \end{cases} \tag{5}$$

Through the application of the fundamental theorem of fractional calculus, we lead to:

$$x(t) - x(0) = \frac{1-\alpha}{ABC(\alpha)} E(t,\ x(t)) + \frac{\alpha}{\Gamma(\alpha) ABC(\alpha)} \int_0^t E(\tau,\ x(\tau))(t-\tau)^{\alpha-1} d\tau, \tag{6}$$

$$y(t) - y(0) = \frac{1-\alpha}{ABC(\alpha)} H(t,\ y(t)) + \frac{\alpha}{\Gamma(\alpha) ABC(\alpha)} \int_0^t H(\tau,\ y(\tau))(t-\tau)^{\alpha-1} d\tau, \tag{7}$$

$$z(t) - z(0) = \frac{1-\alpha}{ABC(\alpha)} P(t,\ z(t)) + \frac{\alpha}{\Gamma(\alpha) ABC(\alpha)} \int_0^t P(\tau,\ z(\tau))(t-\tau)^{\alpha-1} d\tau. \tag{8}$$

At the point $t_{n+1},\ n = 0, 1, 2 \ldots,$, the above equation is reformulated as

$$\begin{aligned} x(t_{n+1}) - x(0) =& \frac{1-\alpha}{ABC(\alpha)} E(t_n,\ x(t_n)) + \frac{\alpha}{ABC(\alpha)\Gamma(\alpha)} \int_0^{t_{n+1}} E(\tau,\ x(\tau))(t_{n+1}-\tau)^{\alpha-1} d\tau \\ =& \frac{1-\alpha}{ABC(\alpha)} E(t_n,\ x(t_n)) + \frac{\alpha}{\Gamma(\alpha) ABC(\alpha)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} E(\tau,\ x(\tau))(t_{n+1}-\tau)^{\alpha-1} d\tau. \end{aligned} \tag{9}$$

$$
\begin{aligned}
y(t_{n+1})-y(0)=&\frac{1-\alpha}{ABC(\alpha)}H(t_n,\ x(t_n))+\frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}\int_0^{t_{n+1}}H(\tau,\ y(\tau))(t_{n+1}-\tau)^{\alpha-1}d\tau\\
=&\frac{1-\alpha}{ABC(\alpha)}H(t_n,\ y(t_n))+\frac{\alpha}{\Gamma(\alpha)ABC(\alpha)}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}}H(\tau,\ y(\tau))(t_{n+1}-\tau)^{\alpha-1}d\tau.
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
z(t_{n+1})-z(0)=&\frac{1-\alpha}{ABC(\alpha)}P(t_n,\ z(t_n))+\frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}\int_0^{t_{n+1}}P(\tau,\ x(\tau))(t_{n+1}-\tau)^{\alpha-1}d\tau\\
=&\frac{1-\alpha}{ABC(\alpha)}P(t_n,\ z(t_n))+\frac{\alpha}{\Gamma(\alpha)ABC(\alpha)}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}}P(\tau,\ z(\tau))(t_{n+1}-\tau)^{\alpha-1}d\tau.
\end{aligned}
\tag{11}
$$

within $[t_k,\ t_{k+1}]$, the functions $E(t_n,\ x(t_n))$, $H(t_n,\ y(t_n))$ and $P(t_n, z(t_n))$ can be approximated using two-step Lagrange polynomial interpolation:

$$P_{1k}(\tau)\ \simeq\ \frac{E(t_k,\ x_k)}{h}(\tau-t_{k-1})-\frac{E(t_{k-1},\ x_{k-1})}{h}(\tau-t_k), \tag{12}$$

$$P_{2k}(\tau)\ \simeq\ \frac{H(t_k,\ y_k)}{h}(\tau-t_{k-1})-\frac{H(t_{k-1},\ y_{k-1})}{h}(\tau-t_k), \tag{13}$$

$$P_{3k}(\tau)\ \simeq\ \frac{P(t_k,\ z_k)}{h}(\tau-t_{k-1})-\frac{P(t_{k-1},\ z_{k-1})}{h}(\tau-t_k). \tag{14}$$

The above approximation can therefore be included in (12), (13) and (14) to produce.

$$
\begin{aligned}
x_{n+1}=&x_0+\frac{(1-\alpha)}{ABC(\alpha)}E(t_n,\ x(t_n))\\
&+\frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}\sum_{k=0}^{n}\left(\begin{array}{c}\frac{E(t_k,\ x_k)}{h}\int_{t_k}^{t_{k+1}}(\tau-t_{k-1})(t_{n+1}-\tau)^{\alpha-1}d\tau\\ -\frac{E(t_{k-1},\ x_{k-1})}{h}\int_{t_k}^{t_{k+1}}(\tau-t_k)(t_{n+1}-\tau)^{\alpha-1}d\tau\end{array}\right)
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
y_{n+1}=&y_0+\frac{(1-\alpha)}{ABC(\alpha)}H(t_n,\ y(t_n))\\
&+\frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}\sum_{k=0}^{n}\left(\begin{array}{c}\frac{H(t_k,\ y_k)}{h}\int_{t_k}^{t_{k+1}}(\tau-t_{k-1})(t_{n+1}-\tau)^{\alpha-1}d\tau\\ -\frac{H(t_{k-1},\ y_{k-1})}{h}\int_{t_k}^{t_{k+1}}(\tau-t_k)(t_{n+1}-\tau)^{\alpha-1}d\tau\end{array}\right)
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
z_{n+1}=&z_0+\frac{(1-\alpha)}{ABC(\alpha)}P(t_n,\ z(t_n))\\
&+\frac{\alpha}{ABC(\alpha)\Gamma(\alpha)}\sum_{k=0}^{n}\left(\begin{array}{c}\frac{P(t_k,\ z_k)}{h}\int_{t_k}^{t_{k+1}}(\tau-t_{k-1})(t_{n+1}-\tau)^{\alpha-1}d\tau\\ -\frac{P(t_{k-1},\ z_{k-1})}{h}\int_{t_k}^{t_{k+1}}(\tau-t_k)(t_{n+1}-\tau)^{\alpha-1}d\tau\end{array}\right)
\end{aligned}
\tag{17}
$$

The following numerical scheme is obtained after solving the integrals on the right:

$$\begin{aligned}x_{n+1} =& x_0 + \frac{(1-\alpha)}{ABC(\alpha)}E(t_n,\ x(t_n))\\&+\frac{\alpha}{ABC(\alpha)}\sum_{k=0}^{n}\Big[\frac{h^\alpha E(t_k,\ x_k)}{\Gamma(\alpha+2)}\left((n+1-k)^\alpha(n-k+2+\alpha)-(n-k)^\alpha(n-k+2+2\alpha)\right)\\&-\frac{h^\alpha E(t_{k-1},\ x_{k-1})}{\Gamma(\alpha+2)}\Big((n+1-k)^{\alpha+1}-(n-k)^\alpha(n-k+1+\alpha)\Big)\Big]\end{aligned} \tag{18}$$

$$\begin{aligned}y_{n+1} =& y_0 + \frac{(1-\alpha)}{ABC(\alpha)}H(t_n,\ y(t_n))\\&+\frac{\alpha}{ABC(\alpha)}\sum_{k=0}^{n}\Big[\frac{h^\alpha H(t_k,\ y_k)}{\Gamma(\alpha+2)}\left((n+1-k)^\alpha(n-k+2+\alpha)-(n-k)^\alpha(n-k+2+2\alpha)\right)\\&-\frac{h^\alpha H(t_{k-1},\ y_{k-1})}{\Gamma(\alpha+2)}\Big((n+1-k)^{\alpha+1}-(n-k)^\alpha(n-k+1+\alpha)\Big)\Big]\end{aligned} \tag{19}$$

$$\begin{aligned}z_{n+1} =& z_0 + \frac{(1-\alpha)}{ABC(\alpha)}P(t_n,\ z(t_n))\\&+\frac{\alpha}{ABC(\alpha)}\sum_{k=0}^{n}\Big[\frac{h^\alpha P(t_k,\ z_k)}{\Gamma(\alpha+2)}\left((n+1-k)^\alpha(n-k+2+\alpha)-(n-k)^\alpha(n-k+2+2\alpha)\right)\\&-\frac{h^\alpha P(t_{k-1},\ z_{k-1})}{\Gamma(\alpha+2)}\Big((n+1-k)^{\alpha+1}-(n-k)^\alpha(n-k+1+\alpha)\Big)\Big]\end{aligned} \tag{20}$$

Now, we will discuss the application of the newly developed numerical scheme for the purpose to solve fractional differential equations.

4. Applications of the ABC-FD Scheme

In this section, we emphasize the practicality and applicability of the ABC-FD scheme in solving numerical problems, with a focus on using computational simulations to assess its effectiveness in addressing specific test cases or scenarios.

Problem 1. Our first problem revolves around tackling the fractional Bernoulli equation.

$$\begin{aligned}{}^{ABC}_{\ \ 0}D_0^\alpha y(t) =& y(t)+y(t)^2+1, \quad 0<\alpha\leq 1,\\ y(0) =& 0.\end{aligned} \tag{21}$$

where ${}^{ABC}_{\ 0} D_0^{\alpha}$ is the Atangana-Baleanu Caputo fractional derivative operator, given in Eq. (2). The exact solution of the Bernoulli Eq. (21) shows in [45], is.

$$y(t) = \frac{1}{2}\left(\sqrt{3}\tan\left(\frac{1}{6}(3\sqrt{3}t+\pi)-1\right)\right) \tag{22}$$

The fractional Bernoulli equation (22) has an exact solution, according to the proposed ABC-FD scheme we show the results in Tables 1 and 2. Table 1 provides numerical results from our new ABC-FD scheme for the fractional Bernoulli equation Eq. (21) when $\alpha = 1$ at $t = 0.5,\ 0.7$ and 0.9 and when $\alpha = 0.95$ at $t = 0.5,\ 0.7$ and 0.9 in Table 2.

The numerical numbers we provided matched the exact result, and the step size h is tiny enough. It is observed that accuracy increases with decreasing step size h. We can observe the numerical stability feature of the ABC-FD scheme according to the convergence of the numerical data in Table 1 and Table 2.

Table 1: Solutions of Equation (21) where $\alpha = 0.95$.

h	$t = 0.5$	$t = 0.7$	$t = 0.9$
1/320	0.727894644424645	1.334617072292245	2.655787522792942
1/640	0.727919150293686	1.334627794673236	2.655815432421105
1/1280	0.727968161290851	1.334637692200060	2.655841194787946
1/2560	0.727989165701209	1.334646856529120	2.655865048517234
1/5120	0.727997567414455	1.334655366222233	2.655887198138255
1/10240	0.727993511418589	1.333663289004453	2.655907819965030
Exact	0.727423275682290	1.333623995935720	2.653204588689912

Table 2: Solutions of Equation (21) where $\alpha = 0.95$.

h	$t = 0.5$	$t = 0.7$	$t = 0.9$
1/320	1.011233122696145	2.254033666282457	4.447754640984550
1/640	1.011244500276947	2.254080841564283	4.455050084792259
1/1280	1.011255002684663	2.254124389955685	4.451529969778694
1/2560	1.011264727101195	2.254164714268654	4.452791939534118
1/5120	1.011273756967766	2.254202159766417	4.453802791022117
1/10240	1.011282164123349	2.254237024056137	4.457911499398058

Figure 1 compares Eq. (21) the exact and numerical solutions of Eq. (22). When $t = 1000$, we plot the numerical solutions of Eq. (21) with different values of α, and the novel scheme simulates the problem perfectly. Moreover, we notice that the numerical solutions are close to the exact solution.

Problem 2. Our second problem covers a basic system that displays chaotic behavior is the Rössler system [33, 34] is a multidimensional system that is comparable to the

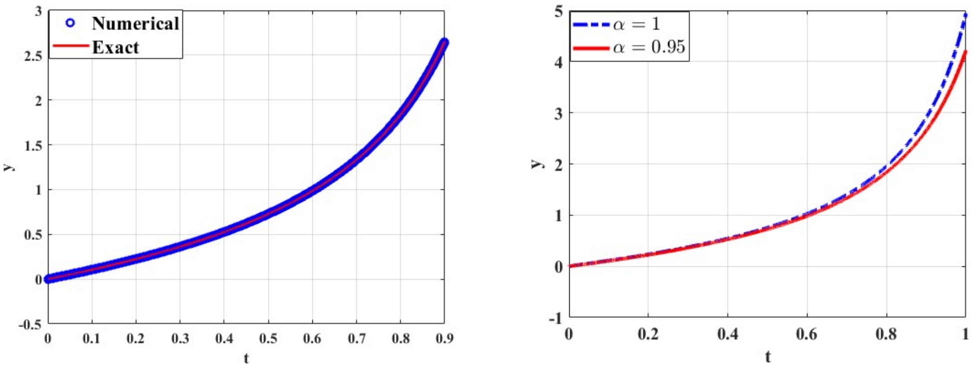

Figure 1: Comparison of numerical and exact solutions of Equation (21) with different t and α.

Lorenz system [32] in a few characteristics. The equations that define the Rössler system are as follows:

$$\begin{cases} \frac{dx}{dt} = -y - z, \\ \frac{dy}{dt} = x + ay, \\ \frac{dz}{dt} = b + z(x - c). \end{cases} \tag{23}$$

For parameter values $a = 0.2$, $b = 0.2$ and $c = 5.7$ and initial conditions $(x_0,\ y_0,\ z_0) = (1,\ 1,\ 1.05)$ show in [1]. Here, we consider the Rössler model in the case when the integer order derivatives is replaced by fractional order. We introduce the fractional Rössler oscillator model of with $0 < \alpha < 1$ as follows.

$$\begin{cases} {}^{ABC}_{\quad 0} D_0^{\alpha} x = -y - z, \\ {}^{ABC}_{\quad 0} D_0^{\alpha} y = x + ay, \\ {}^{ABC}_{\quad 0} D_0^{\alpha} z = b + z(x - c), \end{cases} \tag{24}$$

where ${}^{ABC}_{\quad 0} D_0^{\alpha}(.)$ is the ABC-FD.

The chaotic attractor from the fractional Rössler oscillator model is highly useful in Secured Communication applications. Attractors is a new research concept with enormous potential in the field of secure communication.

In Table 3 below, we provide the numerical solution using the the ABC-FD scheme to Eq. (24) when $\alpha = 1$, and $t = 0.1$. Table 4 provides the numerical solution for the value of $\alpha = 0.95$, and $t = 1$. It should be observed that accuracy increases as step size h is reduced. We can see the numerical stability characteristic of the ABC-FD scheme based on the convergence of the numerical data in Table 3 and Table 4.

Table 3: Solutions of Equation (24) where $\alpha = 1$ and $t = 0.1$.

h	x	y	z
1/320	0.177898180755700	1.407072166904383	0.116315513109570
1/640	0.176707768361964	1.407429186405125	0.115968853477924
1/1280	0.176112624976242	1.407607713412835	0.115796493319784
1/2560	0.175914252956577	1.407667225032148	0.115739183282915
1/5120	0.175815068638023	1.407696981337520	0.115710555116532
1/10240	0.175755558585964	1.407714835280020	0.115693386806025

Table 4: Solutions of Equation (24) where $\alpha = 0.95$ and $t = 1$.

h	x	y	z
1/320	-0.615483039781965	1.322607056797672	0.055611317422203
1/640	-0.616417804431375	1.322218290929184	0.055536319297694
1/1280	-0.616885335290380	1.322023886548481	0.055499005900917
1/2560	-0.617041200990387	1.321959082068103	0.05548659544584
1/5120	-0.617119138320106	1.321926679486526	0.055480395055542
1/10240	-0.617165902091590	1.321907237739616	0.055476676628662

Figure 2 Demonstrates the chaotic attractors plot of Eq. (24) in the case when $\alpha = 0.95$ and $t = 1000$. We present, in Figure 3, the chaotic attractors plot of Eq. (24) when $\alpha = 0.99$ and $t = 1000$.

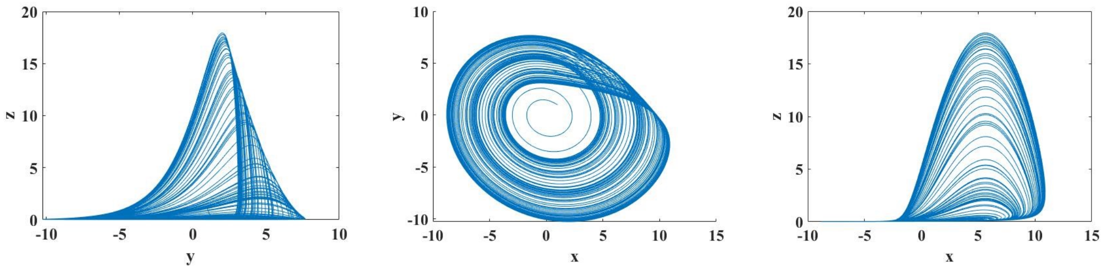

Figure 2: The chaotic attractors plot from Eq. (24), when $\alpha = 0.95$ and $t = 1000$.

These Figures show the projections of the fractional Rössler model Eq. (24) attractors that were obtained by using the novel scheme (ABC-FD). Identifying chaotic behavior in the fractional system in Eq. (24) by using a unique scheme (ABC-FD) allows researchers to acquire insights into the system's fundamental dynamics; this understanding is essential in various scientific areas, including physics, engineering, and mathematics.

5. Numerical Results and discussions

In this section, we present a comparison between our suggested method and the Runge-Kutta 4th-order approach (RK4).

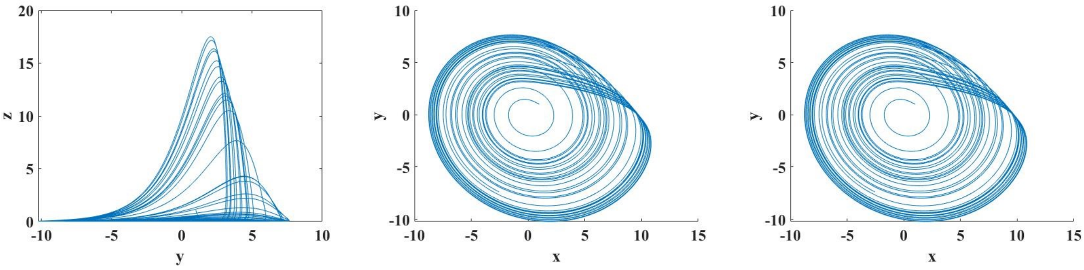

Figure 3: The chaotic attractors plot from Eq. (24), when $\alpha = 0.99$ and $t = 1000$.

Table 5: Comparison between the ABC-FD scheme and RK4 Method for $x(t)$ to the fractional model equation Eq. (24).

t	ABC-FD($\alpha = 0.95$)	ABC-FD($\alpha = 1$)	RK4
0	0.1	0.1	0.1
0.1	0.710684980261416	0.811287822893630	0.809899121526138
0.2	0.541932682851730	0.641749305322459	0.640477455295522
0.3	0.382747205334253	0.483048734060514	0.481833456041290
0.4	0.228902195409422	0.329148739755869	0.327954972715375
0.5	0.078662493918809	0.176707768361964	0.175517541129673

In Tables 5-7, we show that for $\alpha = 1$, the numerical solution outcomes of the ABC-FD scheme found in Eq. (24)) exhibit remarkable concurrence with the Runge-Kutta 4th Order Method solutions. As consequence of our findings, we are certain that the methodology provided a powerful mathematical instrument for solving equations. Moreover, they can be used to find analytical or approximate solutions to other problems.

6. Conclusion

This study affords a numerical solution for the fractional Bernoulli equation and the fractional Rössler model. This research makes use of an innovative method for the fractional derivative, which is referred to as ABC-FD. In this work, we employ the Runge-Kutta Fourth Order (RK4) approach to evaluate the solutions obtained from these meth-

Table 6: Comparison between the ABC-FD scheme and RK4 Method for $y(t)$ to the fractional model equation Eq. (24).

t	ABC-FD($\alpha = 0.95$)	ABC-FD($\alpha = 1$)	RK4
0	0.1	0.1	0.1
0.1	1.162327452222037	1.110604044223248	1.111410479237619
0.2	1.243679775165603	1.206325497512754	1.207014508204590
0.3	1.307095940600464	1.287466819327066	1.288044529114395
0.4	1.354153393129866	1.354496996922664	1.354964920086584
0.5	1.385585796378668	1.407429186405125	1.407786249680785

Table 7: Comparison between the ABC-FD scheme and RK4 Method for $z(t)$ to the fractional model equation Eq. (24).

t	ABC-FD($\alpha = 0.95$)	ABC-FD($\alpha = 1$)	RK4
0	0.1	0.1	0.1
0.1	0.563027164530943	0.668146806337842	0.665758002503422
0.2	0.385951374918309	0.422012789499755	0.420504778950151
0.3	0.271239731606305	0.268070090773060	0.267135297212523
0.4	0.195819117416147	0.173389179063020	0.172818474082330
0.5	0.145751174250475	0.115968853477924	0.115624863785721

ods. To facilitate the ABC-FD approach of solution comparison, we offered a numerical strategy that made use of the resources that were included in the MATLAB software package. In terms of how well they keep their numerical stability, the numerical results suggest that our technology can perform fractional processes that meet the requirements that were set for them. The fact that the numerical figures that were produced were accurate is evidence that this is the case. A comparison of the solutions of these methods with those obtained by the Runge-Kutta Fourth Order (RK4) method is done. Using MATLAB software package tools, we provided a numerical strategy to aid the ABC-FD method in comparing the solutions. The numerical findings demonstrate that our method performs fractional procedures which satisfy expectations for how successfully they maintain their numerical stability. The accuracy of the numerical results produced serves as evidence for this. Proficiency in fractional differential equation solving and precise modeling of chaotic systems has applications in signal processing, control theory, communication systems, and other fields. Engineers may be able to improve the performance of current systems and create more resilient designs that can manage nonlinear dynamics by utilizing the ABC-FD method. We suggested using this approach to tackle brand-new fractional problems [7, 10, 41] and contrasting numerical solutions with other approaches [6, 19, 26–28].

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References

[1] A. Belazi, R. Rhouma, and S. Belghith. A novel approach to construct s-box based on rössler system. *International Wireless Communications and Mobile Computing Conference (IWCMC)*, Aug. 2015.

[2] Abu-Ghuwaleh M., Saadeh R. and Qazza A. General master theorems of integrals with applications. *Mathematics*, 10(19):3547, 2022.

[3] R. Adams. *Sobolev Spaces.* Academic Press, New York, 1975.

[4] Almutairi D. K., Abdoon M. A., Berir M., Saadeh R., and Qazza A. A numerical confirmation of a fractional seitr for influenza model efficiency. *Applied Mathematics Information Sciences*, 17(25):741–749, 2023.

[5] M. Almutairi D. K., Abdoon M. A., Salih S. Y. M., Elsamani S. A., Guma F. E., Berir. Modeling and analysis of a fractional visceral leishmaniosis with caputo and caputo–fabrizio derivatives. *Journal of the Nigerian Society of Physical Sciences*, 5(3):1453, 2023.

[6] Abdulrahman BM Alzahrani, Rania Saadeh, Mohamed A Abdoon, Mohamed Elbadri, Mohammed Berir, and Ahmad Qazza. Effective methods for numerical analysis of the simplest chaotic circuit model with atangana–baleanu caputo fractional derivative. *Journal of Engineering Mathematics*, 144(1):9, 2024.

[7] A Atangana and D Baleanu. New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model. *Therm. Sci.*, 20(2):763–769, 2016.

[8] Abdon Atangana and Sania Qureshi. Modeling attractors of chaotic dynamical systems with fractal–fractional operators. *Chaos, solitons & fractals*, 123:320–337, 2019.

[9] Bingi K., Rajanarayan Prusty B., and Pal Singh A. A review on fractional-order modelling and control of robotic manipulators. *Fractal and Fractional.*, 7(1):77, 2023.

[10] Dumitru D. Description of strange attractors using invariants of phase-plane. recent research in computational techniques. *Non-Linear Systems and Control*, 2011.

[11] D Dudkowski, S Jafari, T Kapitaniak, N V Kuznetsov, G A Leonov and A Prasad. Hidden attractors in dynamical systems. Physics Reports vol 637, 2016.

[12] E Salah, A Qazza, R Saadeh and A. El-Ajou. A hybrid analytical technique for solving multi-dimensional time-fractional navier-stokes system. *AIMS Mathematics*, 8(1):1713–1736, 2023.

[13] Abdoon Mohamed A., et al. Analysis, modeling and simulation of a fractional-order influenza model. *Alexandria Engineering Journal*, 74:231–240, 2023.

[14] Ahmed S., et al. An efficient method for the fractional electric circuits based on fibonacci wavelet. *Results in Physics.*, 52:106–753, 2023.

[15] Elbadri M., et al. A numerical solution and comparative study of the symmetric rossler attractor with the generalized caputo fractional derivative via two different methods. *Mathematics.*, 11(13):1–11, 2023.

[16] Elbadri M., et al. A symmetry chaotic model with fractional derivative order via two different methods. *Symmetry.*, 15(6):1151, 2023.

[17] Guma F. EL., et al. Numerical analysis of fractional-order dynamic dengue disease epidemic in sudan. *Journal of the Nigerian Society of Physical Sciences.*, 5(2):1464, 2023.

[18] Qazza A., et al. A new scheme for solving a fractional differential equation and a chaotic system. *European Journal of Pure and Applied Mathematics.*, 16(2):1128–1139, 2023.

[19] Abdoon M. A., Hasan FL. Advantages of the differential equations for solving problems in mathematical physics with symbolic computation. *Math. Model. Eng. Probl.*, 9(1):268–276, 2022.

[20] G Xu, Y Shekofteh, A Akg¨ul, C Li and S. Panahi. A new chaotic system with a self-excited attractor: entropy measurement, signal encryption, and parameter estimation. *Entropy*, 20(2):86, 2018.

[21] B Ghanbari and D A Atangana. some new edge detecting techniques based on fractional derivatives with nonlocal and non-singular kernels. *Adv. Diff. Equa.*, 2020:435, 2020.

[22] Gorenflo R, and Mainardi F. Fractional calculus: integral and differential equations of fractional order, in: A. carpinteri, f. mainardi (eds). *Fractals and Fractional Calculus in Continuum Mechanics. Springer-Verlag, Newyork*, pages 223–276, 1997.

[23] Guma F. E., Badawy O. M., Musa A. G., Mohammed B. O., Abdoon M. A., Berir M., and Salih S. Y. M. Risk factors for death among covid-19 patients admitted to isolation units in gedaref state, eastern sudan: a retrospective cohort study. *Journal of Survey in Fisheries Sciences*, 10(3s):712–722, 2023.

[24] Hamadneh T., Hioual A., Saadeh R., Abdoon Mohamed A., Almutairi D. K., Khalid T. A., and Ouannas A. General methods to synchronize fractional discrete reaction–diffusion systems applied to the glycolysis model. *Fractal and Fractional*, 7(11):828, 2023.

[25] Z Hammouch and T Mekkaoui. Traveling-wave solutions of the generalized zakharov equation with time-space fractional derivatives. *Math Eng Sci Aerosp MESA*, 5(4):1–11, 2014.

[26] Faeza Hasan, Mohamed A Abdoon, Rania Saadeh, Mohammed Berir, and Ahmad Qazza. A new perspective on the stochastic fractional order materialized by the exact solutions of allen-cahn equation. *International Journal of Mathematical, Engineering and Management Sciences*, 8(5):912, 2023.

[27] Faeza L Hasan and Mohamed A Abdoon. The generalized (2+ 1) and (3+ 1)-dimensional with advanced analytical wave solutions via computational applications. *International Journal of Nonlinear Analysis and Applications*, 12(2):1213–1241, 2021.

[28] M A Abdoon, F L Hasan and N E Taha. Computational technique to study analytical solutions to the fractional modified kdv-zakharov-kuznetsov equation. *Abstract and Applied Analysis*, 2022:p. 9, 2022.

[29] G Samko, A Kilbas and O Marichev. *Fractional Integrals and Derivatives: Theory and Applications.* Gordon and Breach, Amsterdam, 1993.

[30] A. A Klibanov, M. V., Timonov. A comparative study of two globally convergent numerical methods for acoustic tomography. *arXiv preprint arXiv:2301.05817.*, 2023.

[31] Elbadri M. Initial value problems with generalized fractional derivatives and their solutions via generalized laplace decomposition method. *Adv. Math. Phys.*, 2022:1–7, 2022.

[32] Edward Lorenz N. Deterministic non-periodic flows. *Journal of Atmospheric Science*, 1963.

[33] Rössler O.E. Different types of chaos in two simple differential equations. *Zeitschrift fur Naturforschung*, a 31(12):1664–1670, 1976.

[34] Rössler O.E. An equation for continuous chaos. *Physics Letters*, A 57(5):397–398, 1976.

[35] I Petras. Control of fractional-order chua's system. *J. Elec. Engi.*, 53:219–222, 2002.

[36] I Petras. A note on the fractional-order chua's system. *Chaos, Solitons and Fractals*, 38:140–147, 2008.

[37] I Podlubny. *Fractional Differential Equations.* Academic Press, New York, 1999.

[38] Q Lai, A Akgul, C Li, G Xu and U Cavusoglu. A new chaotic system with multiple attractors: Dynamic analysis, circuit realization and s-box design. *Entropy*, 20(1):12, 2018.

[39] Qazza A., Abdoon M., Saadeh R., and Berir M. Anew scheme for solving a fractional differential equation and a chaotic system. *European Journal of Pure and Applied Mathematics.*, 16(2):1128–1139, 2023.

[40] S Kumar, R Kumar, C Cattani and B Samet. Chaotic behaviour of fractional predator-prey dynamical system. *Chaos Soli. Fract.*, 135:p. 12, 2020.

[41] Saadeh R., Abdoon M. A., Qazza A., and Berir M. A numerical solution of generalized caputo fractional initial value problems. *Fractal and Fractional*, 7(4):332, 2023.

[42] Oldham KB, Spanier J. *The Fractional Calculus.* Academic Press, New York, London, 1974.

[43] M Toufik and A Atangana. New numerical approximation of fractional derivative with non-local and non-singular kernel: application to chaotic models. *Eur Phys J Plus*, 132(10):444, 2017.

[44] Wassim A., Alexan N., and Gabr M. Multiple-layer image encryption utilizing fractional-order chen hyperchaotic map and cryptographically secure prngs. *Fractal and Fractional.*, 7(4):287, 2023.

[45] Yang X.-F., Deng Z.-C., and Wei Y. A riccati-bernoulli sub-ode method for nonlinear partial differential equations and its application. *Advances in Difference Equations*, 117(2015), 2015.

[46] Xuefeng Z., Boutat B., and Liu D. Applications of fractional operator in image processing and stability of control systems. *Fractal and Fractional.*, 7(5):359, 2023.

[47] Z. Zhang, X., Li, X., Yang, K., Wang. Lithium-ion battery modeling and state of charge prediction based on fractional-order calculus. *Mathematics.*, 11(15):3401, 2023.