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Extending Abelian Rings: A Generalized Approach

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Abstract. We introduce a novel framework for assessing the centrality of idempotents within a ring by presenting a general concept that assigns a degree of centrality. This approach aligns with the previously established notions of semicentral and q-central idempotents by Birkenmeier and Lam. Specifically, we define an idempotent e in a ring R to be n-central, where n is a positive integer, if $[e, R]^n e = 0$, where $[x, y]$ represents the additive commutator $xy-yx$. If every idempotent in a ring R is n-central, we refer to R as n-Abelian. Our study lays the groundwork by presenting foundational results that support this concept and examines key features of n-central idempotents essential for appropriately categorizing n-Abelian rings among various generalizations of Abelian rings introduced in prior literature. We provide examples of n-central idempotents that do not fall under the categories of semicentral or q-central. Furthermore, we demonstrate that the ring of upper matrices $\mathbb{T}_n(R)$, where R is Abelian, is an n-abelian. We also prove that a ring where all of its idempotents are n-central is an exchange ring if and only if the ring is clean.

2020 Mathematics Subject Classifications: 16U60, 16U70, 16U80, 16E50, 16U80

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1. Introduction

By the term “ring”, we mean an associative ring with nonzero identity. Further, $\mathcal{Z}(R)$, $\mathcal{I}(R)$, $\mathcal{U}(R)$, and $\mathcal{N}(R)$ are used for the set central elements, the set of idempotents (that is $e^2 = e$), the set of invertible (unit) elements, and the set of nilpotent elements of a ring R. An idempotent e of a ring R is called *central* if $e \in \mathcal{Z}(R)$. The set $\mathcal{B}(R)$ denotes the set of all central idempotents of R. A ring R is called *Abelian* if all idempotents of R are central; that $I(R) = \mathcal{B}(R)$. Throughout this paper, we will always notate the ring of $n \times n$ upper triangular matrices over a ring R by $\mathbb{T}_n(R)$.

The concept of semicentrality of idempotents was first introduced by Birkenmeier in 1983 [1] as a form of one-sided centrality to generalize some results on von Neumann regular rings. The semicentral idempotent has since been used in extensions of rings and

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modules by Birkenmeier and others. An idempotent e of a ring R is considered left (resp. right) semicentral if $ae = eae$ (resp., $ea = eae$) for all $a \in R$. The sets of left and right semicentral idempotents are denoted $\mathcal{S}_l(R)$ and $\mathcal{S}_r(R)$, respectively. Those interested in delving deeper into semicentral idempotents and q-central properties can find valuable insights in modern references, such as [2, 3, 7, 13, 17, 26]. An idempotent of a ring R is considered central if it is both left and right semicentral, and the set of central idempotents is denoted $\mathcal{B}(R) = \mathcal{S}_l(R) \cap \mathcal{S}_r(R)$. An idempotent is called semicentral if it is either left or right semicentral. A ring R is called semi-Abelian if all idempotents of R are semicentral.

The notion of idempotent centrality, which generalizes that of semicentral idempotents, was introduced by Lam in [11] as q-central idempotents. An idempotent e of a ring R is called q-central if $eR(1-e)Re = 0$, and the set of all q-central idempotents of R is denoted **q-idem**(R). If every idempotent of a ring R is q-central, then R is called q-Abelian. This condition has been introduced in several works, such as [22, 24], but Lam was the first to name and study this property as an elemental property. A ring R is called q-Abelian if every element of R is q-central, which is referred to as a quasi-normal ring and defined in [23]. It is worth noting that every semicentral idempotent is q-central; therefore, every semi-Abelian ring is q-Abelian.

This paper is structured into three primary sections. The initial segment is dedicated to a comprehensive elucidation of the concept of n-central idempotents. Within this section, we undertake a rigorous exploration to establish the existence of n-central idempotents distinct from semicentral and q-central idempotents, as substantiated by illustrative examples referenced as 1. Furthermore, Proposition 4 rigorously establishes that every n-central idempotent within a semiprime ring unequivocally assumes a central position. The culmination of this section lies in the proof establishing that if idempotents e and f are conjugate, then e assumes centrality if and only if f does so.

The subsequent section of the paper is dedicated to defining n-Abelian rings. Proposition 9 rigorously demonstrates that every n-central idempotent enjoys a state of direct finiteness. Drawing from Lam's seminal work [12], where he established the semiabelian nature of 2×2 upper triangular matrices denoted as $\mathbb{T}_2(R)$, we further extend this understanding. Specifically, Theorem 4 posited within this section firmly establishes the n-Abelian property for the set $\mathbb{T}_n$ across all values of n.

2. n-central idempotents

This section introduces the concept of n-central idempotents for a ring R. The definition of n-central idempotents involves a recursive sequence of sets of R. Let e be an idempotent of R. We define a collection of right ideals $[e]_n$ of R, for $n \geq 0$, recursively as follows:

$$[e]_0 = eR, \quad [e]_1 = (1-e)ReR, \qquad \text{and for } i \geq 2, \qquad [e]_i = [e]_{i-2}(1-e)ReR.$$

Using the notation introduced earlier, if there exists some k such that $[e]_k = [e]_{k+1}$, then for every $i \geq k$, we have $[e]_i = 0$. Therefore, the sequence $([e]_k)$ is eventually-zero, as shown in Example 1. On the other hand, if there exists some k such that $[e]_k = [e]_{k+2}$, then we have $[e]_{i+2} = [e]_i$ for every $i \geq k$, as in Example 2. Note that every two consecutive sets in the sequence $([e]_n)$ have zero intersection. We shall refer to $[e]_n$ as the *right centralizer of e with degree n.*

The *additive commutator* of elements x and y in a ring R is denoted by $[x, y]$ and defined as $xy - yx$. For an idempotent e and a subset S of a ring R, we use the notation $[e, S]$ to represent the subset $\{es - se|s \in S\}$ of R. The following lemma provides an alternative definition for the centralizer $[e]_n$ introduced earlier, using additive commutators. This definition allows for the centralizers to be extended to rings without identity. However, our focus in this research is still on unital rings.

Lemma 1. *For every idempotent e of R and non-negative integer n, we have $[e]_n = [e, R]^n eR$.*

Proof.
We will prove the relation using the mathematical induction on n.

Claim 1. $[e]_1 = [e, R]eR$.
For every, $r, s \in R$, we have $[e, r]es = (er - re)es = eres - res = (1 - e)(-r)es \in [e]_1$ and $[e, R]eR \subseteq [e]_1$. Conversely, for every $a \in [e]_1$, there exist $x, y \in R$ such that $a = (1 - e)xey$. Hence, $a = xey - exey = (xe - ex)ey = [e, x]e(-y) \in [e, R]eR$. Therefore, $[e]_1 = [e, R]eR$.

Claim 2. $e[e, R] = [e, R](1 - e)$ *and* $[e, R]e = (1 - e)[e, R]$.
For every $r \in R$, we have $e[e, r] = e(er - re) = er - ere = er(1 - e) = er(1 - e) - re(1 - e) = (er - re)(1 - e) = [e, r](1 - e)$ and $e[e, R] = [e, R](1 - e)$. Similarly, $[e, R]e = (1 - e)[e, R]$.

Claim 3. $[e, R]^2 e = eR(1 - e)Re$ *and* $[e, R]^2(1 - e) = eR(1 - e)R(1 - e)$.
For every $r, s \in R$, we have $[e, r][e, s]e = (er - re)(es - se)e = (er - re)(ese - se) = (er - re)(1 - e)(-se) = er(1 - e)(-s)e \in eR(1 - e)Re$. Also, $er(1 - e)se = er(1 - e)se = (er - re)(1 - e)se = [e, r](se - ese) = [e, r](se - es)e = [e, r][e, -s]e \in eR(1 - e)Re$. Thus $[e, R]^2 e = eR(1 - e)Re$ and similarly one can prove that $[e, R]^2(1 - e) = eR(1 - e)R(1 - e)$

Indeed, $[e]_n = [e, R]^n eR$, for $n = 0$ and $n = 1$ from Claim 1. Assume that $[e]_k = [e, R]^k eR$, for some integer $k \geq 0$. If k is even, then

$$[e, R]^{k+2} eR = [e, R]^2 [e, R]^k eR = [e, R]^2 e[e, R]^k eR = eR(1 - e)Re[e]_k = [e]_{k+2},$$

using the result of Claims 2 and 3. If k is odd, then

$$[e, R]^{k+2} eR = [e, R]^2 [e, R]^k eR = [e, R]^2 (1 - e)[e, R]^k eR = (1 - e)ReR(1 - e)[e]_k = [e]_{k+2}.$$

The next definition uses our notation to give a generalized centrality for the idempotents of a ring.

Definition 1. *An idempotent e of a ring R is said to be n-central, for some positive integer n, if $[e]_n = 0$. Moreover, e is called complementary n-central if $1 - e$ is n-central and dual n-central if it is both n-central and complementary n-central.*

The following examples serve to showcase idempotents which possess the characteristic of being n-central, yet they do not fall within the categories of semicentral or q-central.

Example 1. *Let $R = \mathbb{T}_3(F)$ for some field F. For the idempotent $e = \mathbf{diag}(1,0,1)$, we have*

$$([e]_n) = \left(\begin{bmatrix} F & F & F \\ 0 & 0 & 0 \\ 0 & 0 & F \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & F \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, 0, 0, 0, \cdots \right);$$

which means that e is 3-central. Note that straightforward calculations can show that e is neither semicentral nor q-central.

The example below illustrates that there are idempotents that cannot be classified as n-central for any n.

Example 2. *In the ring $S = \mathbb{M}_4(F)$ of 4×4 matrices over the filed F, the idempotent* $f = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ *has the chain $([f]_n) = (A, B, C, B, C, B, C, \cdots)$, where*

$$A = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ a_5 & a_6 & a_7 & a_8 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix} \mid a_i \in F \right\},$$

$$B = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ -a_5 & -a_6 & -a_7 & -a_8 \end{bmatrix} \mid a_i \in F \right\},$$

$$C = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ a_5 & a_6 & a_7 & a_8 \\ -a_5 & -a_6 & -a_7 & -a_8 \end{bmatrix} \mid a_i \in F \right\}.$$

We make $\mathcal{C}_n(R)$, $\overline{\mathcal{C}}_n(R)$, and $\widehat{\mathcal{C}}(R)$ denote, respectively, the sets of n-central, complementary n-central, and dual n-central idempotents of a ring R. The definitions show that left semicentral, right semicentral, central, and quarter-central idempotents coincide with 1-central, complement 1-central, dual 1-central, and 2-central, respectively. In other words, $\mathcal{S}_l(R) = \mathcal{C}_1(R)$, $\mathcal{S}_r(R) = \overline{\mathcal{C}}_r(R)$, $\mathcal{B}(R) = \widehat{\mathcal{C}}_1(R)$, and $\mathbf{q\text{-}idem}(R) = \mathcal{C}_2(R)$, of every ring R. Notice that every n-central is m-central if $n \leq m$.

Observe that Lemma 1 provides an alternative definition for the n-centralizer of an idempotent e in a ring R, which is independent of the existence of identity in R. This definition allows us to extend the notion of n-central idempotents to rings without unity or near-rings. Nevertheless, for the purpose of this paper, we restrict our attention to associative and unital rings.

The following proposition presents several equivalent characterizations of n-central idempotents.

Proposition 1. *For a ring R and idempotent e of R, the following statements are equivalent.*

(i) $e \in \mathcal{C}_n(R)$.

(ii) $[e]_{n-1}$ *is an ideal of* R.

(iii) $[e, R]^n e = 0$.

(iv) $e[e, R]^n = 0$ *if* n *is even,* $(1-e)[e, R]^n = 0$ *if* n *is odd.*

Proof. (i)⇒(ii): If n is odd, then $(1-e)R[e]_{n-1} = 0$ and $R[e]_{n-1} = eR[e]_{n-1} = eReR[e]_{n-2} \subseteq eR[e]_{n-2} = [e]_{n-1}$ and $[e]_{n-1}$ is a two-sided ideal of R. Similarly, for the even case.

(ii)⇒(i): If $[e]_{n-1}$ is an ideal of R and assume that n is odd. So that $[e]_n = (1-e)R[e]_{n-1} \subseteq (1-e)[e]_{n-1} = 0$ and e is n-central. Similarity, for the even case.

(i)⇔(iii): It is direct by Lemma 1.

(i)⇔(iv) is clear from Claim 2.

Corollary 1. *For a ring R and idempotent e of R, $e \in \widehat{\mathcal{C}}_n(R)$ if and only if $[e, R]^n = 0$.*

Corollary 2 ([4], Proposition 1.2.2)**.** *For an idempotent e of a ring R, the following conditions are equivalent:*

(i) $e \in \mathcal{S}_l(R)$;

(ii) eR *is an ideal of* R;

(iii) $1 - e \in \mathcal{S}_r(R)$;

(iv) $R(1-e)$ *is an ideal of* R;

(v) $(1-e)Re = 0$;

The following proposition provides a generalization of [7, Proposition 2.1.] and [11, Proposition 2.1], which presents sufficient conditions for verifying the n-centrality of an idempotent. In this context, $1+\mathcal{N}(R)$ and $\mathcal{N}_2(R)$ respectively denote the sets of *unipotent* elements of R (i.e., the elements of the form $1 + a$ for a nilpotent element $a \in R$) and the set of all *square-zero* elements of R (i.e., the nilpotent elements of index 2). Note that although Lemma 1 provides an equivalent definition of the n-centralizer of an idempotent e in a ring R that is independent of the identity element of R, this paper considers only associative and unital rings.

Proposition 2. *For an idempotent e of a ring R the following statements are equivalent:*

(i) $e \in \mathcal{C}_n(R)$;

(ii) $[e, \mathcal{U}(R)]^n e = 0$;

(iii) $[e, 1 + \mathcal{N}(R)]^n e = 0$;

(iv) $[e, \mathcal{N}(R)]^n e = 0$;

(v) $[e, \mathcal{N}_2(R)]^n e = 0$;

(vi) $[e, \mathcal{I}(R)]^n e = 0$;

Proof. (i)$\Rightarrow$(ii) is obvious from Proposition 1.

(ii)$\Rightarrow$(iii) and (iv)$\Rightarrow$(v) are obvious.

(iii)$\Rightarrow$(iv): It is direct since $[a, 1 + b] = [a, b]$, for every $a, b \in R$

(v)$\Rightarrow$(vi): For all $f_i \in \mathcal{I}(R)$ (for $i = 1, \cdots, n$), the elements $ef_i(1-e)$ and $(1-e)f_ie$ are square-zero. But $[e, ef_i(1-e)] = ef_i(1-e) = [e, f_i](1-e)$ and similarity $[e, (1-e)f_ie] = -[e, f_i]e$. So that

$$\begin{aligned} [e, f_1] \cdots [e, f_{n-2}][e, f_{n-1}][e, f_n]e &= [e, f_1] \cdots [e, f_{n-2}][e, f_{n-1}][e, f_n]e^2 \\ &= [e, f_1] \cdots [e, f_{n-2}][e, f_{n-1}](1-e)[e, f_n]e \\ &= -[e, f_1] \cdots [e, f_{n-2}][e, f_{n-1}](1-e)[e, (1-e)f_ne] \\ &= -[e, f_1] \cdots [e, f_{n-2}]e[e, f_{n-1}](1-e)[e, (1-e)f_ne] \\ &= -[e, f_1] \cdots [e, f_{n-2}]e[e, (1-e)f_{n-1}e][e, (1-e)f_ne]. \end{aligned}$$

Continuing, we get $[e, f_1] \cdots [e, f_n]e = (-1)^{\frac{n}{2}} e[e, (1-e)f_1e] \cdots [e, (1-e)f_ne]$ if n is even, and $[e, f_1] \cdots [e, f_n]e = (-1)^{\frac{n+1}{2}} (1-e)[e, ef_1(1-e)] \cdots [e, (1-e)f_ne]$ if n is odd. Using the Claim 2 to e and $1-e$ in the left of the previous two equations to transfer them to the right as e in both cases, we get $[e, \mathcal{I}(R)]^n e \subseteq [e, \mathcal{N}(R)]^n e = 0$ and the result follows.

(vi)$\Rightarrow$(i): For every $r \in R$, the element $e+(1-e)re$ is idempotent and $[e, e+(1-e)re] = -[e, r]e$. Applying the same technique of proving (v)$\Rightarrow$(vi), we get $[e, R]^n e = 0$ and e is n-central.

A ring R is called 2-*primal* if $\mathcal{N}(R) = \mathcal{P}(R)$ where $\mathcal{P}(R)$ the prime radical of R. Hence, we have the next corollary.

Corollary 3. *For a* 2*-primal ring* R *and* $e \in \mathcal{I}(R)$*,* e *is* n*-central if and only if* $[e, \mathcal{P}(R)]^n e = 0$*.*

The argument presented below, which connects the n-centrality of idempotents with consecutive degrees by utilizing the minimality of some centralizer as a one-sided ideal, is inspired by the work of Lam in [11, Proposition 2.10].

Proposition 3. *If* $e \in \mathcal{C}_n(R)$ *(for* $n \geq 2$*) and* $[e]_{n-2}$ *is a minimal right ideal in* R*, then* $e \in \overline{\mathcal{C}}_{n-1}(R) \cup \mathcal{C}_{n-2}(R)$*. (Here,* $\mathcal{C}_0(R) = \{0\}$*)*

Proof. For $e \in \mathcal{C}_n(R)$, we have $[1-e]_{n-1} = [e]_{n-2}(1-e)R \subseteq [e]_{n-2}$. So either $[1-e]_{n-1} = 0$ (and $e \in \mathcal{C}_{n-1}(R)$) or $[1-e]_{n-1} = [e]_{n-2}$, from the minimality of $[1-e]_{n-1}$. If $[1-e]_{n-1} = [e]_{n-2}$, then $0 = [e]_n = [1-e]_{n-1}eR = [e]_{n-2}eR = [e]_{n-2}$ and $e \in \mathcal{C}_{n-2}(R)$.

Corollary 4 ([11], Proposition 2.10)**.** *If $e \in \mathbf{q}\text{-}\mathbf{idem}(R)$ and eR is a minimal right ideal in R, then e is right semicentral.*

Every idempotent element that is central is n-central for all n. Nevertheless, there exist n-central idempotents that are not central, such as the idempotent e in Example 1, which is 2-central but not central. The subsequent proposition provides a sufficient condition for a ring R to have the sets $\mathcal{C}_n(R)$ coincide with $\mathcal{B}(R)$ for all n.

Proposition 4. *Every n-central idempotent of a semiprime ring is central, for every n.*

Proof. Let e be an n-central idempotent of a semiprime ring R for some odd n (that does not loss of generality of n), then $[e]_n = 0$ and $((1-e)ReR)^{\frac{n+1}{2}} = 0$. So, $(1-e)Re = 0$ and $e \in \mathcal{S}_l(R)$. Also, $ReR(1-e)R$ is a nilpotent ideal. Hence $eR(1-e) = 0$ and $e \in \mathcal{S}_l(R)$; that e is central.

As per Theorem [10, Example 10.17], both $R/P(R)$ and $R/J(R)$ are semiprime rings, where $\mathcal{J}(R)$ denotes the Jacobson radical of R. Therefore, we can derive the following corollary.

Corollary 5. *Every $e \in \mathcal{C}_n(R)$ maps onto a central idempotent in $R/P(R)$ and $R/\mathcal{J}(R)$. Therefore, if R is a n-Abelian ring, then every idempotent of R maps onto a central idempotent in R/P(R) and R/rad (R).*

Indeed, indecomposable rings have no nontrivial idempotents. So, we have the next corollary

Corollary 6. *If R is an indecomposable semiprime ring, then $\mathcal{C}_n(R) = \{0, 1\}$, for every n.*

Here, we have a necessary and sufficient condition making $\mathcal{B}(R) = \mathcal{C}_n(R)$ for some n.

Proposition 5. *An idempotent e of a ring R is central if $e \in \mathcal{C}_n(R)$, for some n, and $\mathcal{C}_n(R)$ is commutating.*

Proof. Let $e \in \mathcal{C}_n(R)$ and define the idempotent element $f = e + er(1-e)$, for arbitrary $r \in R$. But $f \in eR$ and $1 - f \in R(1-e)$. Hence, $[f]_n \subseteq [e]_n = 0$ and $f \in \mathcal{C}_n(R)$. So that e and f are commutating from the assumption and $f = ef = fe = e$. Therefore, $eR(1-e)$ and $e \in \mathcal{S}_r(R)$. Similarly, one can show that $e \in \mathcal{S}_l(R)$ and hence e is central.

Here, we give the same result of [11, Proposition 2.6] with a generalized condition.

Proposition 6. *Let $e \in \mathcal{C}_n(R)$, for some n, such that $ReR = R$, then $e = 1$.*

Proof. Indeed, $e \in \mathcal{C}_n(R)$ and $[e]_n = 0$. So $((1-e)R)^m = 0$ where $m = \lceil \frac{n}{2} \rceil$. So that $1 - e = 0$ and $e = 1$.

In the context of a ring R, recall that two idempotents e and f are said to be *isomorphic* if eR and fR are isomorphic as right R-modules. Equivalently, e and f are isomorphic if there exist $a, b \in R$ such that $e = ab$ and $f = ba$. Idempotents e and f are said to be *isomorphic complements* if $1-e$ and $1-f$ are isomorphic. Additionally, e and f are said to be *conjugate* (resp. *right associate*, *left associate*) if there exists a unit $u \in R$ such that $uf = eu$ (resp. $f = eu$, $f = ue$).

The following proposition establishes that if two idempotents are isomorphic, isomorphic complements, or conjugate, right associate, or left associate, then their n-centrality for some n is equivalent.

Proposition 7. *Let R be a ring and $e, f \in I(R)$. Then we have the following identities.*

(i) If e and f are conjugate, then $e \in \mathcal{C}_n(R)$ if and only if $f \in \mathcal{C}_n(R)$.

(ii) If e and f are right associate, then $e \in \mathcal{C}_n(R)$ if and only if $f \in \mathcal{C}_n(R)$.

(iii) If e and f are left associate, then $e \in \mathcal{C}_n(R)$ if and only if $f \in \mathcal{C}_n(R)$.

(iv) If e and f are isomorphic and isomorphic complements, then $e \in \mathcal{C}_n(R)$ if and only if $f \in \mathcal{C}_n(R)$.

Proof. (i) If e and f are conjugate, then $f = u^{-1}fu$ for some unit u in R. By simple calculations, one can find that $[e, r] = u[f, u^{-1}ru]$ and $[f, r] = u[f, uru^{-1}]$, for every $r \in R$. By Proposition 1, the result follows.

(ii) If e and f are right associate, then $ef = f$ and $fe = e$, by [5, Lemma 6.2]. So, $(1-e)(1-f) = 1-e$ and $(1-f)(1-e) = 1-f$. Therefore, $[e]_n \subseteq (1-e)[f]_n$ and $[f]_n \subseteq (1-f)[f]_n$, for every odd n, while $[e]_n = [f]_n$, for every even n.

(iii) is proved as in (ii).

(iv) is clear from (i) [5, Lemma 6.2].

It is important to note that the isomorphism of two idempotents is not always enough to transfer the n-centrality of one to the other. For instance, consider a non-directly finite ring R (i.e., $ab = 1$ does not imply $ba = 1$ for some $a, b \in R$). Such a ring can have an n-central idempotent, for any n, while its isomorphic idempotent may not be n-central. To see why, suppose $ab = 1$ for $a, b \in R$ with R not directly finite. Then, ba is a non-trivial idempotent, and for any m, $([ba, a][ba, b])^m = (-1)^m(1-ba) \neq 0$. Hence, ba cannot be k-central for any k.

3. n-Abelian Rings

This section presents a generalization of idempotent centrality, with the introduction of a ring that contains only idempotents which are n-central for some n. The following definition serves as an introduction to this ring. It states that the idempotents in a ring R are all n-central if and only if they are all complementary n-central if and only if they are all dual n-central.

Definition 2. *A ring R is called n-*Abelian *if every idempotent of R is n-central; that $\mathcal{I}(R) = \mathcal{C}_n(R)$.*

Based on the definition, it can be observed that an n-Abelian ring is also an m-Abelian ring if $n \leq m$. Furthermore, the classes of Abelian rings and q-Abelian rings are equivalent to the classes of 1-Abelian and 2-Abelian rings, respectively. It is worth noting that every Abelian ring is n-Abelian for all values of n. However, the ring in Example 2 cannot be n-Abelian for any n. The following proposition provides an alternative definition for n-Abelian rings.

Proposition 8. *The following conditions are equivalent for a ring R and positive integer n.*

(i) *R is n-Abelian.*

(ii) *$[e, R]^n = 0$, for every $e \in \mathcal{I}(R)$.*

(iii) *$[e]_{n-1}$ is an ideal of R, for every $e \in \mathcal{I}(R)$.*

Proof. The proof is straightforward from the definition and Proposition 1.

Recall, an idempotent e of a ring R is said to be *directly finite* if the eRe is directly finite. The next proposition shows that every n-central idempotent for any n is directly finite, and therefore every n-Abelian ring for any n is directly finite.

Proposition 9. *If e is an n-central idempotent of a ring for some n, then e is directly finite.*

Proof. Let R be a ring e an n-central idempotent of a ring R, for some n. Assume that $ab = e$, for some $a, b \in eRe$. Hence, ba is an idempotent in eRe and $[ba, a][ba, b] = ba - e$. Without loss of generality we assume that n is even, then $0 = ([ba, a][ba, b])^{\frac{n}{2}} = (-1)^{\frac{n}{2}}(e - ba)$ and $ba = e$. Thus, eRe is differently finite.

Corollary 7. *Every n-Abelian ring is directly finite, for every n.*

Corollary 8 ([22],Theorem 2.4)**.** *Quasi-normal rings are directly finite.*

According to to [21], an element a of a ring R is called *left minimal* if Ra is a minimal left ideal of R and R is called *left min-Abel* if each left minimal idempotent left semicentral. The next proposition states that a ring R is left min-Abel whenever it is n-Abelian for some n.

Proposition 10. *If R is n-Abelian for some n, then R is left min-Abel.*

Proof. Let e be a nonzero left minimal idempotent element of an n-Abelian ring R. If e is not left semicentral, then $(1 - e)ae \neq 0$ for some $a \in R$ and $0 \neq R(1 - e)ae \subseteq Re$. Hence $R(1-e)ae = Re$ because Re is minimal left ideal of R. So $(Re)^m = (R(1-e)ae)^m \subseteq R[e]_{2m-1}$, for every $m > 0$. Choosing $m \geq \frac{n+1}{2}$, we get $e = 0$, which is a contradiction. Hence $(1 - e)ae = 0$ for all $a \in R$ and R is left min-Abel.

Corollary 9 ([23],Theorem 2.5)**.** *Every quasi-normal ring is left min-Abel.*

The previous corollary presented in [23] has a counterexample, which also serves as a counterexample for Proposition 10. In accordance with [25], a ring R is referred to as *weakly normal* if $ae = 0$ implies $Rera$ is a nil left ideal of R, where $a, r \in R$ and $e \in \mathcal{I}(R)$. It is worth noting that every quasi-normal (or q-Abelian) ring is weakly normal, as shown in [25, Corollary 2.3 (1)]. The following proposition extends this result to a broader scope.

Proposition 11. *If R is an n-Abelian ring for some n, then R is weakly normal.*

Proof. It is direct by [25, Theorem 2.2].

In [8], a ring R is called *left* (resp. *right*) *idempotent-reflexive* if $aRe = 0$ (resp. $eRa = 0$) implies $eRa = 0$ (resp, $aRe = 0$) for every $a \in R$ and $e \in \mathcal{I}(R)$. The following proposition shows that the sets of n-central idempotents and complementary n-central idempotents of a ring one-sided idempotent-reflexive ring R coincide.

Proposition 12. *For every left (or right) idempotent-reflexive ring R, we have $\mathcal{C}_n(R) = \widehat{\mathcal{C}}_n(R)$ if n is odd and $\mathcal{C}_n(R) = \widehat{\mathcal{C}}_{n-1}(R)$ if n is even.*

Proof. If $e \in \mathcal{C}_n(R)$ for some odd n, then $((1-e)ReR)^{\frac{n+1}{2}} = 0$ and therefore $\left(((1-e)ReR)^{\frac{n-1}{2}}(1-e)\right) Re = 0$. If R is left idempotent-reflexive, then $(eR(1-e)R)^{\frac{n+1}{2}} = 0$ and $e \in \overline{\mathcal{C}}_n(R)$. In case of n is even, we have $eR((1-e)ReR)^{\frac{n-2}{2}} = 0$ and $0 = (R(1-e)Re)^{\frac{n-2}{2}} Re = (R(1-e)Re)^{\frac{n-2}{2}}$. Therefore, $e \in \mathcal{C}_{n-1}(R)$. But $n-1$ is odd and consequently $\mathcal{C}_n(R) = \widehat{\mathcal{C}}_{n-1}$, from the previous result.

Every subring S (which is not necessarily with identity) of an n-Abelian ring R is also n-Abelian since for every $e \in \mathcal{I}_n(S)$, we have $e \in \mathcal{C}_n(R)$ and $[e, S]^n e \subseteq [e, R]^n e = 0$ and $e \in \mathcal{C}_n(S)$. However, the n-Abelianity of a subring is not necessarily extended to the ring itself.

Example 3. *In the ring R of Example 1, the element $e = \mathbf{diag}(1, 0, 1)$ is not 1-central in R. However, e is an idempotent of subring $S = \begin{bmatrix} F & 0 & F \\ 0 & 0 & 0 \\ 0 & 0 & F \end{bmatrix} \cong \mathbb{T}_2(F)$ and it is 1-central. Also, $\mathbb{F}$ is a subring of R which is commutative and in particular, is 1-Abelian while R is not 1-Abelian.*

Proposition 13. *Every subdirect product of a family n-Abelian rings is also n-Abelian, for every n.*

Proof. Let $R_i = R/A_i$, for some ideals $\{A_i\}_{i\in\Lambda}$ of a ring R with $\bigcap_{i\in\Lambda} A_i = 0$. For every $e \in \mathcal{I}(R)$, we have $e_i = e + A_i \in \mathcal{I}(A_i)$ where $i \in \Lambda$. If each R_i is n-Abelian for some n, then $[e_i, R]^n e_i R = 0$ for every $i \in \Lambda$. So $0 = [e_i, R]^n e_i R = [e + A_i, R]^n e_i R \subseteq [e, R]^n (e+A_i)R + [A_i, R]^n (e+A_i)R \subseteq [e, R]^n e + [e, R]^n A_i + [A_i, R]^n (e+A_i)R \subseteq [e, R]^n e + A_i$ and $[e, R]^n e$ for every $i \in \Lambda$. Therefore, $[e, R]^n e \in \bigcap_{i\in\Lambda} A_i = 0$ and e is n-central.

Motivated by [11, Section 3], which discusses how the Abelianity and q-Abelianity of rings can be determined by their upper triangular matrices, we present a generalization of [11, Theorem 3.4]. Our approach combines the order of the upper triangular matrix with the order of Abelianity. We begin by establishing the following lemma, which will be used in the proof of our main result.

Lemma 2. *Let the ring* $T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$*, for any rings* R*, and* S*, and a unital* (R, S)*-bimodule* M*, be* n*-Abelian, for some* n*. If* R *and* S *are respectively* n_1*-Abelian and* n_2*-Abelian rings, then the ring* T *is* $(n_1 + n_2)$*-Abelian. While, if* T *is* n*-Abelian, then both* R *and* S *are* $(n-1)$*-Abelian.*

Proof. First, assume that R and S are n_1-Abelian and n_2-Abelian, for some n_1 and n_2, respectively. For an arbitrary idempotent $\varepsilon = \begin{bmatrix} e & s \\ 0 & f \end{bmatrix}$ of T, $e, f \in \mathcal{I}(R)$ and $es + sf = s$. Indeed, $[\varepsilon, T] \subseteq \begin{bmatrix} [e,R] & M \\ 0 & [f,S] \end{bmatrix}$ and $[\varepsilon, T]^{n_1+n_2} \subseteq \begin{bmatrix} 0 & M \\ 0 & [f,S]^{n_1} \end{bmatrix} \begin{bmatrix} [e,R]^{n_2} & M \\ 0 & 0 \end{bmatrix} = 0$ and $\varepsilon \in \overline{\mathcal{C}}_{(n_1+n_2)}(T)$. Since $\varepsilon \in \mathcal{I}(T)$ is arbitrary, this shows that T is $(n_1 + n_2)$-Abelian.

Secondly, if T is n-Abelian and $e^2 = e, r, r_1, r_2, \cdots, r_{n-1} \in R$, define the elements $\epsilon = \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix}$, $t = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $t_k = \begin{bmatrix} r_k & 0 \\ 0 & 0 \end{bmatrix}$, for $k = 1, 2, \cdots, n-1$ in T. If T is n-Abelian, then $0 = [\epsilon, t_1] \cdots [\epsilon, t_{n-1}][\epsilon, t] = \begin{bmatrix} [e, r_1] & 0 \\ 0 & 0 \end{bmatrix} \cdots \begin{bmatrix} [e, r_{n-1}] & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & e-1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & [e, r_1] \cdots [e, r_{n-1}](e-1) \\ 0 & 0 \end{bmatrix}$. So, $[e, r_1] \cdots [e, r_{n-1}](e-1) = 0$ and $[e, R]^{n-1}(1-e) = 0$. Therefore, $e \in \overline{C}_{n-1}(R)$ and R is $(n-1)$-Abelian since e is arbitrary in $\mathcal{I}(R)$. Similarly, one can verify that S is also $(n-1)$-Abelian.

In Example 4.7 of Lam's work [12], it is demonstrated that $\mathbb{T}_3(R)$ fails to exhibit semi-abelian characteristics. Expanding upon the findings elucidated in Lam's prior works, specifically Theorem 3.4 in [11] and Theorem 2.4 in [12], we have undertaken a comprehensive exploration leading to a noteworthy generalization, which we present in the subsequent theorem. Our derived result establishes a fundamental link between the Abelian nature of a ring and the Abelianity observed within its corresponding upper triangular matrix extension.

Theorem 4. *For a ring* R*, the following conditions are equivalent.*

(i) R *is Abelian.*

(ii) $\mathbb{T}_n(R)$ *is* n*-Abelian, for every* n*.*

(iii) $\mathbb{T}_n(R)$ *is* n*-Abelian, for some* n*.*

Proof. (i)⇒(ii): Assume that R is Abelian. Hence, $\mathbb{T}_2(R)$ is 2-Abelian, by the previous lemma. Also, $\mathbb{T}_3(R) = \begin{bmatrix} R & \mathbb{R}^2 \\ 0 & \mathbb{T}_2(R) \end{bmatrix}$, where $\mathbb{R}^2$ is a unital $(R, \mathbb{T}_2(R))$-bimodule. Again, $\mathbb{T}_3(R)$ is 3-Abelian form the previous lemma. Continuing, we have $\mathbb{T}_n(R)$ is n-Abelian, for every n.

(ii)⇒(iii) is direct.

(iii)⇒(i): Let $\mathbb{T}_n(R)$ be n-Abelian, for some n. $\mathbb{T}_n(R) = \begin{bmatrix} R & \mathbb{R}^{n-1} \\ 0 & \mathbb{T}_{n-1}(R) \end{bmatrix}$. So, R and $\mathbb{T}_{n-1}(R)$ are $(n-1)$-Abelian. Inducting on n gives R is 1-Abelian,

Corollary 10 ([11], Theorem 3.4)**.** *A ring R is Abelian if and only if $\mathbb{T}_2(R)$ is q-Abelian.*

4. Applications

In ring theory, several types of regularity are defined for elements in a ring. An element a in a ring R is called regular (in the sense of von Neumann) if there exists an element $b \in R$ such that $a = aba$. Here, the element b is called an inner inverse of a, and $\Im(a)$ denotes the set of all inner inverses of a in R. We can also define the left regularity and right regularity of an element a in a similar manner. If an element is both left and right regular, it is called strongly regular. A ring R is called regular (resp. strongly regular) if all its elements are regular (resp. strongly regular). It is well-known that a ring R is strongly regular if and only if it is regular and Abelian.

In addition to regularity, we can define π-regularity for elements in a ring. An element $a \in R$ is called π-regular if $a \in a^n R^n$ for some positive integer n depending on a. If an element a is both π-regular and strongly regular, it is called strongly π-regular. A ring R is called π-regular (resp. strongly π-regular) if all its elements are π-regular (resp. strongly π-regular). It is worth noting that a ring R is strongly π-regular if and only if it is Abelian and π-regular.

Previous research by Wei and Li in [23] showed that for a regular (resp. π-regular) ring, being quasi-Abelian (2-Abelian) is enough to become strongly regular (strongly π-regular). Moreover, Lam in [10] showed that if the idempotent ba of a regular element $a = aba$ is q-central (2-central), then a is strongly regular. In the following, we present a weak condition that guarantees a regular element to be strongly regular.

Theorem 5. *Let a be a regular element of a ring R. Then the following statements are satisfied.*

(i) If ab is n-central for some n and $b \in \Im(a)$, then ac is n-central for every $c \in \Im(a)$

(ii) If ba is n-central for some n and $b \in \Im(a)$, then ca is n-central for every $c \in \Im(a)$

(iii) If ab is n-central for some n and $b \in \Im(a)$, then a is right regular

(iv) If ba is n-central for some n and $b \in \Im(a)$, then a is left regular

(v) If both ab_1 and b_2a are n-central for some n and $b_1, b_2 \in \Im(a)$, then a is strongly regular.

Proof. (i) Easy calculations show that the idempotents of $a\Im(a)$ are isomorphic and isomorphic complements. So, Therefore, ac is n-central for every $c \in \Im(a)$.

(ii) Similarly as (i).

(iii) Write $e = ab$ and consider n is even which does not lose the geniality of n. Write $n = 2k$, for integer $k \geq 1$ and $eR((1-e)ReR)^k = 0$. So, $0 = e(a(1-e)be)^k = e(abe - aebe)^k = e(e - aebe)^k = (e - aebe)^k$. Expanding and using the fact $ea = a$, we get $e \in aeRe$. So that $a = ea \in aeRa = a^2bRa \subseteq a^2R$ and a is left regular.

(iv) Similar as (iii).

(iii) Let a be a regular element a of R with $a = aba$, for some $b \in R$. We prove the case of the idempotent $e = ba$ is n-central for some n. The case of $n = 2$ has been shown in [11, Theorem C.] and considering n is even does not lose the geniality of n. Write $n = 2k+2$, for integer $k \geq 0$ and $((1-e)ReR)^k = 0$ from the n-centrality of e. So, $eR((1-e)ReR)^k = 0$ and $(eb(1-e)a)^k = 0$, since $ae = a$. But $eb(1-e)a = eba - ebea = e - ebea$. Hence, $(e - ebea)^k = 0$ and the expanding gives $e \in eRea$. So that $a = ae \in e \in aeRea \subseteq Ra^2$ and a is right regular.

(iv) can be proved similarly as in (iii).

(v) is direct from (iii) and (iv).

Corollary 11. *A ring R is strongly regular if and only if R is regular and n-Abelian for some n.*

Corollary 12. *Let R be a regular ring. Then the following statements are equivalent:*

(i) R is reduced.

(ii) R is Abelian.

(iii) R is n-Abelian for every n.

(iv) R is n-Abelian for some n.

Proposition 14. *Let R be an n-Abelian ring, for some n. If R is π-regular, then R is Abelian (consequently R is strongly π-regular).*

Proof. Straightforward.

By the previous proposition, we can get the results [23, Theorems 3.8 and 3.10] using the n-centrality with any degree.

In [20], Warfield called a ring R an *exchange ring* if ${}_RR$ has the finite exchange property. An equivalent idempotent-wise definition of exchange rings was introduced in [6, 14]; that a ring R is exchange if and only if for every $a \in R$, there exists an idempotent e of R such that $e \in aR$ and $1 - e \in (1-a)R$. According to [14], a ring R is said to be *clean* if every element $a \in R$ can be written as a sum of a unit and an idempotent.

The next theorem gives an equivalent definition of n-Abelian exchange rings, drawing inspiration from [11, Theorem 5.10.] and extending it to a more general context.

Theorem 6. *The following statements are equivalent for any ring R and positive integer n:*

(i) R is a n-Abelian exchange ring.

(ii) R is a n-Abelian clean ring.

(iii) Every element in R is the sum of a unit and an n-central idempotent.

(iv) For any $a \in R$, there exists $e \in \mathcal{C}_n(R)$ such that $e \in aR$ and $1-e \in (1-a)R$.

Proof. (i)⇒ (ii): By utilizing the exchange property of R, any idempotent $e \in R/\mathcal{J}(R)$ can be raised to an idempotent in R. Using Corollary 5 leads to the conclusion that e must be central in $R/\mathcal{J}(R)$. Consequently, $R\mathcal{J}(R)$ is demonstrated to be an Abelian exchange ring. By [14, Proposition 1.8], $R/J(R)$ is a clean ring and we can find $e \in \mathcal{I}(R)$ such that $\overline{a-e}$ is a unit in $R/\mathcal{J}(R)$, for every $a \in R$. So, a-e must also be a unit in R since $e \in \mathcal{C}_n(R)$ is a n-central idempotent according to the given assumption.

(ii)⇒ (iii) is direct from the definitions.

(iii)⇒ (iv): If $a \in R$, then $1-a = e+u$, for some $e \in \mathcal{C}_n(R)$ and $u \in \mathcal{U}(R)$ for the assumption. Define the idempotent $f = ueu^{-1}$ which is conjugate to e and consequently $f \in \mathcal{C}_n(R)$, by Proposition 7. Now, $f = ueu^{-1} = (1-a-e)eu^{-1} = -aeu^{-1} \in aR$ and $1-f = 1-ueu^{-1} = uu^{-1} - ueu^{-1} = u(1-e)u^{-1} = u(1-e)u^{-1} = (1-a-e)(1-e)u^{-1} = (1-a)(1-e)u^{-1} \in (1-a)R$; it follows.

(iv)⇒ (i): The condition shows that R is exchange and it is enough to show that R is n-Abelian. For every $e \in \mathcal{I}(R)$, the assumption yields that there exist $f \in \mathcal{C}_n(R)$ such that $f \in eR$ and $1-f \in (1-e)R$. So $f = ef$ and $1-f = (1-e)(1-f) = 1-e-f+ef = 1-$. Hence, $e = f \in\in \mathcal{C}_n(R)$ and R is n-Abelian.

Vaserstein [19] defines a ring R to have stable range 1 if for any $a, b \in R$ with $aR+bR = R$, there exists $y \in R$ such that $a+by$ is right invertible. R has stable range 1 if and only if $R/\mathcal{J}(R)$ has stable range 1. The next corollary demonstrates that exchange rings with n-central idempotents, for some n have stable range 1.

Corollary 13. *Every n-Abelain exchange rings, for some n, has stable range 1.*

Proof. Corollary 5 and [27, Theorem 6] jointly yield the result.

The inherent generality of hyperrings as an extension of rings prompts a pertinent consideration: the exploration of the concepts of n-central idempotents and n-Abelian rings within the domain of hyperrings. This avenue of inquiry holds promise in further elucidating the structural properties and algebraic characteristics inherent in hyperring theory.

For readers intrigued by the realm of hyperrings and desiring a deeper understanding, an extensive exploration can be found in the following scholarly sources: [9, 15, 16, 18]. These references offer comprehensive insights into hyperrings, serving as valuable resources for those engaged in advanced studies or research endeavors within this domain.

5. Conclusion

Within this paper, we present a series of comprehensive findings. Initially, we establish that every n-central idempotent within a semiprime ring is central. Additionally, we demonstrate that a full idempotent e is an n-central idempotent if and only if e equals zero. Moreover, we showcase that if e and f represent isomorphic idempotents with isomorphic complements, then e being n-central is equivalent to f being n-central.

Furthermore, we provide proof indicating that if e denotes an n-central idempotent within a ring R, then the left ideal eR is directly finite. Additionally, we illustrate that the condition mandating all idempotents to be n-central extends to upper triangular matrix rings, thereby compelling a von Neumann ring to attain the status of strong regularity.

Lastly, we conclude our findings by demonstrating that a ring wherein all idempotents are n-central achieves the status of an exchange ring if and only the ring is clean.

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References

[1] G. F. Birkenmeier. Idempotents and completely semiprime ideals. *Commun. Algebra*, 11:567–58, 1983.

[2] G. F. Birkenmeier, H. Heatherly, J. Y. Kim, and J. K. Park. Algebras generated by semicentral idempotents. *Acta Math. Hungar.*, 95(1-2):101–114, 2002.

[3] G. F. Birkenmeier, J. Y. Kim, and J. K. Park. Semicentral reduced algebras. In *International Symposium on Ring Theory*, pages 67–84. Springer, 2001.

[4] G. F. Birkenmeier, J. K. Park, and S. T. Rizvi. *Extensions of rings and modules.* Springer, New York Heidelberg Dordrecht London, 2013.

[5] A. J. Diesl, S. J. Dittmer, and P. P Nielsen. Idempotent lifting and ring extensions. *J. Algebra Appl.*, 15(06):1650112, 2016.

[6] K. R. Goodearl and R. B. Warfield Jr. Algebras over zero-dimensional rings. *Math. Annal.*, 223(2):157–168, 1976.

[7] J. Han, Y. Lee, and S. Park. Semicentral idempotents in a ring. *J. Korean Math. Soc.*, 51(3):463–472, 2014.

[8] J. Y. Kim and J. U. Baik. On idempotent reflexive rings. *Kyungpook Math. J.*, 46:597–601, 2006.

[9] Z. Kou, S. Kosari, M. Monemrad, M. Akhoundi, and S. Omidi. A note on the connection between ordered semihyperrings. *Symmetry*, 13(11):2035, 2021.

[10] T. Y. Lam. *A first course in noncommutative rings*, volume 131. Springer Science & Business Media, New York Heidelberg Dordrecht London, 2013.

[11] T. Y. Lam. An introduction to q-central idempotents and q-abelian rings. *Commun. Algebra*, pages 1–18, 2022.

[12] T. Y. Lam. On some generalizations of abelian rings. *J. Algebra Appl.*, page 2550146, 2023.

[13] C. Lomp and J. Matczuk. A note on semicentral idempotents. *Commun. Algebra*, 45(6):2735–2737, 2017.

[14] W. K. Nicholson. Lifting idempotents and exchange rings. *Trans. Am. Math. Soc.*, 229:269–278, 1977.

[15] Y. Rao, S. Kosari, Z. Shao, M. Akhoundi, and S. Omidi. A study on -hyperideals and -hyperfilters in ordered -semihypergroups. *Discrete Dynamics in Nature and Society*, 2021:6683910, 2021.

[16] Y. Rao, S. Kosari, Z. Shao, and S. Omidi. Some properties of derivations and m-k-hyperideals in ordered semihyperrings. *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys*, 83(3):87–96, 2021.

[17] M. Saad. Rings in which every semicentral idempotent is central. *Korean J. Math.*, 31(4):405–417, 2023.

[18] Z. Shao, X. Chen, S. Kosari, and S. Omidi. On some properties of right pure (bi-quasi-) hyperideals in ordered semihyperrings. *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys*, 83(4):95–104, 2021.

[19] L. N. Vaserstein. Bass's first stable range condition. *J. Pure Appl. Algebra*, 34(2-3):319–330, 1984.

[20] R. B. Warfield Jr. Exchange rings and decompositions of modules. *Math. Annal.*, 199(1):31–36, 1972.

[21] J. Wei. Certain rings whose simple singular modules are nil-injective. *Turk. J. Math.*, 32(4):393–408, 2008.

[22] J. Wei and J. Chen. Nil- injective rings. *Inter. Electron. J. Algebra*, 2(2):1–21, 2007.

[23] J. Wei and L. Li. Quasi-normal rings. *Commun. Algebra*, 38(5):1855–1868, 2010.

[24] J. Wei and L Li. Nilpotent elements and reduced rings. *Turk. J. Math.*, 35(2):341–353, 2011.

[25] J. Wei and L. Li. Weakly normal rings. *Turk. J. Math.*, 36(1):47–57, 2012.

[26] J. Wei and N. Li. Some notes on semiabelian rings. *J. Math. Math. Sci.*, 2011, 2011.

[27] H-P. Yu. Stable range one for exchange rings. *J. Pure Appl. Algebra*, 98(1):105–109, 1995.