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Utilization of the modified Adomian decomposition method on the Bagley-Torvik equation amidst Dirichlet boundary conditions

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Abstract. The Bagley-Torvik equation is an imperative differential equation that considerably arises in various branches of mathematical physics and mechanics. However, very few methods exist for the treatment of the model analytically; in fact, researchers frequently shop for semi-analytical and numerical methods in their studies. Therefore, the main goal of this research is to find the exact analytical solution for the fractional Bagley-Torvik equation fitted with Dirichlet boundary data, as well as a system of fractional Bagley-Torvik equations. Thus, this research aims to show that the modified Adomian decomposition method (MADM) via the proposed two algorithms is a very effective method for treating a class of Bagley-Torvik equations endowed with Dirichlet boundary data. Certainly, MADM is a very powerful approach for solving dissimilar functional equations without the need for either linearization, discretization, perturbation, or even unnecessary restraining postulations. Additionally, the method reveals exact analytical solutions whenever obtainable or closed-form series solutions whenever exact solutions are not feasible. Lastly, some illustrative test problems of the governing model are examined to demonstrate the superiority of the proposed algorithms.

2020 Mathematics Subject Classifications: 26A33, 34A08, 34B15.

Key Words and Phrases: Fractional calculus, Bagley-Torvik equation, Dirichlet boundary condition, Modified Adomian decomposition method, Boundary-value problem.

1. Introduction

Fractional calculus is an aged area of research that recently reemerged more strongly with burning applications that cuts across all aspects of life. Indeed, the area started off by the ignition put forward by Leibniz (1695) and Euler (1730) [28, 30], and since then keeps propelling to date. Notably, various real-life models are discovered to be perfectly captured through the application of fractional differential equations (FDEs). These FDEs have, in recent times gained considerable relevance in modeling different

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emerging problems arising in, for instance, electrical networks, electromagnetic theory, fractals theory, viscoelasticity, control theory, material science, chemistry, potential theory, fluid flow, biology, and statistics to mention but a few [5, 6, 10, 11, 24, 26, 27, 33, 34, 39, 41]. In light of this, different researchers have in the past and recent times proposed a variety of methods, including analytical, semi-analytical, and computational to deal with FDEs. In this regard, the present paper shops for an elegant semi-analytical method that is founded on the utilization of the famous Adomian decomposition method (ADM) [1, 3, 4, 15–17, 32, 38, 47, 49].

On the other hand, the boundary-value problems (BVPs) featuring fractional-order derivatives have - in recent years - fascinated or rather shaped the thoughts of various theoreticians and experimentalists in diverse stems of applied and pure sciences. In particular, we make mention of the Bagley-Torvik equation, being an imperative differential equation that arises in various branches of mathematical physics and mechanics [46]. This equation is, however, used in modeling various processes, including viscoelasticity, the submergence of solid structures in fluids, and the interaction of solid media with fluids among others; for more on the uniqueness and existence results of the model when prescribed with Dirichlet boundary data, an interested reader can consult [8, 25] and the references therewith. Moreover, as it is always thorny to tackle FDEs analytically, many mathematicians have introduced several efficient computational schemes based on various concepts to computationally treat the class of Bagley-Torvik equations. Here, we make mention of such approaches that are applied on the Bagley-Torvik equation in recent years to comprise the quadratic spline solution [50], the cubic spline polynomials [51], the Chebyshev wavelet method [35], the shifted Legendre polynomials [45], the Taylor's method [40], the exponential spline technique [7], the Chelyshkov-Tau approach [20], the Chebyshev collocation method [44], the quintic B-spline polynomial [23], the exponential spline approximation [21], the Green's function iterative approach [22] and the shifted Chebyshev operational matrix [29] to review but just a few; yet, read [14] for a mesmerizing study on the Bagley-Torvik equation with the aid of the differential transform approach.

However, we, in the current paper aim to make use of the modified Adomian decomposition method (MADM) [2, 9, 12, 36, 37, 42, 48] by proposing two different algorithms to treat the Bagley-Torvik equation with Dirichlet boundary condition. MADM is a semi-analytical approach that was improved upon the classical ADM [3]-[38] to easily reveal exact analytical solutions whenever obtainable or closed-form series solutions whenever exact solutions are not feasible. In fact, the approach is very powerful in solving dissimilar functional equations without the need for either of linearization, discretization, perturbation, or even unnecessary restraining postulations. Besides, the method has been successfully used in the literature to solve various real-life models. In addition, we organize the paper in the following pattern: Section 2 gives certain fundamental definitions of features with regard to fractional calculus, Section 3 gives the procedures of the devised MADM algorithms on BVP for the Bagley-Torvik equation, while Section 4 demonstrates the applicability of the devised algorithms, and Section 5 provides some concluding points.

2. Fractional calculus

The current section introduces certain fundamental definitions and features for fractional calculus, consisting mainly of the fractional derivatives and fractional integrals that were put forward based on the Riemann-Liouville fractional (RLF) and the Caputo fractional (CF) integrals/derivatives. For more on some basics related to this work; an interested reader can further read the famous book by Kilbas et al. [13].

2.1. Preliminaries

Here, we review some definitions of the fractional-order derivatives and integrals based on the definitions put forward by Riemann-Liouville and Caputo, respectively. Certainly, we will be considering the set $\chi = [a, b] \in \mathbb{R}$, such that $a < b$, a finite closed interval on $\mathbb{R}$, the set of real numbers.

Definition 1. (RLF integrals): *The RLF right-sided $I^{\alpha}_{b^-}y$ and left-sided $I^{\alpha}_{a^+}y$ integrals of order $\alpha \in \mathbb{R}$ are respectively defined as follows*

$$(I^{\alpha}_{b^-}y)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{y(t)}{(t-x)^{-\alpha+1}} dt, \qquad (x < b;\ \alpha > 0),$$

and

$$(I^{\alpha}_{a^+}y)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{y(t)}{(x-t)^{-\alpha+1}} dt, \qquad (x > a;\ \alpha > 0).$$

Definition 2. (RLF derivatives): *The RLF right-sided $D^{\alpha}_{b^-}y$ and left-sided $D^{\alpha}_{a^+}y$ derivatives of order $\alpha \in \mathbb{R}$ are respectively defined as follows*

$$\begin{aligned} D^{\alpha}_{b^-}y(x) &:= \frac{-d^n}{dx^n}(I^{n-\alpha}_{b^-}y)(x), \\ &:= \frac{1}{\Gamma(n-\alpha)} \frac{-d^n}{dx^n} \int_x^b \frac{y(t)}{(t-x)^{\alpha+1-n}} dt, \qquad (x > b;\ \alpha \geq 0;\ n = [\alpha]+1), \end{aligned}$$

and

$$\begin{aligned} D^{\alpha}_{a^+}y(x) &:= \frac{d^n}{dx^n}(I^{n-\alpha}_{a^+}y)(x), \\ &:= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{y(t)}{(x-t)^{\alpha+1-n}} dt, \qquad (x > a;\ \alpha \geq 0;\ n = [\alpha]+1), \end{aligned}$$

with $[\alpha]$ representing the integer part of the fractional-order α.

Definition 3. (CF derivatives): *The CF right-sided ${}^cD^{\alpha}_{b^-}y(x)$ and left-sided ${}^cD^{\alpha}_{a^+}y(x)$ derivatives of order $\alpha \in \mathbb{R}^+$ on $[a, b]$ are respectively defined via the RLF derivatives as follows*

$${}^cD^{\alpha}_{b^-}y(x) := \left(D^{\alpha}_{b^-}\left[y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(b)}{k!}(b-t)^k\right]\right)(x), \tag{1}$$

and

$$ {}^{c}D^{\alpha}_{a^+}y(x) := \left(D^{\alpha}_{a^+}\left[y(t) - \sum_{k=0}^{n-1}\frac{y^{(k)}(a)}{k!}(t-a)^k\right]\right)(x), \tag{2} $$

with $n = [\alpha] + 1$, *when* $\alpha \notin \mathbb{N}$, *and* $n = \alpha$, *for* $\alpha \in \mathbb{N}$; *with* $\mathbb{N}$ *denoting the set of positive whole numbers.*

Note that, as a peculiar case, when $0 < \alpha < 1$, *the formulae given in* (2) *and* (1) *could be re-expressed as follows*

$$ {}^{c}D^{\alpha}_{b^-}y(x) = (D^{\alpha}_{b^-}[y(t) - y(b)])(x), $$

and

$$ {}^{c}D^{\alpha}_{a^+}y(x) = (D^{\alpha}_{a^+}[y(t) - y(a)])(x). $$

2.2. Some useful properties

This subsection recalls some useful properties of the aforementioned RLF integrals/derivatives and CF derivatives that will be greatly utilized in the course of the governing model.

Lemma 1. *Assume* $\alpha > 0$, *and further assume* $n = [\alpha] + 1$, *when* $\alpha \notin \mathbb{N}$, *and* $n = \alpha$, *when* $\alpha \in \mathbb{N}$. *Then, if* $y(x) \in C^n[a,b]$ *or* $y(x) \in AC^n[a,b]$, *we accordingly have as follows*

$$ (I^{\alpha}_{b^-}\,{}^{c}D^{\alpha}_{b^-}y)(x) = y(x) - \sum_{k=0}^{n-1}\frac{(-)^k y^{(k)}(b)}{k!}(b-x)^k, \tag{3} $$

and

$$ (I^{\alpha}_{a^+}\,{}^{c}D^{\alpha}_{a^+}y)(x) = y(x) - \sum_{k=0}^{n-1}\frac{y^{(k)}(a)}{k!}(x-a)^k. \tag{4} $$

Consequently, when $0 < \alpha \leqq 1$ *and* $y(x) \in C[a,b]$ *or* $y(x) \in AC[a,b]$, *the above results respectively reduce to*

$$ (I^{\alpha}_{b^-}\,{}^{c}D^{\alpha}_{b^-}y)(x) = y(x) - y(b), $$

and

$$ (I^{\alpha}_{a^+}\,{}^{c}D^{\alpha}_{a^+}y)(x) = y(x) - y(a). $$

Lemma 2. *The following results for the fractional integral/derivative hold [45]*

(i)

$$ I^{\alpha}[c] = \frac{cx^{\alpha}}{\Gamma(\alpha+1)}, \quad \text{if } c \text{ is constant}, \tag{5} $$

(ii)

$$ D^{\alpha}x^n = \frac{\Gamma(n+1)x^{n-\alpha}}{\Gamma(n+1-\alpha)}. \tag{6} $$

(iii)

$$ I^{\alpha}[x^n] = \frac{\Gamma(n+1)x^{\alpha+n}}{\Gamma(\alpha+n+1)}, \tag{7} $$

3. Treatment of Bagley-Torvik BVPs via MADM

The current section mainly makes use of MADM [2, 9, 12, 36, 37, 42, 48] to acquire the exact analytical solution of the Bagley-Torvik BVP where possible; moreover, when the acquisition of such an exact analytical solution is not feasible, a closed-form series solution of the governing model will be acquired. In fact, we will be making consideration to the BVP of the nonhomogeneous Bagley-Torvik equation endowed with Dirichlet boundary conditions as follows [51]-[29]

$$\begin{aligned} D^2y(x) + D^{3/2}y(x) + y(x) = g(x), \quad a < x < b, \quad a, b \in \mathbb{R}^+, \\ y(a) = \lambda_1, \qquad y(b) = \lambda_2, \end{aligned} \tag{8}$$

with $g(x)$ denoting the nonhomogeneous/source term, λ_1 and λ_2 are constants; while the function $y(x)$ is the solution that we intend to determine. Certainly, the differential equation in (8) is a fractional-order differential equation defined in CF derivative sense with the highest integer-order of 2, and then followed by the CF-order of 3/2. In addition, this fractional-order model arises in many physical processes, and further has vast relevance in the study of fluid flows and viscoelastic materials [14], among others; more so, the highest-order of the equation remains 2, the integer-order, then follows by the fractional-order derivative of 3/2.

Further, the Bagley-Torvik equation expressed above can equally be expressed in an operator notation upon making use of the differential operator notation D as follows

$$Ly = D^2y = g(x) - D^{3/2}y(x) - y(x), \tag{9}$$

where $L = D^2 = \frac{d^2}{dx^2}$.

Furthermore, to determine the explicit exact analytical solution of the Bagley-Torvik BVP expressed in (8) - using the version expressed in (9) through the differential operator - the MADM [36]-[42] will be utilized. In fact, two algorithms based on MADM will be proposed in this section for the governing model in what follows.

3.1. Algorithm 1

The first algorithm can be summed up in the following bulleted lists:

- Applying the inverse operator

$$L^{-1}(.) = \int_a^x \int_a^x (.)dxdx,$$

  on (9), one gets

$$y(x) = y(a) + xy'(a) + L^{-1}[g(x)] - L^{-1}[D^{3/2}y(x)] - L^{-1}[y(x)]. \tag{10}$$

  Remarkably, we will further put $y'(a) = c_1$; because we do not have the value of this condition at the present, thereafter, the prescribed boundary data will help in getting a hold of the real value of c_1.

- Application of MADM requires that the term

$$-pL^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big] + L^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big] \tag{11}$$

should be added to (10), with p as a synthetic parameter, and $a_i,\ i \geq 0$ are obscure coefficients. Moreover, recall that the classical ADM [3]-[38] expresses the solution function $y(x)$ as $y(x) = \sum_{n=0}^{\infty} y_n(x)$. Consequently, one gets

$$\begin{aligned} \sum_{n=0}^{\infty} y_n(x) \ &= \ \lambda_1 + c_1 x - pL^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big] + L^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big] + L^{-1}[g(x)] \\ &\quad - L^{-1}\Big[D^{3/2}\sum_{n=0}^{\infty} y_n(x)\Big] - L^{-1}\Big[\sum_{n=0}^{\infty} y_n(x)\Big]. \end{aligned} \tag{12}$$

- As a result, one may deduce the resulting recursive scheme from (12) as follows

$$\begin{aligned} y_0(x) &= \lambda_1 + c_1 x + L^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big], \\ y_1(x) &= L^{-1}[g(x)] - pL^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big] - L^{-1}[D^{3/2} y_0(x)] - L^{-1}[y_0(x)], \\ &\vdots \\ y_{n+1}(x) &= -L^{-1}[D^{3/2} y_n(x)] - L^{-1}[y_n(x)], \quad n \geqslant 0. \end{aligned} \tag{13}$$

- Computation of the coefficients a_i, for $i \geq 0$ by taking $y_1 = 0$ and subsequently fixing $p = 1$ reveals the solution of (8) in form $y(x) = y_0(x)$.

- Lastly, substituting the values of a_i into $y_0(x)$, and further upon utilizing the second boundary condition $y(1) = \lambda_2$, the constant c_1 is thus revealed.

3.2. Algorithm 2

Algorithm 2 is equally based on MADM, which possesses the following steps:

- We begin by implementing the inverse operator $L^{-1}(.)$ that takes following representation [19, 31]

$$L^{-1}(.) = \int_a^x dx' \int_a^{x''} (.)dx'' - \frac{x-a}{b-a}\int_a^b dx \int_a^{x''} (.)dx'', \tag{14}$$

on (9), which reveals

$$y(x) - y(a) - xy(b) + xy(a) = L^{-1}[g(x)] - L^{-1}[D^{3/2} y(x)] - L^{-1}[y(x)]. \tag{15}$$

- MADM requires the addition of the expression given in (11) to (15) to yield the following equation

$$\begin{aligned}\sum_{n=0}^{\infty} y_n(x) &= \lambda_1 + \lambda_2 x - \lambda_1 x - pL^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big] + L^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big] + L^{-1}[g(x)] \\ &- L^{-1}\Big[D^{3/2}\sum_{n=0}^{\infty} y_n(x)\Big] - L^{-1}\Big[\sum_{n=0}^{\infty} y_n(x)\Big].\end{aligned} \tag{16}$$

- Then, the formal recursive relation of the governing model is thus determined in this regard as follows

$$\begin{aligned} y_0(x) &= \lambda_1 + (\lambda_2 - \lambda_1)x + L^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big], \\ y_1(x) &= L^{-1}[g(x)] - pL^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big] - L^{-1}[D^{3/2} y_0(x)] - L^{-1}[y_0(x)], \\ &\vdots \\ y_{n+1}(x) &= -L^{-1}[D^{3/2} y_n(x)] - L^{-1}[y_n(x)], \quad n \geqslant 0. \end{aligned} \tag{17}$$

  Therefore, the values of the coefficients a_i, for $i \geq 0$ are then calculated by taking $y_1 = 0$ and setting $p = 1$. Moreover, these values are then substituted into $y_0(x)$ to obtain $y(x) = y_0(x)$.

4. Applications

This section applies the proposed MADM on several test models, featuring various forms of BVP of the Bagley-Torvik equation. Precisely, the proposed MADM through the devised algorithms 1 and 2 will be applied to the governing model and its variant, including the coupled system of the fractional differential equation, which emanates from the Bagley-Torvik equation. Moreover, the test models of interest will be considered from the open literature in order to draw a firm conclusion with well-known exact solutions.

Example 1. *Consider the BVP for the nonhomogeneous Bagley-Torvik equation as follows [43]*

$$D^2 y(x) + D^{3/2} y(x) + y(x) = x^3 + 5x + \frac{8}{\sqrt{\pi}} x^{3/2}, \quad y(0) = 0, \quad y(1) = 0. \tag{18}$$

The exact analytical solution of the above BVP is thus found to be $y(x) = x^3 - x$.

***Algorithm 1**:*

Write (18) *in an operator form as follows*

$$Ly = D^2 y = x^3 + 5x + \frac{8}{\sqrt{\pi}} x^{3/2} - D^{3/2} y(x) - y(x). \tag{19}$$

Here, we define $L^{-1}(.) = \int_0^x \int_0^x (.)dxdx$ *to be the inverse operator. Therefore, applying* L^{-1} *on both sides of* (19) *such that* $y'(0) = c_1$ *gives*

$$y(x) = c_1 x - L^{-1}[D^{3/2}y(x)] - L^{-1}[y(x)] + L^{-1}[x^3 + 5x + \frac{8}{\sqrt{\pi}}x^{3/2}].$$

Then, on using MADM as explained in the procedure, we write

$$\begin{aligned} \sum_{n=0}^{\infty} y_n(x) &= c_1 x - pL^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big] + L^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big] - L^{-1}\Big[D^{3/2}\sum_{n=0}^{\infty} y_n(x)\Big] \\ &- L^{-1}\Big[\sum_{n=0}^{\infty} y_n(x)\Big] + L^{-1}[x^3 + 5x + \frac{8}{\sqrt{\pi}}x^{3/2}], \end{aligned}$$

thereby taking

$$y_0(x) = c_1 x + L^{-1}[a_0 + a_1 x + a_2 x^2 + ...], \tag{20}$$

and

$$y_1(x) = -pL^{-1}[a_0 + a_1 x + a_2 x^2 + ...] + L^{-1}[x^3 + 5x + \frac{8}{\sqrt{\pi}}x^{3/2}] - L^{-1}[D^{3/2}y_0(x)] - L^{-1}[y_0(x)]. \tag{21}$$

Computing the values of $y_0(x)$ *and* $y_1(x)$ *from* (20) *and* (21), *respectively by using* L^{-1} *and* (6), *one gets*

$$y_0(x) = c_1 x + \frac{1}{2}a_0 x^2 + \frac{1}{6}a_1 x^3 + \frac{1}{12}a_2 x^4 + ...$$

$$\begin{aligned} y_1(x) &= -p[\frac{1}{2}a_0 x^2 + \frac{1}{6}a_1 x^3 + \frac{1}{12}a_2 x^4 + ...] + \frac{1}{20}x^5 + \frac{5}{6}x^3 + \frac{32}{35\sqrt{\pi}}x^{7/2} - \frac{4c}{3\sqrt{\pi}}x^{3/2} \\ &- \frac{8a_0}{15\sqrt{\pi}}x^{5/2} - \frac{16a_1}{105\sqrt{\pi}}x^{7/2} - \frac{64a_2}{945\sqrt{\pi}}x^{9/2} - \frac{c}{6}x^3 - \frac{a_0}{24}x^4 - \frac{a_1}{120}x^5 - \frac{a_2}{360}x^6 + ... \end{aligned}$$

Now, put $y_1(x) = 0$ *and collecting the like terms, we have*

$$\begin{aligned} y_1(x) &= -p\frac{1}{2}a_0 x^2 + [-p\frac{1}{6}a_1 + \frac{5}{6} - \frac{c}{6}]x^3 + [-p\frac{1}{12}a_2 - \frac{a_0}{24}]x^4 + ... \\ &= 0. \end{aligned}$$

Then, on considering $p = 1$ *and equating the coefficients of* x^n *from the both sides of the above equation, one gains* $a_0 = a_2 = 0$, *and* $a_1 = 5 - c_1$. *Next, on substituting the values of* $a_0,\ a_1$ *and* a_2 *in* $y_0(x)$, *where* $y_n(x) = 0$, *for* $n \geqslant 1$, *we obtain*

$$y(x) = y_0(x) = c_1 x + \frac{1}{6}(5 - c_1)x^3 + ...$$

Lastly, with the aid of the second boundary prescription, that is, $y(1) = 0$, *we have* $c_1 = -1$, *which finally yields*

$$y(x) = x^3 - x,$$

that exactly matches the exact analytical solution for (18).

Algorithm 2:

Applying the inverse operator (14) *on the left-hand side of* (19), *one gets*

$$\begin{aligned} L^{-1}Ly(x) &= \int_0^x dx' \int_0^{x'} \frac{d^2y}{dx^2}dx'' - x\int_0^1 dx \int_0^{x'} \frac{d^2y}{dx^2}dx'', \\ &= y(x) - y(0) - xy(1) + xy(0), \\ &= y(x). \end{aligned}$$

So,

$$y(x) = L^{-1}[x^3 + 5x + \frac{8}{\sqrt{\pi}}x^{3/2}] - L^{-1}[D^{3/2}y(x)] - L^{-1}[y(x)].$$

Therefore, upon deploying the MADM procedure, we have

$$\begin{aligned} \sum_{n=0}^{\infty} y_n(x) &= L^{-1}[x^3 + 5x + \frac{8}{\sqrt{\pi}}x^{3/2}] - pL^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big] + L^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big] \\ &- L^{-1}\Big[D^{3/2}\sum_{n=0}^{\infty} y_n(x)\Big] - L^{-1}\Big[\sum_{n=0}^{\infty} y_n(x)\Big], \end{aligned}$$

which leads to the recurrent scheme as follows

$$y_0(x) = L^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big],$$

and

$$y_1(x) = L^{-1}\Big[x^3 + 5x + \frac{8}{\sqrt{\pi}}x^{3/2}\Big] - pL^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big] - L^{-1}[D^{3/2}y_0(x)] - L^{-1}[y_0(x)].$$

In fact, explicit expression for $y_0(x)$ *is found to be*

$$y_0(x) = \frac{a_0}{2}x^2 + \frac{a_1}{6}x^3 + \frac{a_2}{12}x^4 - x\Big(\frac{a_0}{2} + \frac{a_1}{6} + \frac{a_2}{12}\Big).$$

Also, on computing the component of $y_1(x)$ *using the inverse operator* (14), *one explicitly gets*

$$\begin{aligned} y_1(x) &= \frac{32}{35\sqrt{\pi}}x^{7/2} + \frac{1}{20}x^5 + \frac{5}{6}x^3 - \Big(\frac{53}{60} + \frac{32}{35\sqrt{\pi}}\Big)x - p\Big[\frac{a_0}{2}x^2 + \frac{a_1}{6}x^3 + \frac{a_2}{12}x^4 \\ &- x\Big(\frac{a_0}{2} + \frac{a_1}{6} + \frac{a_2}{12}\Big)\Big] - \frac{1}{945\sqrt{\pi}}\Big[64a_2x^{9/2} + 144a_1x^{7/2} + 504a_0x^{5/2} - \Big(105a_2 \\ &+ 210a_1 + 630a_0\Big)x^{3/2}\Big] - \frac{1}{210\sqrt{\pi}}\Big[\frac{82}{9}a_2 + \frac{44}{3}a_1 + 28a_0\Big]x - \frac{a_2}{360}x^6 - \frac{a_1}{120}x^5 \end{aligned}$$

$$- \frac{a_0}{24}x^4 + \Big(\frac{a_2}{72} + \frac{a_1}{36} + \frac{a_0}{12}\Big)x^3 - \Big(\frac{a_2}{90} + \frac{7a_1}{360} + \frac{a_0}{24}\Big)x.$$

More so, upon collecting the related terms of $y_1(x)$ from the above equation at $p = 1$, we get

$$\begin{aligned} y_1(x) &= -\frac{64a_2}{945\sqrt{\pi}}x^{9/2} + \Big(\frac{32}{35\sqrt{\pi}} - \frac{16a_1}{105\sqrt{\pi}}\Big)x^{7/2} - \frac{8a_0}{15\sqrt{\pi}}x^{5/2} + \frac{1}{945\sqrt{\pi}}\Big(105a_2 \\ &+ 210a_1 + 630a_0\Big)x^{3/2} - \frac{a_2}{360}x^6 + \Big(\frac{1}{20} - \frac{a_1}{120}\Big)x^5 - \Big(\frac{a_2}{12} + \frac{a_0}{24}\Big)x^4 \\ &+ \Big(\frac{5}{6} + \frac{a_2}{72} - \frac{5a_1}{36} + \frac{a_0}{12}\Big)x^3 - \frac{a_0}{2}x^2 + \Big[\frac{13a_2}{180} + \frac{53a_1}{360} + \frac{11a_0}{24} \\ &- \frac{1}{210\sqrt{\pi}}\Big(\frac{82a_2}{9} + \frac{44a_1}{3} + 28a_0\Big) - \frac{32}{35\sqrt{\pi}} + \frac{53}{60}\Big]x. \end{aligned}$$

Indeed, on taking $y_1(x) = 0$, and thereafter equating the coefficients of x^n, we find $a_0 = 0$, $a_1 = 6$, and $a_2 = 0$. Hence, substituting these values into $y_0(x)$, gives $y(x) = y_0(x) = x^3 - x$, which matches the exact analytical solution of (18).

Example 2. *Consider the BVP for the nonhomogeneous Bagley-Torvik equation [29]*

$$D^{3/2}y(x) + y(x) = \frac{2x^{1/2}}{\Gamma(3/2)} + x^2 - x, \quad y(0) = 0, \quad y(1) = 0, \tag{22}$$

which admits the exact analytical solution as follows $y(x) = x^2 - x$.

To begin with, we write (22) in an operator form as follows

$$D^{3/2}y(x) = \frac{2x^{1/2}}{\Gamma(3/2)} + x^2 - x - y(x). \tag{23}$$

Next, upon deploying the inverse operator $I^{3/2}$ on both sides of (23) using the property (4) on the left-hand side with $y'(0) = c_1$, one gets

$$\begin{aligned} I^{3/2}[D^{3/2}y(x)] &= I^{3/2}\Big[\frac{2x^{1/2}}{\Gamma(3/2)} + x^2 - x\Big] - I^{3/2}[y(x)], \\ y(x) - \sum_{k=0}^{1} y^k(0)\frac{x^k}{\Gamma(k+1)} &= I^{3/2}\Big[\frac{2x^{1/2}}{\Gamma(3/2)} + x^2 - x\Big] - I^{3/2}[y(x)], \\ y(x) &= c_1 x + I^{3/2}\Big[\frac{2x^{1/2}}{\Gamma(3/2)} + x^2 - x\Big] - I^{3/2}[y(x)]. \end{aligned}$$

Using MADM and the properties mentioned in (5) and (7), we write

$$\sum_{n=0}^{\infty} y_n(x) = c_1 x - pI^{3/2}\Big[\sum_{n=0}^{\infty} a_n x^n\Big] + I^{3/2}\Big[\sum_{n=0}^{\infty} a_n x^n\Big] + I^{3/2}\Big[\frac{2x^{1/2}}{\Gamma(3/2)} + x^2 - x\Big] - I^{3/2}[y_n(x)],$$

upon which we iteratively consider

$$\begin{aligned} y_0(x) &= c_1 x + I^{3/2}\Big[\frac{2x^{1/2}}{\Gamma(3/2)} + x^2 - x\Big] + I^{3/2}\Big[\sum_{n=0}^{\infty} a_n x^n\Big], \\ &= c_1 x + x^2 + \frac{32}{105\sqrt{\pi}}x^{7/2} - \frac{8}{15\sqrt{\pi}}x^{5/2} + \frac{4a_0}{3\sqrt{\pi}}x^{3/2} + \frac{8a_1}{15\sqrt{\pi}}x^{5/2} \\ &+ \frac{32a_2}{105\sqrt{\pi}}x^{7/2} + \frac{64a_3}{315\sqrt{\pi}}x^{9/2} + ... \end{aligned}$$

and

$$\begin{aligned} y_1(x) &= -pI^{3/2}\Big[\sum_{n=0}^{\infty} a_n x^n\Big] - I^{3/2}[y_0(x)], \\ &= -p\Big[\frac{4a_0}{3\sqrt{\pi}}x^{3/2} + \frac{8a_1}{15\sqrt{\pi}}x^{5/2} + \frac{32a_2}{105\sqrt{\pi}}x^{7/2} + \frac{64a_3}{315\sqrt{\pi}}x^{9/2} + ...\Big] - \frac{8c_1}{15\sqrt{\pi}}x^{5/2} \\ &- \frac{32}{105\sqrt{\pi}}x^{7/2} - \frac{1}{60}x^5 + \frac{1}{24}x^4 - \frac{a_0}{6}x^3 - \frac{a_1}{24}x^4 - \frac{a_2}{60}x^5 - \frac{3a_3}{360}x^6 + ... \end{aligned}$$

Now, setting $y_1(x) = 0$ with $p = 1$ and collecting the related terms while equating the coefficients of x^n in both sides of the above expression, one obtains

$$\begin{aligned} y_1(x) &= -\frac{a_0}{6}x^3 + \Big(\frac{1}{24} - \frac{a_1}{24}\Big)x^4 - \Big(\frac{1}{60} + \frac{a_2}{60}\Big)x^5 - \frac{3a_3}{360}x^6 + ... \\ &= 0. \end{aligned}$$

Hence, we acquire $a_0 = a_3 = 0$, $a_1 = 1$, and $a_2 = -1$, which when these values of a_0, a_1, a_2 and a_3 are substituted in $y_0(x)$, where $y_n(x) = 0$, for $n \geqslant 1$ reveals

$$\begin{aligned} y(x) = y_0(x) &= c_1 x + x^2 + \frac{32}{105\sqrt{\pi}}x^{7/2} - \frac{8}{15\sqrt{\pi}}x^{5/2} + \frac{8}{15\sqrt{\pi}}x^{5/2} - \frac{32}{105\sqrt{\pi}}x^{7/2} + ... \\ &= c_1 x + x^2 + ... \end{aligned}$$

Lastly, when cancelling the noise terms, alongside using the second boundary prescription of $y(1) = 0$, one obtains $c_1 = -1$, so that

$$y(x) = x^2 - x,$$

which matches the exact analytical solution for (22)*.*

Example 3. *We make consideration to the BVP for the nonhomogeneous Bagley-Torvik [40]*

$$D^2 y(x) + D^{3/2} y(x) + y(x) = x + 1, \quad y(0) = 1, \quad y(1) = 2, \tag{24}$$

that admits $y(x) = x + 1$ as its exact analytical solution.

Algorithm 1:
We re-write (24) *using operator notation as follows*

$$Ly(x) = x + 1 - D^{3/2}y(x) - y(x). \tag{25}$$

Applying inverse operator $L^{-1}(.) = \int_0^x \int_0^x (.)dxdx$ *on both sides of* (25) *such that* $y'(0) = c_1$ *gives*

$$y(x) = 1 + c_1x - L^{-1}[D^{3/2}y(x)] - L^{-1}[y(x)] + L^{-1}[x + 1].$$

Further, using the MADM procedure along with (6)*, we gain*

$$\begin{aligned}\sum_{n=0}^{\infty} y_n(x) &= 1 + c_1x - pL^{-1}\Big[\sum_{n=0}^{\infty} a_nx^n\Big] + L^{-1}\Big[\sum_{n=0}^{\infty} a_nx^n\Big] - L^{-1}\Big[D^{3/2}\sum_{n=0}^{\infty} y_n(x)\Big] \\ &- L^{-1}\Big[\sum_{n=0}^{\infty} y_n(x)\Big] + L^{-1}[x + 1],\end{aligned}$$

upon which the related iterates $y_0(x)$ *and* $y_1(x)$ *are considered from the latter equation as follows*

$$\begin{aligned}y_0(x) &= 1 + c_1x + L^{-1}\Big[\sum_{n=0}^{\infty} a_nx^n\Big], \\ &= 1 + c_1x + \frac{1}{2}a_0x^2 + \frac{1}{6}a_1x^3 + \frac{1}{12}a_2x^4 + ...\end{aligned}$$

and

$$\begin{aligned}y_1(x) &= L^{-1}[1 + x] - pL^{-1}\Big[\sum_{n=0}^{\infty} a_nx^n\Big] - L^{-1}\Big[D^{3/2}y_0(x)\Big] - L^{-1}[y_0(x)], \\ &= \frac{1}{2}x^2 + \frac{1}{6}x^3 - p\Big[\frac{1}{2}a_0x^2 + \frac{1}{6}a_1x^3 + \frac{1}{12}a_2x^4 + ...\Big] - \frac{4c_1}{3\sqrt{\pi}}x^{3/2} - \frac{8a_0}{15\sqrt{\pi}}x^{5/2}, \\ &- \frac{16a_1}{105\sqrt{\pi}}x^{7/2} - \frac{64a_2}{945\sqrt{\pi}}x^{9/2} - \frac{1}{2}x^2 - \frac{c_1}{6}x^3 - \frac{a_0}{24}x^4 - \frac{a_1}{120}x^5 - \frac{a_2}{360}x^6 + ...\end{aligned}$$

Next, collecting the like terms, and thereafter equating the coefficients of x^n *in both sides of the above expression, where* $y_1(x) = 0$ *with* $p = 1$ *gives*

$$\begin{aligned}y_1(x) &= \frac{1}{6}x^3 - \frac{1}{2}a_0x^2 - \frac{1}{6}a_1x^3 - \frac{1}{12}a_2x^4 - \frac{4c_1}{3\sqrt{\pi}}x^{3/2} - \frac{8a_0}{15\sqrt{\pi}}x^{5/2} \\ &- \frac{16a_1}{105\sqrt{\pi}}x^{7/2} - \frac{64a_2}{945\sqrt{\pi}}x^{9/2} - \frac{c_1}{6}x^3 - \frac{a_0}{24}x^4 - \frac{a_1}{120}x^5 - \frac{a_2}{360}x^6 + ... \\ &= 0.\end{aligned}$$

Therefore, we obtain $a_0 = 0$, $a_1 = 1-c_1$, *and* $a_2 = 0$. *In addition, substituting these values in* $y_0(x)$, *where* $y_n(x) = 0$, *for* $n \geqslant 1$ *yields*

$$y(x) = y_0(x) = 1 + c_1 x + \frac{1}{6}(1-c)x^3.$$

Moreover, on using the subsequent boundary condition $y(1) = 2$, *one acquires* $c_1 = 1$, *so that*

$$y(x) = x + 1,$$

which is indeed the reported exact analytical solution for BVP (24).

Algorithm 2:

Here, we make use of the equation represented in an operator notation (25), *and further define* L^{-1} *as given in* (14). *Thus, implementing the inverse operator on equation* (25) *reveals*

$$y(x) = x + 1 + L^{-1}[x+1] - L^{-1}[D^{3/2}y(x)] - L^{-1}[y(x)].$$

Using MADM and (6)

$$\begin{aligned}\sum_{n=0}^{\infty} y_n(x) &= x + 1 + L^{-1}[x+1] - pL^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big] + L^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big] \\ &- L^{-1}\Big[D^{3/2}\sum_{n=0}^{\infty} y_n(x)\Big] - L^{-1}\Big[\sum_{n=0}^{\infty} y_n(x)\Big].\end{aligned}$$

Now, let

$$y_0(x) = x + 1 + L^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big],$$

and

$$y_1(x) = L^{-1}[x+1] - pL^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big] - L^{-1}[D^{3/2}y_0(x)] - L^{-1}[y_0(x)],$$

then, we precisely compute the expressions for $y_0(x)$ *and* $y_1(x)$ *via the inverse operator* (14) *as follows*

$$y_0(x) = x + 1 + \frac{a_0}{2}x^2 + \frac{a_1}{6}x^3 + \frac{a_2}{12}x^4 - x\Big(\frac{a_0}{2} + \frac{a_1}{6} + \frac{a_2}{12}\Big),$$

and

$$\begin{aligned}y_1(x) &= \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{2}{3}x - p\Big[\frac{a_0}{2}x^2 + \frac{a_1}{6}x^3 + \frac{a_2}{12}x^4 - x\Big(\frac{a_0}{2} + \frac{a_1}{6} + \frac{a_2}{12}\Big)\Big] \\ &- \frac{1}{945\sqrt{\pi}}\Big[504a_0x^{5/2} + 144a_1x^{7/2} + 64a_2x^{9/2} - \Big(630a_0 + 210a_1 \\ &+ 105a_2 - 1260\Big)x^{3/2}\Big] - \frac{1}{210\sqrt{\pi}}\Big[28a_0 + \frac{44}{3}a_1 + \frac{82}{9}a_2 - 280\Big]x\end{aligned}$$

$$- \frac{a_0}{24}x^4 - \frac{a_1}{120}x^5 - \frac{a_2}{360}x^6 + \Big(\frac{a_0}{12} + \frac{a_1}{36} + \frac{a_2}{72} - \frac{1}{6}\Big)x^3 - \frac{1}{2}x^2$$
$$- \Big(\frac{a_0}{24} + \frac{7a_1}{360} + \frac{a_2}{90} - \frac{2}{3}\Big)x.$$

In fact, the explicit expression for $y_1(x)$ will be found after collecting the like terms from the equation at $p = 1$ as follows

$$\begin{aligned} y_1(x) &= -\frac{64a_2}{945\sqrt{\pi}}x^{9/2} - \frac{16a_1}{105\sqrt{\pi}}x^{7/2} - \frac{8a_0}{15\sqrt{\pi}}x^{5/2} + \frac{1}{945\sqrt{\pi}}\Big(630a_0 + 210a_1 + 105a_2 \\ &- 1260\Big)x^{3/2} - \frac{a_2}{360}x^6 - \frac{a_1}{120}x^5 - \Big(\frac{a_2}{12} + \frac{a_0}{24}\Big)x^4 + \Big(\frac{a_2}{72} - \frac{5a_1}{36} + \frac{a_0}{12}\Big)x^3 - \frac{a_0}{2}x^2 \\ &+ \Big[\frac{13a_2}{180} + \frac{53a_1}{360} + \frac{11a_0}{24} - \frac{1}{210\sqrt{\pi}}\Big(\frac{82a_2}{9} + \frac{44a_1}{3} + 28a_0 - 280\Big)\Big]x. \end{aligned}$$

Finally, setting $y_1(x) = 0$, and thereafter equating the coefficients of x^n, we obtain $a_0 = a_1 = 0 = a_2 = 0$. We, therefore, obtain $y(x) = y_0(x) = x + 1$ as the aiming solution, which is indeed the reported exact analytical solution for (24)*.*

Example 4. *We consider the BVP for the nonhomogeneous Bagley-Torvik equation as follows [35]*

$$D^2y(x) + D^{3/2}y(x) + y(x) = x^2 + 2 + 4\sqrt{\frac{x}{\pi}}, \quad y(0) = 0, \quad y(5) = 25, \tag{26}$$

that admits $y(x) = x^2$ as its exact analytical solution.

Algorithm 1*:*

Illustrating (26) *via operator notation, one writes*

$$Ly(x) = x^2 + 2 + 4\sqrt{\frac{x}{\pi}} - D^{3/2}y(x) - y(x). \tag{27}$$

Next, on enforcing the inverse operator $L^{-1}(.) = \int_0^x \int_0^x (.)dxdx$ on both sides of (27) *such that $y'(0) = c_1$, we get*

$$y(x) = c_1x - L^{-1}[D^{3/2}y(x)] - L^{-1}[y(x)] + L^{-1}\Big[x^2 + 2 + 4\sqrt{\frac{x}{\pi}}\Big].$$

Using MADM with (6)*, we further get*

$$\begin{aligned} \sum_{n=0}^{\infty} y_n(x) &= c_1x - pL^{-1}\Big[\sum_{n=0}^{\infty} a_nx^n\Big] + L^{-1}\Big[\sum_{n=0}^{\infty} a_nx^n\Big] + L^{-1}\Big[x^2 + 2 + 4\sqrt{\frac{x}{\pi}}\Big] \\ &- L^{-1}\Big[D^{3/2}\sum_{n=0}^{\infty} y_n(x)\Big] - L^{-1}\Big[\sum_{n=0}^{\infty} y_n(x)\Big], \end{aligned}$$

and thereafter take into account $y_0(x)$ and $y_1(x)$ as follows

$$\begin{aligned} y_0(x) &= c_1x + L^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big], \\ &= c_1x + \frac{1}{2}a_0x^2 + \frac{1}{6}a_1x^3 + \frac{1}{12}a_2x^4 + ... \end{aligned}$$

and

$$\begin{aligned} y_1(x) &= -pL^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big] + L^{-1}\Big[x^2 + 2 + 4\sqrt{\frac{x}{\pi}}\Big] - L^{-1}\Big[D^{3/2}y_0(x)\Big] - L^{-1}[y_0(x)], \\ &= -p\Big[\frac{1}{2}a_0x^2 + \frac{1}{6}a_1x^3 + \frac{1}{12}a_2x^4 + ...\Big] + \frac{1}{12}x^4 + x^2 + \frac{16}{15\sqrt{\pi}}x^{5/2} - \frac{4c_1}{3\sqrt{\pi}}x^{3/2} \\ &- \frac{8a_0}{15\sqrt{\pi}}x^{5/2} - \frac{16a_1}{105\sqrt{\pi}}x^{7/2} - \frac{64a_2}{945\sqrt{\pi}}x^{9/2} - \frac{c_1}{6}x^3 - \frac{a_0}{24}x^4 - \frac{a_1}{120}x^5 - \frac{a_2}{360}x^6 + ... \end{aligned}$$

More so, collecting the related terms, and subsequently the equating coefficients of x^n in both sides of the above expression, where $y_1(x) = 0$ with $p = 1$, one gets

$$\begin{aligned} y_1(x) &= \Big(-\frac{1}{2}a_0 + 1\Big)x^2 - \Big(\frac{1}{6}a_1 + \frac{c_1}{6}\Big)x^3 + \Big(\frac{1}{12} - \frac{1}{12}a_2 - \frac{a_0}{24}\Big)x^4 \\ &+ \frac{16}{15\sqrt{\pi}}x^{5/2} - \frac{4c_1}{3\sqrt{\pi}}x^{3/2} - \frac{8a_0}{15\sqrt{\pi}}x^{5/2} - \frac{16a_1}{105\sqrt{\pi}}x^{7/2} - \frac{64a_2}{945\sqrt{\pi}}x^{9/2} + ... \\ &= 0. \end{aligned}$$

Therefore, we get $a_0 = 2$, $a_1 = -c_1$, and $a_2 = 0$. More so, putting these values in $y_0(x)$, where $y_n(x) = 0$, for $n \geqslant 1$, we get

$$y(x) = y_0(x) = c_1x + x^2 - \frac{1}{6}c_1x^3.$$

Finally, deploying the second boundary condition $y(5) = 25$, we get $c_1 = 0$, which leads to the acquisition of $y(x) = x^2$, as the reported exact analytical solution.

Algorithm 2:

As represented in (27), we define the inverse operator L^{-1} in (14). Further, we deploy $L^{-1}(.) = \int_0^x dx' \int_0^{x''}(.)dx'' - \frac{x}{5}\int_0^5 dx \int_0^{x''}(.)dx''$, on both sides of (27) to get

$$y(x) = 5x + L^{-1}\Big[x^2 + 2 + 4\sqrt{\frac{x}{\pi}}\Big] - L^{-1}[D^{3/2}y(x)] - L^{-1}[y(x)]$$

Using MADM and (6)

$$\sum_{n=0}^{\infty} y_n(x) = 5x + L^{-1}\Big[x^2 + 2 + 4\sqrt{\frac{x}{\pi}}\Big] - pL^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big] + L^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big]$$

$$- \quad L^{-1}\Big[D^{3/2}\sum_{n=0}^{\infty} y_n(x)\Big] - L^{-1}\Big[\sum_{n=0}^{\infty} y_n(x)\Big].$$

In addition, the latter equation further results in

$$y_0(x) = 5x + L^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big],$$

and

$$y_1(x) = L^{-1}\Big[x^2 + 2 + 4\sqrt{\frac{x}{\pi}}\Big] - pL^{-1}\Big[\sum_{n=0}^{\infty} a_n x^n\Big] - L^{-1}[D^{3/2}y_0(x)] - L^{-1}[y_0(x)],$$

then, evaluating $y_0(x)$ *using the inverse operator* (14) *yields*

$$y_0(x) = 5x + \frac{a_0}{2}x^2 + \frac{a_1}{6}x^3 + \frac{a_2}{12}x^4 + \frac{a_3}{20}x^5 - x\Big(\frac{5a_0}{2} + \frac{25a_1}{6} + \frac{125a_2}{12} + \frac{125a_3}{4}\Big),$$

while $y_1(x)$ *is found by collecting out the like terms by using the inverse operator* (14) *and setting* $p = 1$ *as follows*

$$\begin{aligned} y_1(x) &= -\frac{a_3}{840}x^7 - \frac{a_2}{360}x^6 - \Big(\frac{a_1}{120} + \frac{a_3}{20}\Big)x^5 + \Big(\frac{1}{12} - \frac{a_2}{12} - \frac{a_0}{24}\Big)x^4 \\ &+ \Big(\frac{125a_3}{24} + \frac{125a_2}{72} - \frac{19a_1}{36} + \frac{5a_0}{12} - \frac{25}{6}\Big)x^3 + \Big(1 - \frac{a_0}{2}\Big)x^2 - \Big[\frac{185}{12} + \frac{80\sqrt{5}}{15\sqrt{\pi}} \\ &+ \frac{1125a_3}{14} + \frac{875a_2}{36} + \frac{575a_1}{72} + \frac{65a_0}{24} - \frac{652}{6} + \frac{1}{3150\sqrt{\pi}}\Big(2100\sqrt{5}a_0 + 5500\sqrt{5}a_1 \\ &+ \frac{51250\sqrt{5}}{3}a_2 + \frac{643750\sqrt{5}}{11}a_3 - 105000\sqrt{5}\Big)\Big]x - \frac{128a_3}{3465\sqrt{\pi}}x^{11/2} - \frac{64a_2}{945\sqrt{\pi}}x^{9/2} \\ &- \frac{16a_1}{105\sqrt{\pi}}x^{7/2} + \Big(\frac{16}{15\sqrt{\pi}} - \frac{8a_0}{15\sqrt{\pi}}\Big)x^{5/2} + \frac{1}{10395\sqrt{\pi}}\Big(433125a_3 + 144375a_2 \\ &+ 57750a_1 + 34650a_0 - 346500\Big)x^{3/2}. \end{aligned}$$

Moreover, setting $y_1(x) = 0$ *while equating the coefficients of* x^n*, we acquire* $a_0 = 2$*, and* $a_1 = a_2 = a_3 = 0$*. Hence,* $y(x) = y_0(x) = x^2$*, which happens to be the reported exact analytical solution of* (26).

Example 5. *Consider the BVP* (24) *after being transformed into a coupled system via [18] as follows*

$$\begin{cases} D^{1.5}y_1 = y_2, \;\; y_1(0) = 1, \;\; y_1(1) = 2, \\ D^{0.5}y_2 = -y_2 - y_1 + 1 + x, \;\; y_2(0) = 0. \end{cases} \tag{28}$$

Here, we apply the inverse operator on both sides of the equations in (28) *to get*

$$\begin{cases} I^{1.5}[D^{1.5}y_1] = I^{1.5}[y_2], \\ I^{0.5}[D^{0.5}y_2] = I^{0.5}[-y_2 - y_1 + 1 + x]. \end{cases} \tag{29}$$

More so, applying the property of the fractional derivative mentioned in (4) *on the left-hand sides of the latter equations with* $y_1'(0) = c_1$*, we get*

$$\begin{cases} y_1(x) = 1 + c_1 x + I^{1.5}[y_2], \\ y_2(x) = -I^{0.5}[y_2] - I^{0.5}[y_1] + I^{0.5}[1+x]. \end{cases}$$

Next, upon using MADM, one gets

$$\begin{aligned} \sum_{n=0}^{\infty} y_{1,n}(x) &= 1 + c_1 x + I^{1.5}[\sum_{n=0}^{\infty} y_{2,n}] - pI^{1.5}[\sum_{n=0}^{\infty} a_n x^n] + I^{1.5}[\sum_{n=0}^{\infty} a_n x^n], \\ \sum_{n=0}^{\infty} y_{2,n}(x) &= -I^{0.5}[\sum_{n=0}^{\infty} y_{2,n}] - I^{0.5}[\sum_{n=0}^{\infty} y_{1,n}] + I^{0.5}[1+x] - pI^{0.5}[\sum_{n=0}^{\infty} a_n x^n] \\ &+ I^{0.5}[\sum_{n=0}^{\infty} a_n x^n], \end{aligned}$$

that expands to

$$\begin{cases} y_{1,0}(x) = 1 + c_1 x + I^{1.5}[\sum_{n=0}^{\infty} a_n x^n], \\ y_{1,1}(x) = -pI^{1.5}[\sum_{n=0}^{\infty} a_n x^n] + I^{1.5}[y_{2,0}], \\ \\ y_{2,0}(x) = I^{0.5}[1+x] + I^{0.5}[\sum_{n=0}^{\infty} a_n x^n], \\ y_{2,1}(x) = -pI^{0.5}[\sum_{n=0}^{\infty} a_n x^n] - I^{0.5}[y_{2,0}] - I^{0.5}[y_1]. \end{cases} \tag{30}$$

Additionally, upon deploying (7)*, one can iteratively solve the equations in* (30) *concurrently as follows*

$$\begin{aligned} y_{1,0}(x) &= 1 + c_1 x + \frac{4a_0}{3\sqrt{\pi}} x^{3/2} + \frac{8a_1}{15\sqrt{\pi}} x^{5/2} + \frac{32a_2}{105\sqrt{\pi}} x^{7/2} + \frac{64a_3}{315\sqrt{\pi}} x^{9/2} + ... \\ y_{1,1}(x) &= -p\Big[\frac{4a_0}{3\sqrt{\pi}} x^{3/2} + \frac{8a_1}{15\sqrt{\pi}} x^{5/2} + \frac{32a_2}{105\sqrt{\pi}} x^{7/2} + \frac{64a_3}{315\sqrt{\pi}} x^{9/2} + ...\Big] + \frac{1}{2}x^2 \\ &\quad + \frac{1}{6}x^3 + \frac{a_0}{2}x^2 + \frac{a_1}{6}x^3 + \frac{a_2}{12}x^4 + \frac{a_3}{20}x^5 + ... \end{aligned}$$

and

$$\begin{aligned} y_{2,0}(x) &= \frac{2}{\sqrt{\pi}} x^{1/2} + \frac{4}{3\sqrt{\pi}} x^{3/2} + \frac{2a_0}{\sqrt{\pi}} x^{1/2} + \frac{4a_1}{3\sqrt{\pi}} x^{3/2} + \frac{16a_2}{15\sqrt{\pi}} x^{5/2} + \frac{32a_3}{35\sqrt{\pi}} x^{7/2} + ... \\ y_{2,1}(x) &= -p\Big[\frac{2a_0}{\sqrt{\pi}} x^{1/2} + \frac{4a_1}{3\sqrt{\pi}} x^{3/2} + \frac{16a_2}{15\sqrt{\pi}} x^{5/2} + \frac{32a_3}{35\sqrt{\pi}} x^{7/2} + ...\Big] - x - \frac{1}{2}x^2 \\ &\quad - a_0 x - \frac{a_1}{2}x^2 - \frac{a_2}{3}x^3 - \frac{a_3}{12}x^4 - I^{0.5}[y_1]. \end{aligned}$$

Now, to compute the expression for the term $I^{0.5}[y_1]$ *in the last equation, we have to find* $y_1(x)$ *first. So, let* $y_{1,1}(x) = 0$ *with* $p = 1$*, and equating coefficients of* $x^{n/2}$*, we get* $a_0 = a_1 = a_2 = a_3 = 0$*. Then,*

$$y_1(x) = y_{1,0}(x) = 1 + c_1 x = 1 + x, \tag{31}$$

after using the second boundary condition $y_1(1) = 2$*. Further, we get*

$$I^{0.5}[y_1] = I^{0.5}[1 + x] = \frac{2}{\sqrt{\pi}} x^{1/2} + \frac{4}{3\sqrt{\pi}} x^{3/2}.$$

Next, substituting $I^{0.5}[y_1]$ *in* $y_{2,1}(x)$*, and further letting* $y_{2,1}(x) = 0$ *with* $p = 1$*, the resulting coefficients of* $x^{n/2}$ *are equated to obtain* $a_0 = a_1 = -1$*, and* $a_2 = a_3 = 0$*. Hence, substituting these values in* $y_{2,0}(x)$*, one gets*

$$\begin{aligned} y_2(x) = y_{2,0}(x) &= \frac{2}{\sqrt{\pi}} x^{1/2} + \frac{4}{3\sqrt{\pi}} x^{3/2} - \frac{2}{\sqrt{\pi}} x^{1/2} - \frac{4}{3\sqrt{\pi}} x^{3/2} + 0, \\ &= 0. \end{aligned} \tag{32}$$

We, therefore, conclude from (31) *and* (32) *that the coupled system expressed in* (28) *admits the following solution*

$$\begin{aligned} y_1(x) &= 1 + x, \\ y_2(x) &= 0, \end{aligned} \tag{33}$$

which is indeed the exact analytical solution for (24)*.*

5. Conclusion

In conclusion, we obtained the exact analytical solution of the fractional Bagley-Torvik equation, as well as a system of fractional Bagley-Torvik equations fitted with Dirichlet boundary data. we introduced two algorithms based on MADM to semi-analytically treat the class of fractional Bagley-Torvik equation endowed with Dirichlet boundary data. The algorithms were founded by the inverse linear operator theorems and infused in MADM to calculate only the first and second components y_0 and y_1; indeed, this is one of the advantages of MADM over the classical ADM, which computes several components most at times to arrive as the optimal solution. Further, to demonstrate the effectiveness and application of the new algorithms, Maple software 2023 has been used for the computational simulation of different nonhomogeneous BVP for Bagley-Torvik equations, which are indeed more complex than their homogeneous counterparts. Additionally, based on the results obtained with regard to the examined examples, one can easily infer that the devised algorithms are efficient and reliable methods for solving BVPs with Dirichlet boundary conditions in particular, and in general, they can be extended to tackling different types of BVPs, such as fractional PDEs and fractional integral IDEs.

6. Declaration

- **Authors' contribution**
  The authors collectively worked on the manuscript, and they read and approved the final draft.
- **Availability of data and materials**
  Not applicable.
- **Competing interests**
  The authors declare that they have no competing interests.
- **Funding**
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