Fermatean Neutrosophic INK-Algebras

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Abstract. This paper introduces the concept of the direct product of sets that involve Fermatean neutrosophic (FN) elements in structures called INK-algebras. It defines terms like the direct product of Fermatean neutrosophic INK-ideals (FNINK-Is) in INK-algebras and Fermatean neutrosophic sets (FNSs), FNINK-Is, and Fermatean neutrosophic closed INK-ideals (FNCINK-Is). The proof of theorems illustrating the relationships between these ideas is included in the paper. It also defines the INK-subalgebra embedded in an INK-algebra and gives a theorem elucidating the connection between the direct product of FNINK-Is and the images of these subalgebras. In essence, the paper investigates and establishes connections between different mathematical ideas concerning INK-algebras and FNSs.

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1. Introduction

Zadeh [1] seminal work on fuzzy sets established the basis for fuzzy logic. Fuzzy sets provide a more flexible representation of uncertainty by assigning degrees of membership to elements. Atanassov [2] paper explores intuitionistic fuzzy sets, which include degrees of membership and degrees of non-membership. This gives a more complete picture of uncertainty in decision-making. Neutrosophic logic has been introduced by Smarandache which involves various disciplines of philosophy and mathematics that studies indeterminacy, uncertainty, and contradictions. Neutrosophic logic is a three-valued logic system

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that includes the truth values “true” and “false,” as well as a third value termed indeterminate, which represents uncertain or ambiguous information. This method is especially beneficial for dealing with difficulties involving inadequate or inconsistent data, which are widespread in domains such as artificial intelligence, decision making, philosophy, and cognitive science.

Abdel-Bassset et al. [3] present a decision-making framework for professional selection based on bipolar neutrosophic sets. The approach seeks to address uncertainty and imprecision in decision-making processes, his work expands on intuitionistic fuzzy sets to include neutrosophic sets, allowing for indeterminacy, uncertainty, and contradictory information [4]. Jun [5] research on neutrosophic subalgebras in BCK/BCI-algebras advances our understanding of algebraic structures with neutrosophic elements. Kaviyarasu et al. [6] investigates fuzzy subalgebras and fuzzy INK-ideals in INK-algebras, providing insights into the integration of fuzzy logic in algebraic structures. Additionally, Kaviyarasu and Indhira present a review of BCI/BCK-algebras and discuss their development, contributing to the understanding of these specific algebraic structures [7], investigate fuzzy p-ideals in INK-algebras [8], adding to the understanding of fuzzy ideals in the context of specific algebraic structures. Jun et al.[9] collaboration aims to integrate neutrosophic N-structures into BCK/BCI-algebras.

Jun et al.[10] investigate neutrosophic positive implicative N-ideals in the context of BCK-algebras, advancing our understanding of neutrosophic structures in algebraic systems. Ozturk and Jun [11] investigates neutrosophic ideals in BCK/BCI algebras based on neutrosophic points, broadening the application of neutrosophic concepts to algebraic structures. Songsaeng and Iampan [12] introduce neutrosophic set theory to UP-algebra and demonstrate its utility in a specific algebraic context. Kaviyarasu, Indhira and Chandrasekaran [[13], [14], [15]] investigate the direct product of intuitionistic fuzzy INK-ideals, providing insights into the interaction of different algebraic structures; discuss intuitionistic fuzzy translation in INK-Algebra, contributing to the understanding of translation operations in algebraic structures; and apply neutrosophic sets in INK-Algebra, extending the study of neutrosophic concepts to a specific algebraic context. As an extension of partial algebra, Smarandache [16] presents the theory of neutro algebra, which advances the development of algebraic structures in addition to neutro and anti-algebraic structures, which provide additional insights into mathematical structures. Abdel-Bassset et al. proposed a novel plithogenic model for supply chain problem solving, which incorporates neutrosophic and plithogenic sets into optimization theory [17]. Making contributions to the fields of environmental technology and innovation, Mohamed and Abdel [[18], [19], [20]] introduce an integrated plithogenic MCDM approach for assessing the financial performance of manufacturing industries, utilizing a combination of mathematical decision-making methods, as well as a novel framework for assessing the innovation value proposition for smart product-service systems. Neutrosophic Vague Binary BCK/BCI-algebra, which explores the use of vague and neutrosophic notions in a binary algebraic framework, is covered by Remya and Francina Shalini [21]. Muralikrishna and Manokaran [22] introduce MBJ- neutrosophic B-ideals in B-algebras, which adds to the understanding of neutrosophic ideals in specific algebraic structures. For more results on algebraic structures with uncertainty (see works
This paper presents a new concept based on two distinct sets, called FNSs, and investigates their direct product in the framework of INK-algebra. Specifically, it looks at the relation between FNINK-Ss and FNINK-Is, as well as the conditions that hold for this relation.

1.1. Motivation

- It aims to provide a new perspective on FN elements in mathematical structures.
- Ensures a systematic and coherent discussion of these mathematical concepts.
- Validating the proposed connections rigorously through theorems adds credibility and reliability.
- Exploration of Substructures contributes to a more comprehensive understanding of the intricate relationships within the algebraic framework.

1.2. Novelty

- The paper aims to present a novel mathematical framework by introducing the concept of the direct product, which involves sets with FN elements within the domain of INK-algebras.
- The goal is to create a precise mathematical language by defining terms like direct product of FNINK-Is, FN-Ss and FNCINK-Is, which will improve discourse clarity.
- The study illuminates complex mathematical relationships through rigorous theorem proofs, delving into interconnected ideas in INK-algebras and FN-Ss.

1.3. Structure of the paper

The paper begins with an introductory exploration of the novel concept of the direct product within structures known as INK-algebras, which include sets enriched with FN elements. It then defines key terms like the direct product of FNINKs, FN-Ss and FNCINK-Is. The narrative then proceeds to provide a comprehensive exposition, including proofs of theorems that intricately illustrate the relationships between these defined concepts. Furthermore, it broadens its scope to define an INK-subalgebra embedded within an INK-algebra, as well as a theorem that explains the relationship between the direct product of FNINKs and the images of these sub algebras. In essence, the paper culminates in a thorough investigation, establishing connections and interrelations between diverse mathematical ideas concerning both INK-algebras.
2. Basic Definitions

In the beginning the research, the definition and beneficial properties of INK-algebras will be explained.

**Definition 1** ([15]). An INK-algebra is a mathematical structure with specific rules; it is represented by the notation \((\chi, \cdot, 0)\). For any elements \(\vartheta, \eta, z \in \chi\)

1. \(((\vartheta \cdot \eta) \cdot (\vartheta \cdot z)) \cdot (z \cdot \eta) = 0,\)
2. \(((\vartheta \cdot z) \cdot (\eta \cdot z)) \cdot (\vartheta \cdot \eta) = 0,\)
3. \((\vartheta \cdot 0) = 0,\)
4. \((\vartheta \cdot \eta) = 0\) and \(\eta \cdot \vartheta = 0\) imply \(\eta = \vartheta\).

The operation \(\cdot\) denotes a binary operation and 0 is a constant belonging to the set \(\chi\).

**Definition 2** ([6]). A non-empty subset \(S\) of an INK-algebra \((\chi, \cdot, 0)\) is considered as an INK-subalgebra of \(\chi\), if for every elements \(\vartheta\) and \(\eta \in \chi\), the result of the operation \((\vartheta \cdot \eta)\) is also an element of \(S\).

**Definition 3** ([6]). Let \((\chi, \cdot, 0)\) be an INK-algebra. An ideal of \(\chi\) is defined as a non-empty subset \(\Im\) of \(\chi\) such that it satisfies the following conditions, \(\forall \vartheta, \eta \in \chi\)

1. \(0 \in \Im,\)
2. \((\vartheta \cdot \eta) \in \Im\) and \(\eta \in \Im\) imply \(\vartheta \in \Im\).

**Definition 4** ([6]). Let an INK-algebra \(\chi\) have a non-empty subset \(\Im\). If all of the following hold for every \(\vartheta, \eta, z \in \chi\), then \(\Im\) is called an INK-ideal of \(\chi\).

1. \(0 \in \Im,\)
2. \((z \cdot \vartheta) \cdot (z \cdot \eta) \in \Im\) and \(z \in \Im\) imply \(\vartheta \in \Im\).

**Definition 5** ([22]). The structure of a FNS \(\mathfrak{M}\) defined on a nonempty set \(\chi\) can be expressed as: \(\mathfrak{M} = \left\{ \left(\vartheta, \rho_{\mathfrak{M}}^F(\vartheta), \rho_{\mathfrak{M}}^D(\vartheta), \rho_{\mathfrak{M}}^3(\vartheta)\right) \mid \vartheta \in \chi \right\}\), where \(\rho^F : \chi \to [0, 1]\) is a membership function \(\rho^D : \chi \to [0, 1]\) is a non-membership function and \(\rho^3 : \chi \to [0, 1]\) is a non-membership function and these three functions are satisfying the inequalities; \(0 \leq (\rho_{\mathfrak{M}}^F(\vartheta))^3 + (\rho_{\mathfrak{M}}^D(\vartheta))^3 \leq 1, 0 \leq (\rho_{\mathfrak{M}}^D(\vartheta))^3 \leq 1\) and \(0 \leq (\rho_{\mathfrak{M}}^F(\vartheta))^3 + (\rho_{\mathfrak{M}}^D(\vartheta))^3 + (\rho_{\mathfrak{M}}^3(\vartheta))^3 \leq 2\).

Here, \(\rho_{\mathfrak{M}}^F(\vartheta)\) and \(\rho_{\mathfrak{M}}^3(\vartheta)\) are dependent components and \(\rho_{\mathfrak{M}}^3(\vartheta)\) is an independent component.

Throughout the current research article, we shall use \(\mathfrak{M} = \left\{ \left(\vartheta, \rho_{\mathfrak{M}}^F(\vartheta), \rho_{\mathfrak{M}}^D(\vartheta), \rho_{\mathfrak{M}}^3(\vartheta)\right) \mid \vartheta \in \chi \right\}\).

**Definition 6** ([22]). If \(\mathfrak{M} = \left\{ \left(\rho_{\mathfrak{M}}^F(\vartheta), \rho_{\mathfrak{M}}^D(\vartheta), \rho_{\mathfrak{M}}^3(\vartheta)\right) \right\}\) and \(\mathfrak{M} = \left\{ \left(\rho_{\mathfrak{M}}^F(\vartheta), \rho_{\mathfrak{M}}^D(\vartheta), \rho_{\mathfrak{M}}^3(\vartheta)\right) \right\}\) be two FNSs, then \(\forall \vartheta \in \chi\)
Example 1. If $\chi \in \{0, x, y, z\}$ is a set with a binary operation $\cdot$ given by the following Table:

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Thus, $(\chi, \cdot, 0)$ is an INK-algebra. Consider a FNS $\mathfrak{M}$ in $\chi$, where $\rho_{\mathfrak{M}}^\varphi(0) = 0.8$, $\rho_{\mathfrak{M}}^\varphi(x) = 0.4$, $\rho_{\mathfrak{M}}^\varphi(y) = 0.5$, $\rho_{\mathfrak{M}}^\varphi(z) = 0.2$, $\rho_{\mathfrak{M}}^\varphi(0) = 0.7$, $\rho_{\mathfrak{M}}^\varphi(x) = 0.5$, $\rho_{\mathfrak{M}}^\varphi(y) = 0.3$, and $\rho_{\mathfrak{M}}^\varphi(0) = 0.1$, $\rho_{\mathfrak{M}}^\varphi(x) = 0.4$, $\rho_{\mathfrak{M}}^\varphi(z) = 0.3$. Then, $\mathfrak{M}$ is a FN-I of $\chi$, which is easily verified.

Definition 9. A FNs $\mathfrak{M}$ of $\chi$ is considered as a FNINK-I of $\chi$ if it meets the described conditions, $\forall \vartheta, \eta, z \in \chi$

(1) $\rho^\varphi(0) \leq \rho^\varphi(\vartheta), \rho^\varrho(0) \geq \rho^\varrho(\vartheta), \rho^\delta(0) \geq \rho^\delta(\vartheta),$

(2) $\rho^\varphi(\vartheta) \leq \min \{\rho^\varphi(\vartheta), \rho^\varrho(\eta), \rho^\delta(\eta)\},$

(3) $\rho^\varrho(\vartheta) \geq \max \{\rho^\varrho(\vartheta), \rho^\varrho(\eta), \rho^\delta(\eta)\},$

(4) $\rho^\delta(\vartheta) \geq \max \{\rho^\delta(\vartheta), \rho^\delta(\eta)\}.$
3. Formation of Direct Product: FNINK-Ss and FNINK-Is

Definition 10. INK-algebras $\chi_1$ and $\chi_2$ contain two FNSs, $\mathfrak{M}$ and $\mathfrak{N}$. The structure

$$\mathfrak{M} \times \mathfrak{N} = \left\{ \rho_{(\mathfrak{M} \times \mathfrak{N})}, \rho_{(\mathfrak{M} \times \mathfrak{N})}, \rho_{(\mathfrak{M} \times \mathfrak{N})} \right\}$$

is defined as the direct product of FNSs $\mathfrak{M}$ and $\mathfrak{N}$, specified by $\forall (\vartheta, \eta) \in \chi_1 \times \chi_2$

(1) $\rho_{(\mathfrak{M} \times \mathfrak{N})}(\vartheta, \eta) = \min \left\{ \rho_{\mathfrak{M}}(\vartheta), \rho_{\mathfrak{N}}(\eta) \right\}$,

(2) $\rho_{(\mathfrak{M} \times \mathfrak{N})}(\vartheta, \eta) = \max \left\{ \rho_{\mathfrak{M}}(\vartheta), \rho_{\mathfrak{N}}(\eta) \right\}$,

(3) $\rho_{(\mathfrak{M} \times \mathfrak{N})}(\vartheta, \eta) = \max \left\{ \rho_{\mathfrak{M}}(\vartheta), \rho_{\mathfrak{N}}(\eta) \right\}$.

Definition 11. The direct product of FNINK-Ss of $\chi_1 \times \chi_2$ is a FNSs

$$\mathfrak{M} \times \mathfrak{N} = \left\{ \rho_{(\mathfrak{M} \times \mathfrak{N})}, \rho_{(\mathfrak{M} \times \mathfrak{N})}, \rho_{(\mathfrak{M} \times \mathfrak{N})} \right\}$$

of $\chi_1$ and $\chi_2$ if $\forall (\vartheta_1, \eta_1), (\vartheta_2, \eta_2) \in \chi_1 \times \chi_2$

(1) $\rho_{(\mathfrak{M} \times \mathfrak{N})}(\vartheta_1, \eta_1) \cdot (\vartheta_2, \eta_2) \leq \min \left\{ \rho_{(\mathfrak{M} \times \mathfrak{N})}(\vartheta_1, \eta_1), \rho_{(\mathfrak{M} \times \mathfrak{N})}(\vartheta_2, \eta_2) \right\}$,

(2) $\rho_{(\mathfrak{M} \times \mathfrak{N})}(\vartheta_1, \eta_1) \cdot (\vartheta_2, \eta_2) \geq \max \left\{ \rho_{(\mathfrak{M} \times \mathfrak{N})}(\vartheta_1, \eta_1), \rho_{(\mathfrak{M} \times \mathfrak{N})}(\vartheta_2, \eta_2) \right\}$,

(3) $\rho_{(\mathfrak{M} \times \mathfrak{N})}(\vartheta_1, \eta_1) \cdot (\vartheta_2, \eta_2) \geq \max \left\{ \rho_{(\mathfrak{M} \times \mathfrak{N})}(\vartheta_1, \eta_1), \rho_{(\mathfrak{M} \times \mathfrak{N})}(\vartheta_2, \eta_2) \right\}$.

Definition 12. The direct product of FNINK-I of $\chi_1 \times \chi_2$ is a FNSs

$$\mathfrak{M} \times \mathfrak{N} = \left\{ \rho_{(\mathfrak{M} \times \mathfrak{N})}, \rho_{(\mathfrak{M} \times \mathfrak{N})}, \rho_{(\mathfrak{M} \times \mathfrak{N})} \right\}$$

of $\chi_1$ and $\chi_2$ if $\forall (\vartheta_1, \eta_1), (\vartheta_2, \eta_2), (\vartheta_3, \eta_3) \in \chi_1 \times \chi_2$

(1) $\rho_{(\mathfrak{M} \times \mathfrak{N})}(0,0) \leq \rho_{(\mathfrak{M} \times \mathfrak{N})}(\vartheta, \eta), \rho_{(\mathfrak{M} \times \mathfrak{N})}(0,0) \geq \rho_{(\mathfrak{M} \times \mathfrak{N})}(\vartheta, \eta), \rho_{(\mathfrak{M} \times \mathfrak{N})}(0,0) \geq \rho_{(\mathfrak{M} \times \mathfrak{N})}(\vartheta, \eta)$,

(2) $\rho_{(\mathfrak{M} \times \mathfrak{N})}(\vartheta_1, \eta_1) \leq \min \left\{ \rho_{(\mathfrak{M} \times \mathfrak{N})}(\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1), \rho_{(\mathfrak{M} \times \mathfrak{N})}(\vartheta_2, \eta_2) \right\}$,

(3) $\rho_{(\mathfrak{M} \times \mathfrak{N})}(\vartheta_1, \eta_1) \geq \max \left\{ \rho_{(\mathfrak{M} \times \mathfrak{N})}(\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1), \rho_{(\mathfrak{M} \times \mathfrak{N})}(\vartheta_2, \eta_2) \right\}$,

(4) $\rho_{(\mathfrak{M} \times \mathfrak{N})}(\vartheta_1, \eta_1) \geq \max \left\{ \rho_{(\mathfrak{M} \times \mathfrak{N})}(\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1), \rho_{(\mathfrak{M} \times \mathfrak{N})}(\vartheta_2, \eta_2) \right\}$.

Definition 13. The direct product of FNCINK-I of $\chi_1 \times \chi_2$ is a FNSs

$$\mathfrak{M} \times \mathfrak{N} = \left\{ \rho_{(\mathfrak{M} \times \mathfrak{N})}, \rho_{(\mathfrak{M} \times \mathfrak{N})}, \rho_{(\mathfrak{M} \times \mathfrak{N})} \right\}$$

of $\chi_1$ and $\chi_2$ if it meets (2), (3) and (4) of Definition 12) and the following inequalities, $\forall (\vartheta, \eta) \in \chi_1 \times \chi_2$
Then, the direct product $M \times N$, defined by
\[
M \times N = \left\{ \rho_{\mathbb{M} \times \mathbb{N}}(\vartheta_1, \eta_1), \rho_{\mathbb{M} \times \mathbb{N}}(\vartheta_2, \eta_2) \right\},
\]
is a FNINK-S of $\chi_1 \times \chi_2$.

**Proof.** Assume that $\mathbb{M}$ and $\mathbb{N}$ are two FNINK-Ss. Let $(\vartheta_1, \eta_1), (\vartheta_2, \eta_2) \in \chi_1 \times \chi_2$. Then,
\[
\rho_{\mathbb{M} \times \mathbb{N}}^\chi((\vartheta_1, \eta_1), (\vartheta_2, \eta_2)) = \left\{ \rho_{\mathbb{M} \times \mathbb{N}}^\chi(\vartheta_1 \cdot \vartheta_2), (\eta_1 \cdot \eta_2) \right\} = \min \left\{ \rho_{\mathbb{M} \times \mathbb{N}}^\chi(\vartheta_1), \rho_{\mathbb{M} \times \mathbb{N}}^\chi(\vartheta_2) \right\} \leq \min \left\{ \min \left\{ \rho_{\mathbb{M} \times \mathbb{N}}^\chi(\vartheta_1), \rho_{\mathbb{M} \times \mathbb{N}}^\chi(\vartheta_2) \right\}, \min \left\{ \rho_{\mathbb{M} \times \mathbb{N}}^\chi(\eta_1), \rho_{\mathbb{M} \times \mathbb{N}}^\chi(\eta_2) \right\} \right\} = \min \left\{ \rho_{\mathbb{M} \times \mathbb{N}}^\chi(\vartheta_1, \eta_1), \rho_{\mathbb{M} \times \mathbb{N}}^\chi(\vartheta_2, \eta_2) \right\},
\]
and
\[
\rho_{\mathbb{M} \times \mathbb{N}}^3((\vartheta_1, \eta_1), (\vartheta_2, \eta_2)) = \left\{ \rho_{\mathbb{M} \times \mathbb{N}}^3(\vartheta_1 \cdot \vartheta_2), (\eta_1 \cdot \eta_2) \right\} = \max \left\{ \rho_{\mathbb{M} \times \mathbb{N}}^3(\vartheta_1), \rho_{\mathbb{M} \times \mathbb{N}}^3(\vartheta_2) \right\} \geq \max \left\{ \max \left\{ \rho_{\mathbb{M} \times \mathbb{N}}^3(\vartheta_1), \rho_{\mathbb{M} \times \mathbb{N}}^3(\vartheta_2) \right\}, \max \left\{ \rho_{\mathbb{M} \times \mathbb{N}}^3(\eta_1), \rho_{\mathbb{M} \times \mathbb{N}}^3(\eta_2) \right\} \right\} = \max \left\{ \rho_{\mathbb{M} \times \mathbb{N}}^3(\vartheta_1, \eta_1), \rho_{\mathbb{M} \times \mathbb{N}}^3(\vartheta_2, \eta_2) \right\},
\]
and
\[
\rho_{\mathbb{M} \times \mathbb{N}}^\xi((\vartheta_1, \eta_1), (\vartheta_2, \eta_2)) = \left\{ \rho_{\mathbb{M} \times \mathbb{N}}^\xi(\vartheta_1 \cdot \vartheta_2), (\eta_1 \cdot \eta_2) \right\} = \max \left\{ \rho_{\mathbb{M} \times \mathbb{N}}^\xi(\vartheta_1), \rho_{\mathbb{M} \times \mathbb{N}}^\xi(\vartheta_2) \right\} \geq \max \left\{ \max \left\{ \rho_{\mathbb{M} \times \mathbb{N}}^\xi(\vartheta_1), \rho_{\mathbb{M} \times \mathbb{N}}^\xi(\vartheta_2) \right\}, \max \left\{ \rho_{\mathbb{M} \times \mathbb{N}}^\xi(\eta_1), \rho_{\mathbb{M} \times \mathbb{N}}^\xi(\eta_2) \right\} \right\} = \max \left\{ \rho_{\mathbb{M} \times \mathbb{N}}^\xi(\vartheta_1, \eta_1), \rho_{\mathbb{M} \times \mathbb{N}}^\xi(\vartheta_2, \eta_2) \right\}.
\]
\[
= \max \left\{ \rho_{(2n \times 2n)}^\varphi(\vartheta_1, \eta_1), \rho_{(2n \times 2n)}^{\varphi_1}(\vartheta_2, \eta_2) \right\}.
\]

Hence, \(\mathfrak{M} \times \mathfrak{N} = \left\langle \rho_{(2n \times 2n)}^\varphi, \rho_{(2n \times 2n)}^{\varphi_1}, \rho_{(2n \times 2n)}^{\varphi_2} \right\rangle\) is a FNINK-S of \(\chi_1 \times \chi_2\).

**Theorem 2.** Let \(\mathfrak{M} = \left\langle \rho_{\varphi_1}, \rho_{\varphi_2}, \rho_{\varphi_3} \right\rangle\) and \(\mathfrak{N} = \left\langle \rho_{\varphi_1}, \rho_{\varphi_2}, \rho_{\varphi_3} \right\rangle\) be two FNINK-Is of \(\chi_1\) and \(\chi_2\), respectively. Then, the direct product \(\mathfrak{M} \times \mathfrak{N}\), defined by

\[
\mathfrak{M} \times \mathfrak{N} = \left\langle \rho_{(2n \times 2n)}^\varphi, \rho_{(2n \times 2n)}^{\varphi_1}, \rho_{(2n \times 2n)}^{\varphi_2} \right\rangle,
\]

is a FNINK-I of \(\chi_1 \times \chi_2\).

**Proof.** For any \((\vartheta, \eta) \in \chi_1 \times \chi_2\). We have

\[
\rho_{(2n \times 2n)}^\varphi(0, 0) = \min \left\{ \rho_{2n}^\varphi(0), \rho_{2n}^{\varphi_1}(0) \right\} \\
\leq \min \left\{ \rho_{2n}^\varphi(\vartheta), \rho_{2n}^{\varphi_1}(\eta) \right\} = \rho_{(2n \times 2n)}^\varphi(\vartheta, \eta),
\]

\[
\rho_{(2n \times 2n)}^{\varphi_1}(0, 0) = \max \left\{ \rho_{2n}^{\varphi_1}(0), \rho_{2n}^{\varphi_1}(0) \right\} \\
\geq \max \left\{ \rho_{2n}^{\varphi_1}(\vartheta), \rho_{2n}^{\varphi_1}(\eta) \right\} = \rho_{(2n \times 2n)}^{\varphi_1}(\vartheta, \eta)
\]

and

\[
\rho_{(2n \times 2n)}^{\varphi_2}(0, 0) = \max \left\{ \rho_{2n}^{\varphi_2}(0), \rho_{2n}^{\varphi_2}(0) \right\} \\
\geq \max \left\{ \rho_{2n}^{\varphi_2}(\vartheta), \rho_{2n}^{\varphi_2}(\eta) \right\} = \rho_{(2n \times 2n)}^{\varphi_2}(\vartheta, \eta).
\]

Also, for any \((\vartheta_1, \eta_1), (\vartheta_2, \eta_2), (\vartheta_3, \eta_3) \in \chi_1 \times \chi_2\). We have

\[
\rho_{(2n \times 2n)}^\varphi(\vartheta_1, \eta_1) = \min \left\{ \rho_{2n}^\varphi(\vartheta_1), \rho_{2n}^\varphi(\eta_1) \right\} \\
\leq \min \left\{ \min \left\{ \rho_{2n}^\varphi((\vartheta_3, \vartheta_2) \cdot (\vartheta_3, \vartheta_1)), \rho_{2n}^{\varphi_1}(\vartheta_1) \right\}, \left\{ \rho_{2n}^\varphi((\eta_3, \eta_1) \cdot (\eta_3, \eta_2)), \rho_{2n}^{\varphi_1}(\eta_1) \right\} \right\} \\
= \min \left\{ \min \left\{ \rho_{2n}^\varphi((\vartheta_3, \vartheta_2) \cdot (\vartheta_3, \vartheta_1)), \rho_{2n}^{\varphi_1}(\vartheta_1) \right\}, \left\{ \rho_{2n}^\varphi((\eta_3, \eta_1) \cdot (\eta_3, \eta_2)), \rho_{2n}^{\varphi_1}(\eta_1) \right\} \right\} \\
= \min \left\{ \rho_{2n \times 2n}^\varphi(((\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1)), ((\vartheta_3, \eta_3) \cdot (\vartheta_2, \eta_2))), \rho_{2n \times 2n}^\varphi((\vartheta_1, \eta_1)), \rho_{2n \times 2n}^\varphi((\vartheta_2, \eta_2)) \right\},
\]

\[
\rho_{(2n \times 2n)}^{\varphi_1}(\vartheta_1, \eta_1) = \max \left\{ \rho_{2n}^{\varphi_1}(\vartheta_1), \rho_{2n}^{\varphi_1}(\eta_1) \right\}
\]
\[ \rho_{2M \times N}^{3}(\vartheta, \eta) = \max \left\{ \rho_{2M}^{3}((\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1)), ((\vartheta_3, \eta_3) \cdot (\vartheta_2, \eta_2)), \rho_{2M \times N}^{3}(\vartheta, \eta) \right\} \]

and

\[ \rho_{2M \times N}^{3}(\vartheta, \eta) = \max \left\{ \rho_{2M}^{3}(\vartheta, \eta), \rho_{2N}^{3}(\vartheta, \eta) \right\} \]

Hence, \( M \times N = \left( \rho_{(2M \times N)}^{3}, \rho_{(2N \times M)}^{3}, \rho_{(2N \times N)}^{3} \right) \) is a FNINK-I of \( \chi_1 \times \chi_2 \).

**Theorem 3.** Let \( M = \left( \rho_{(2M \times N)}^{3}, \rho_{2M}^{3}, \rho_{2N}^{3} \right) \) and \( N = \left( \rho_{(2M \times N)}^{3}, \rho_{2M}^{3}, \rho_{2N}^{3} \right) \) be two FNINK-Is of \( \chi_1 \) and \( \chi_2 \), respectively. Then, the direct product \( M \times N \), defined by

\[ M \times N = \left( \rho_{(2M \times N)}^{3}, \rho_{(2N \times M)}^{3}, \rho_{(2N \times N)}^{3} \right), \]

is a FNINK-I of \( \chi_1 \times \chi_2 \).

**Proof.** By applying Theorem 2, the FNS \( M \times N = \left( \rho_{(2M \times N)}^{3}, \rho_{(2N \times M)}^{3}, \rho_{(2N \times N)}^{3} \right) \) is a FNINK-I of \( \chi_1 \times \chi_2 \). Now, \( \forall (\vartheta, \eta) \in \chi_1 \times \chi_2 \), we have

\[ \rho_{2M \times N}^{3}((0, 0), (\vartheta, \eta)) \leq \rho_{2M \times N}^{3}((0, 0), (\vartheta, \eta)) \]

\[ = \min \left\{ \rho_{2M}^{3}(0, \vartheta), \rho_{2N}^{3}(0, \eta) \right\} \]

\[ \leq \min \left\{ \rho_{2M}^{3}(\vartheta), \rho_{2N}^{3}(\eta) \right\} \]

\[ = \rho_{2M \times N}^{3}(\vartheta, \eta), \]

\[ \rho_{2M \times N}^{3}((0, 0), (\vartheta, \eta)) \geq \rho_{2M \times N}^{3}((0, 0), (\vartheta, \eta)) \]

\[ = \max \left\{ \rho_{2M}^{3}(0, \vartheta), \rho_{2N}^{3}(0, \eta) \right\} \]

\[ \geq \max \left\{ \rho_{2M}^{3}(\vartheta), \rho_{2N}^{3}(\eta) \right\} \]

and

\[ \rho_{2M \times N}^{3}((0, 0), (\vartheta, \eta)) \geq \rho_{2M \times N}^{3}((0, 0), (\vartheta, \eta)) \]
\[
\begin{align*}
\rho(0, 0) & \leq \rho(0, \eta) \\
1 - \rho(0, 0) & \geq 1 - \rho(0, \eta) \\
\rho(1, 0) & \geq \max \left\{ 1 - \rho((0, \eta), (0, \eta)), 1 - \rho((0, \eta), (0, \eta)) \right\} \\
\rho(1, 0) & \geq \max \left\{ 1 - \rho((0, \eta), (0, \eta)), 1 - \rho((0, \eta), (0, \eta)) \right\}.
\end{align*}
\]

Hence, \(\mathfrak{M} \times \mathfrak{N} = \left\langle \rho(0, 0), \rho(0, \eta) \right\rangle\) is a FNINK-I of \(\chi_1 \times \chi_2\).

**Theorem 4.** Let \(\mathfrak{M} = \left\langle \rho(0, 0), \rho(0, \eta) \right\rangle\) and \(\mathfrak{N} = \left\langle \rho(0, 0), \rho(0, \eta) \right\rangle\) be two FNINK-Is of \(\chi_1\) and \(\chi_2\), respectively. Then, \(\mathfrak{M} \times \mathfrak{N} = \left\langle \rho(0, 0), \rho(0, \eta) \right\rangle\) is a FNINK-I of \(\chi_1 \times \chi_2\), where \(\rho(0, 0) = 1 - \rho(0, 0)\).

**Proof.** According to Theorem 2, \(\mathfrak{M} \times \mathfrak{N} = \left\langle \rho(0, 0), \rho(0, \eta) \right\rangle\) is a FNINK-I of \(\chi_1 \times \chi_2\). Then,

\[
\begin{align*}
\rho(0, 0) & \leq \rho(0, \eta) \\
1 - \rho(0, 0) & \geq 1 - \rho(0, \eta) \\
\rho(1, 0) & \geq \max \left\{ 1 - \rho((0, \eta), (0, \eta)), 1 - \rho((0, \eta), (0, \eta)) \right\} \\
\rho(1, 0) & \geq \max \left\{ 1 - \rho((0, \eta), (0, \eta)), 1 - \rho((0, \eta), (0, \eta)) \right\}.
\end{align*}
\]

Hence, \(\mathfrak{M} \times \mathfrak{N} = \left\langle \rho(0, 0), \rho(0, \eta) \right\rangle\) is a FNINK-I of \(\chi_1 \times \chi_2\).

**Theorem 5.** Let \(\mathfrak{M} = \left\langle \rho(0, 0), \rho(0, \eta) \right\rangle\) and \(\mathfrak{N} = \left\langle \rho(0, 0), \rho(0, \eta) \right\rangle\) be two FNINK-Is of \(\chi_1\) and \(\chi_2\), respectively. Then, \(\mathfrak{M} \times \mathfrak{N} = \left\langle \rho(0, 0), \rho(0, \eta) \right\rangle\) is a FNINK-I of \(\chi_1 \times \chi_2\), where \(\rho(0, 0) = 1 - \rho(0, 0)\).

**Proof.** According to Theorem 2, \(\mathfrak{M} \times \mathfrak{N} = \left\langle \rho(0, 0), \rho(0, \eta) \right\rangle\) is a FNINK-I of \(\chi_1 \times \chi_2\). Then,

\[
\begin{align*}
\rho(0, 0) & \geq \rho(0, \eta) \\
1 - \rho(0, 0) & \leq 1 - \rho(0, \eta)
\end{align*}
\]
\[ \rho_{M_1 \times M_2}(0, 0) \geq \rho_{M_1 \times M_2}(\vartheta, \eta). \]

Now, for any \((\vartheta_1, \eta_1), (\vartheta_2, \eta_2), (\vartheta_3, \eta_3) \in \chi_1 \times \chi_2\). We have

\[
\rho_{(M_1 \times M_2)}(\vartheta_1, \eta_1) = \max \left\{ \rho_{(M_1 \times M_2)}((\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1)) \cdot (\vartheta_2, \eta_2)), \rho_{(M_1 \times M_2)}(\vartheta_2, \eta_2) \right\}
\]

\[
1 - \rho_{(M_1 \times M_2)}(\vartheta_1, \eta_1) \leq 1 - \max \left\{ \rho_{(M_1 \times M_2)}((\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1)) \cdot (\vartheta_2, \eta_2)), \rho_{(M_1 \times M_2)}(\vartheta_2, \eta_2) \right\}
\]

\[
\overline{\rho}_{(M_1 \times M_2)}(\vartheta_1, \eta_1) \leq \min \left\{ 1 - \rho_{(M_1 \times M_2)}((\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1)) \cdot (\vartheta_2, \eta_2)), 1 - \rho_{(M_1 \times M_2)}(\vartheta_2, \eta_2) \right\}
\]

\[
\overline{\rho}_{(M_1 \times M_2)}(\vartheta_1, \eta_1) \leq \min \left\{ \overline{\rho}_{(M_1 \times M_2)}((\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1)) \cdot (\vartheta_2, \eta_2)), \overline{\rho}_{(M_1 \times M_2)}(\vartheta_2, \eta_2) \right\}.
\]

Hence, \( M_1 \times M_2 = \left\{ \rho_{(M_1 \times M_2)}, \rho_{(M_1 \times M_2)}, \rho_{(M_1 \times M_2)} \right\} \) is a FNINK-I of \( \chi_1 \times \chi_2 \).

**Theorem 6.** Let \( M = \left\{ \rho_{M_1}, \rho_{M_2}, \rho_{M_2} \right\} \) and \( N = \left\{ \rho_{N_1}, \rho_{N_2}, \rho_{N_2} \right\} \) be two FNINK-Is of \( \chi_1 \) and \( \chi_2 \), respectively. Then, \( M \times N = \left\{ \overline{\rho}_{(M_1 \times N_1)}(0, 0), \rho_{(M_1 \times N_1)}, \overline{\rho}_{(M_1 \times N_1)}(0, 0) \right\} \) is a FNINK-I of \( \chi_1 \times \chi_2 \), where \( \overline{\rho}_{(M_1 \times N_1)} = 1 - \rho_{(M_1 \times N_1)} \) and \( \overline{\rho}_{(M_1 \times N_1)} = 1 - \rho_{(M_1 \times N_1)} \).

**Proof.** The proof is produced by using Theorem 4 and Theorem 5 together.

**Theorem 7.** Let \( M \times N = \left\{ \rho_{(M_1 \times N_1)}, \rho_{(M_1 \times N_1)}, \rho_{(M_1 \times N_1)} \right\} \) be a FNINK-I of \( \chi_1 \times \chi_2 \). Then, \((M \times N)^s = \left\{ \rho_{(M_1 \times N_1)^s}, \rho_{(M_1 \times N_1)^s}, \rho_{(M_1 \times N_1)^s} \right\} \) is a FNINK-I of \( \chi_1 \times \chi_2 \).

**Proof.** For any \((\vartheta, \eta) \in \chi_1 \times \chi_2 \). Then,

\[
\rho_{(M_1 \times N_1)}(0, 0) \leq \rho_{(M_1 \times N_1)}(\vartheta, \eta)
\]

\[
\rho_{(M_1 \times N_1)}(0, 0) \leq \rho_{(M_1 \times N_1)}(\vartheta, \eta)
\]

\[
\rho_{(M_1 \times N_1)}(0, 0) \leq \rho_{(M_1 \times N_1)}(\vartheta, \eta)
\]

\[
\rho_{(M_1 \times N_1)}(0, 0) \leq \rho_{(M_1 \times N_1)}(\vartheta, \eta)
\]

\[
\rho_{(M_1 \times N_1)}(0, 0) \leq \rho_{(M_1 \times N_1)}(\vartheta, \eta)
\]

\[
\rho_{(M_1 \times N_1)}(0, 0) \leq \rho_{(M_1 \times N_1)}(\vartheta, \eta)
\]

and

\[
\rho_{(M_1 \times N_1)}(0, 0) \geq \rho_{(M_1 \times N_1)}(\vartheta, \eta)
\]
\[ \begin{align*}
\{ \rho^s_{(\mathcal{X} \times \mathcal{Y})}(0,0) \} & \leq \{ \rho^s_{(\mathcal{X} \times \mathcal{Y})}(\vartheta, \eta) \}^s \\
\{ \rho^s_{(\mathcal{X} \times \mathcal{Y})}(0,0) \} & \geq \{ \rho^s_{(\mathcal{X} \times \mathcal{Y})}(\vartheta, \eta) \}^s \\
\{ \rho^s_{(\mathcal{X} \times \mathcal{Y})}(0,0) \} & \geq \{ \rho^s_{(\mathcal{X} \times \mathcal{Y})}(\vartheta, \eta) \}.
\end{align*} \]

If \((\vartheta_1, \eta_1), (\vartheta_2, \eta_2)\) and \((\vartheta_3, \eta_3)\) \(\in \chi_1 \times \chi_2\), then
\[ \begin{align*}
\{ \rho^s_{(\mathcal{X} \times \mathcal{Y})}(\vartheta_1, \eta_1) \}^s & \leq \min \left\{ \rho^s_{(\mathcal{X} \times \mathcal{Y})}( (\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1) ) \cdot ( (\vartheta_3, \eta_3) \cdot (\vartheta_2, \eta_2) ), \rho^s_{(\mathcal{X} \times \mathcal{Y})}(\vartheta_2, \eta_2) \right\}^s \\
\{ \rho^s_{(\mathcal{X} \times \mathcal{Y})}(\vartheta_1, \eta_1) \}^s & \leq \min \left\{ \rho^s_{(\mathcal{X} \times \mathcal{Y})}( (\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1) ) \cdot ( (\vartheta_3, \eta_3) \cdot (\vartheta_2, \eta_2) ), \rho^s_{(\mathcal{X} \times \mathcal{Y})}(\vartheta_2, \eta_2) \right\}^s, \\
\{ \rho^s_{(\mathcal{X} \times \mathcal{Y})}(\vartheta_1, \eta_1) \}^s & \leq \min \left\{ \rho^s_{(\mathcal{X} \times \mathcal{Y})}( (\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1) ) \cdot ( (\vartheta_3, \eta_3) \cdot (\vartheta_2, \eta_2) ), \rho^s_{(\mathcal{X} \times \mathcal{Y})}(\vartheta_2, \eta_2) \right\},
\end{align*} \]

and
\[ \begin{align*}
\{ \rho^s_{(\mathcal{X} \times \mathcal{Y})}(\vartheta_1, \eta_1) \}^s & \geq \max \left\{ \rho^s_{(\mathcal{X} \times \mathcal{Y})}( (\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1) ) \cdot ( (\vartheta_3, \eta_3) \cdot (\vartheta_2, \eta_2) ), \rho^s_{(\mathcal{X} \times \mathcal{Y})}(\vartheta_2, \eta_2) \right\}^s \\
\{ \rho^s_{(\mathcal{X} \times \mathcal{Y})}(\vartheta_1, \eta_1) \}^s & \geq \max \left\{ \rho^s_{(\mathcal{X} \times \mathcal{Y})}( (\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1) ) \cdot ( (\vartheta_3, \eta_3) \cdot (\vartheta_2, \eta_2) ), \rho^s_{(\mathcal{X} \times \mathcal{Y})}(\vartheta_2, \eta_2) \right\}^s, \\
\{ \rho^s_{(\mathcal{X} \times \mathcal{Y})}(\vartheta_1, \eta_1) \}^s & \geq \max \left\{ \rho^s_{(\mathcal{X} \times \mathcal{Y})}( (\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1) ) \cdot ( (\vartheta_3, \eta_3) \cdot (\vartheta_2, \eta_2) ), \rho^s_{(\mathcal{X} \times \mathcal{Y})}(\vartheta_2, \eta_2) \right\},
\end{align*} \]

Hence, \((\mathcal{X} \times \mathcal{Y})^s = \left\{ \rho_{(\mathcal{X} \times \mathcal{Y})}^s, \rho^s_{(\mathcal{X} \times \mathcal{Y})}, \rho^s_{(\mathcal{X} \times \mathcal{Y})} \right\} \) is a FNINK-I of \(\chi_1 \times \chi_2\).

**Theorem 8.** Let
\[ \mathcal{X} \times \mathcal{Y} = \left\{ \rho_{(\mathcal{X} \times \mathcal{Y})}^s, \rho^s_{(\mathcal{X} \times \mathcal{Y})}, \rho^s_{(\mathcal{X} \times \mathcal{Y})} \right\} \]
and
\[ \mathcal{D} \times \mathcal{E} = \left\{ \rho_{(\mathcal{D} \times \mathcal{E})}^s, \rho^s_{(\mathcal{D} \times \mathcal{E})}, \rho^s_{(\mathcal{D} \times \mathcal{E})} \right\} \]
be two FNINK-Is of \(\chi_1 \times \chi_2\). Then,
\[ (\mathcal{X} \times \mathcal{Y}) \cap (\mathcal{D} \times \mathcal{E}) = \left\{ \rho_{(\mathcal{X} \times \mathcal{Y}) \cap (\mathcal{D} \times \mathcal{E})}^s, \rho^s_{(\mathcal{X} \times \mathcal{Y}) \cap (\mathcal{D} \times \mathcal{E})}, \rho^s_{(\mathcal{X} \times \mathcal{Y}) \cap (\mathcal{D} \times \mathcal{E})} \right\} \]
is a FNINK-I of \(\chi_1 \times \chi_2\).

**Proof.** Since \(\mathcal{X} \times \mathcal{Y}\) and \(\mathcal{D} \times \mathcal{E}\) are two FNINK-Is of \(\chi_1 \times \chi_2\). Then, \(\forall (\vartheta, \eta) \in \chi_1 \times \chi_2\), we have
\[ \rho_{(\mathcal{X} \times \mathcal{Y}) \cap (\mathcal{D} \times \mathcal{E})}(0,0) = \min \left\{ \rho_{(\mathcal{X} \times \mathcal{Y})}(0,0), \rho_{(\mathcal{D} \times \mathcal{E})}(0,0) \right\} \]
\[
\begin{align*}
\rho^2_{\mathcal{M} \times \mathcal{N} \cap (\mathcal{D} \times \mathcal{E})}(0, 0) &= \max \left\{ \rho^2_{\mathcal{M} \times \mathcal{N} \cap (\mathcal{D} \times \mathcal{E})}(0, 0) \right\} \\
&\geq \max \left\{ \rho^2_{\mathcal{M} \times \mathcal{N} \cap (\mathcal{D} \times \mathcal{E})}(\vartheta, \eta), \rho^2_{\mathcal{D} \times \mathcal{E}}(\vartheta, \eta) \right\} \\
&= \rho^2_{\mathcal{M} \times \mathcal{N} \cap (\mathcal{D} \times \mathcal{E})}(\vartheta, \eta)
\end{align*}
\]

and

\[
\begin{align*}
\rho^3_{\mathcal{M} \times \mathcal{N} \cap (\mathcal{D} \times \mathcal{E})}(0, 0) &= \max \left\{ \rho^3_{\mathcal{M} \times \mathcal{N} \cap (\mathcal{D} \times \mathcal{E})}(0, 0) \right\} \\
&\geq \max \left\{ \rho^3_{\mathcal{M} \times \mathcal{N} \cap (\mathcal{D} \times \mathcal{E})}(\vartheta, \eta), \rho^3_{\mathcal{D} \times \mathcal{E}}(\vartheta, \eta) \right\} \\
&= \rho^3_{\mathcal{M} \times \mathcal{N} \cap (\mathcal{D} \times \mathcal{E})}(\vartheta, \eta)
\end{align*}
\]

Now, for any \((\vartheta_1, \eta_1), (\vartheta_2, \eta_2)\) and \((\vartheta_3, \eta_3) \in \chi_1 \times \chi_2\), we have

\[
\begin{align*}
\rho^2_{\mathcal{M} \times \mathcal{N} \cap (\mathcal{D} \times \mathcal{E})}(\vartheta_1, \eta_1) &= \min \left\{ \rho^2_{\mathcal{M} \times \mathcal{N} \cap (\mathcal{D} \times \mathcal{E})}(\vartheta_1, \eta_1), \rho^2_{\mathcal{D} \times \mathcal{E}}(\vartheta_1, \eta_1) \right\} \\
&\leq \min \left\{ \min \left\{ \rho^2_{\mathcal{M} \times \mathcal{N}}(((\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1)) \cdot ((\vartheta_3, \eta_3) \cdot (\vartheta_2, \eta_2)) \right\}, \rho^2_{\mathcal{D} \times \mathcal{E}}((\vartheta_1, \eta_1) \cdot ((\vartheta_3, \eta_3) \cdot (\vartheta_2, \eta_2)) \right\} \\
&= \min \left\{ \min \left\{ \rho^2_{\mathcal{M} \times \mathcal{N}}(((\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1)) \cdot ((\vartheta_3, \eta_3) \cdot (\vartheta_2, \eta_2)) \right\}, \rho^2_{\mathcal{D} \times \mathcal{E}}((\vartheta_1, \eta_1) \cdot ((\vartheta_3, \eta_3) \cdot (\vartheta_2, \eta_2)) \right\} \\
&= \min \left\{ \rho^2_{\mathcal{M} \times \mathcal{N} \cap (\mathcal{D} \times \mathcal{E})}(((\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1)) \cdot ((\vartheta_3, \eta_3) \cdot (\vartheta_2, \eta_2)) \right\}, \rho^2_{\mathcal{D} \times \mathcal{E}}((\vartheta_1, \eta_1) \cdot ((\vartheta_3, \eta_3) \cdot (\vartheta_2, \eta_2)) \right\}
\end{align*}
\]
\[
\rho_{(D \times E)}^3(((\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1)) \cdot ((\vartheta_3, \eta_3) \cdot (\vartheta_2, \eta_2))) \bigg| \max \left\{ \rho_{(\mathcal{M} \times \mathcal{N})}^3(\vartheta_2, \eta_2), \rho_{(D \times E)}^3(\vartheta_2, \eta_2) \right\} \\
= \max \left\{ \rho_{(\mathcal{M} \times \mathcal{N}) \cap (D \times E)}^3(((\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1)) \cdot ((\vartheta_3, \eta_3) \cdot (\vartheta_2, \eta_2))), \rho_{(\mathcal{M} \times \mathcal{N}) \cap (D \times E)}^3(\vartheta_2, \eta_2) \right\},
\]

and

\[
\rho_{(\mathcal{M} \times \mathcal{N}) \cap (D \times E)}^3(\vartheta_1, \eta_1) = \max \left\{ \rho_{(\mathcal{M} \times \mathcal{N})}^3(\vartheta_1, \eta_1), \rho_{(D \times E)}^3(\vartheta_1, \eta_1) \right\} \\
\geq \max \left\{ \max \left\{ \rho_{(\mathcal{M} \times \mathcal{N})}^3(((\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1)) \cdot ((\vartheta_3, \eta_3) \cdot (\vartheta_2, \eta_2))), \rho_{(\mathcal{M} \times \mathcal{N}) \cap (D \times E)}^3(\vartheta_2, \eta_2) \right\}, \rho_{(D \times E)}^3(\vartheta_2, \eta_2) \right\} \\
= \max \left\{ \max \left\{ \rho_{(\mathcal{M} \times \mathcal{N})}^3(((\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1)) \cdot ((\vartheta_3, \eta_3) \cdot (\vartheta_2, \eta_2))), \rho_{(D \times E)}^3(\vartheta_2, \eta_2) \right\} \right\}.
\]

Hence, \((\mathcal{M} \times \mathcal{N}) \cap (D \times E) = \left\{ \rho_{(\mathcal{M} \times \mathcal{N})}^3(\vartheta_1, \eta_1), \rho_{(D \times E)}^3(\vartheta_1, \eta_1) \right\} \) is a FNINK-I of \(\chi_1 \times \chi_2\).

4. Comparison Analysis

A common ground between FNINK-Algebras and Neutrosophic INK-Algebras is INK-algebra, which emphasizes the integration of non-membership, indeterminacy, and uncertainty in algebraic structures. Both approaches are intended for complex system modelling and analysis, where a high prevalence of imprecise and incomplete information exists. Although both approaches provide useful tools for managing uncertainties in algebraic structures, FNINK-Algebras are a better method because of their improved specificity and precision. FNINK-algebras are a more sophisticated and elegant mathematical framework because the incorporation of Fermatean features enables a more detailed representation of indeterminacies and non-memberships. Specialized conditions for FNCINK-Is and direct products add to the robustness and generalizability of the method in different fields. As a result, FNINK-Algebras become the method of choice for accurately and completely representing uncertainties in complex systems.
5. Conclusion

The notion of the direct product of FNSs is used in this paper to discuss an INK-ideal inside an INK-algebra. We present the direct product for FNCINK-Is and Fermatean neutrosophic INK-algebras, examining a number of properties. The direct product of FNSs is shown to satisfy certain requirements in order to be considered a direct product of FNINK-Is inside an INK-algebra.

References


