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Paradistributive Latticoids

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Abstract. In this paper, the concept of Paradistributive Latticoid (PDL) as a generalization of a distributive lattice is introduced and investigated its properties. A set of equivalent conditions for a PDL to become a distributive lattice are given. The notions of an ideal and a filter in a PDL are introduced and studied their properties. Subdirect representation of a PDL is obtained.

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Key Words and Phrases: Paradistributive Latticoid(PDL), Ideal, Filter, Congruence, Subdirectly irreducible.

1. Introduction

Garret Birkhoff's effort in the mid-1930's arouse off with the overall development of lattice theory (see [1]). In a great sequence of works, he signified the circumstances of lattice theory and demonstrated how it provides a conjoining for independent advancements in many arithmetic disciplines. Birkhoff system is an algebra in which two binary operations meet and join, each of which is commutative, associative, and idempotent, and which, when combined, satisfy the relation $x \wedge (x \vee y) = x \vee (x \wedge y)$ (see [3, 4]). This is a weakend version of the absorption law for lattices and was introduced in the year 1948. Lattices and quasilattices are examples of Birkhoff systems, with the latter being the regularization of a variety of lattices. The types of Birkhoff systems that adhere to one or both distributive rules were studied. They were three Birkhoff systems namely meet-distributive, join-distributive, and distributive. The duality between meet and join operations lead to corresponding results for join-distributive Birkhoff systems. A characterisation of these varieties, subvarieties, a duality thesis for distributive Birkhoff systems, a structure theory for meet-distributive Birkhoff systems improved descriptions of some of the subdirectly irreducibles. As one of the standard application of Birkhoff theorem

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is Stone's theorem, here our aim of this paper is to use Birkhoff theorem for subdirectly irreducible PDLs.

George Boole's attempt to formulate propositional logic in the first part of the nineteenth century induced the idea of Boolean algebras (see [2]). At the end of the course, he looked into the axiomatic of Boolean algebras. Distributive lattices had a main role in lattice theory. As lattice theory began with Boolean algebra, the postulation of distributive lattices is the most comprehensive with fulfilling chapter in the history of lattice theory. Many lattice conditions, as well as lattice components and ideals, are debilitate variants of distributivity. As a result, a detailed mastery of distributive lattices is required to perform in lattice theory. Distributive lattices are characterized by their lattice of ideals. Finally, in several applications, the distributivity constraint is enforced on lattices arising in various fields of mathematics, particularly algebra. Certain algebras are referred as distributive quasi lattices, and they are used to generalize distributive lattices. J. A. Kalman given subdirect decomposition of distributive quasilattices (see [5])

U. M. Swamy and G. C. Rao introduced the concept of an almost distributive lattice(ADL) (see [8]). This group of ADLs covers nearly all the existing ring theoretical hypothesis of a Boolean algebra. The class of triple systems has been introduced by Subrahmanyam as a lattice theoretic generalisation of P_1-rings (see [6, 7]). For most of the results that are valid in triple systems the additive semigroup structure in the triple system does not play any role. This motivated them to introduce the class of almost distributive lattices. An ADL is an algebra $(L, \vee, \wedge)$ which fulfills all of the distributive lattice's axioms with 0 except the possibility of commutativity with respect to $\vee$ and $\wedge$. The idea of our paper is to introduce a new algebraic structure as a generalization of distributive lattice and distributive Birkhoff systems. We introduce Paradistributive Latticoids, which is an algebra of type (2,2,0) that assures all the propositions of a distributive lattice with the possible exception of the left distributivity of the operation meet and the commutativity of the operations join and meet. In §2, we discuss the basic definition of a PDL and some preliminary results related to PDL and illustrate few examples. In §3, we introduce the notions of an ideal and a filter in a PDL and investigate its properties. Further in §4, we also provide a subdirect representation theorem for associative PDLs using Birkhoff theorem, which simplifies many results in PDLs.

2. Paradistributive Latticoids

In this section, we define a Paradistributive Latticoid and present some fundamental findings, the most of which require just simple verification.

Definition 1. *An algebra $(V, \vee, \wedge, 1)$ of type (2,2,0) is called a Paradistributive Latticoid, if it assures the subsequent axioms:*
$(LD\vee)$ $p_1 \vee (p_2 \wedge p_3) = (p_1 \vee p_2) \wedge (p_1 \vee p_3)$.
$(RD\vee)$ $(p_1 \wedge p_2) \vee p_3 = (p_1 \vee p_3) \wedge (p_2 \vee p_3)$.
(L_1) $(p_1 \vee p_2) \wedge p_2 = p_2$.
(L_2) $(p_1 \vee p_2) \wedge p_1 = p_1$.

(L_3) $p_1 \vee (p_1 \wedge p_2) = p_1$.
(I_1) $p_1 \vee 1 = 1$.
for all $p_1, p_2, p_3 \in V$.

The independence of the axioms mentioned in the above definition can be verified using non-trivial examples. The example below demonstrates how every non-empty set can be transformed into a PDL with any member that has been arbitrarily preassigned as its unity(greatest) element.

For any $p_1, p_2 \in V$, we say that p_1 is less than or equal to p_2 and write $p_1 \leq p_2$ if $p_1 \wedge p_2 = p_1$ or equivalently $p_1 \vee p_2 = p_2$ and it can be easily observed that $\leq$ is a partial order on V. The element 1, in definition 1, is called the greatest element.

Example 1. *Let* V *be a non-empty set. Fix some element* $y_0 \in V$*. Then, for any* $x, y \in V$ *define* $\vee$ *and* $\wedge$ *on* V *by*

$$x \vee y = \begin{cases} x & y \neq y_0 \\ y_0 & y = y_0 \end{cases}$$

and

$$x \wedge y = \begin{cases} y & y \neq y_0 \\ x & y = y_0 \end{cases}$$

Then $(V, \vee, \wedge, y_0)$ *is a disconnected PDL with* y_0 *as its greatest element.*

Example 2. *Let* $V = \{0, 1, 2, 3, 4\}$ *be a set with binary operations* $\vee$ *and* $\wedge$ *given in the following tables:*

$\vee$	0	1	2	3	4
0	0	1	0	3	3
1	1	1	1	1	1
2	2	1	2	4	4
3	3	1	3	3	3
4	4	1	4	4	4

$\wedge$	0	1	2	3	4
0	0	0	2	0	2
1	0	1	2	3	4
2	0	2	2	0	2
3	0	3	2	3	4
4	0	4	2	3	4

Then $(V;\ \vee\ \wedge\ 1)$ *is a Paradistributive Latticoid. But* V *is not a distributive lattice, since* $0 \wedge (2 \vee 1) = 0 \wedge 1 = 0 \neq 2 = 2 \vee 0 = (0 \wedge 2) \vee (0 \wedge 1), 0 \wedge 2 = 2 \neq 0 = 2 \wedge 0$ *and* $0 \vee 2 = 0 \neq 2 = 2 \vee 0$.

Example 3. *Let* $(V, +, \cdot, 0, 1)$ *be a commutative regular ring with unity and let* x_0 *be the unique idempotent element in* V *such that* $xV = x_0V$*. Now, for any* $x, y \in V$*, define*
(1) $x \vee y = y_0 x$
(2) $x \wedge y = x + y - y_0 x$.
Then $(V, \vee, \wedge, 0)$ *is a PDL.*

We now provide some essential results.

Lemma 1. *For any $p_1, p_2 \in V$, the following holds:*
(1) $p_1 \wedge p_1 = p_1$.
(2) $p_1 \vee p_1 = p_1$.
(L_4) $(p_1 \wedge p_2) \vee p_2 = p_2$.
(L_5) $p_1 \vee (p_2 \wedge p_1) = p_1$.
(L_6) $p_1 \wedge (p_1 \vee p_2) = p_1$.

Proof. Proofs of (1), (2) are obvious. Now $(p_1 \wedge p_2) \vee p_2 = (p_1 \vee p_2) \wedge (p_2 \vee p_2) = (p_1 \vee p_2) \wedge p_2 = p_2$, which proves (L_4). Also $p_1 \vee (p_2 \wedge p_1) = (p_1 \vee p_2) \wedge (p_1 \wedge p_1) = (p_1 \vee p_2) \wedge p_1 = p_1$ which proves (L_5). Lastly, (L_6) follows as $p_1 \wedge (p_1 \vee p_2) = (p_1 \vee p_1) \wedge (p_1 \vee p_2) = p_1 \vee (p_1 \wedge p_2) = p_1$.

Lemma 2. *For any $p_1, p_2 \in V$, $p_1 \vee p_2 = p_2 \vee p_1$ whenever $p_1 \leq p_2$.*

Proof. Let $p_1, p_2 \in V$ and $p_1 \leq p_2$. Then $p_2 \vee p_1 = p_1 = p_1 \vee (p_2 \wedge p_1) = p_1 \vee p_2$.

Lemma 3. *The relation $\leq$ is a partial ordering on V.*

Proof. The reflexivity of $\leq$ follows from Lemma 1. Let $p_1, p_2 \in V$ be such that $p_1 \leq p_2$ and $p_2 \leq p_1$. That is $p_1 \vee p_2 = p_2$ and $p_2 \vee p_1 = p_1$ and hence by Lemma 2, we have $p_1 = p_2$. Thus $\leq$ is anti symmetric. Finally, suppose $p_1, p_2, p_3 \in V$ such that $p_1 \leq p_2 \leq p_3$. Then $p_1 \vee p_3 = (p_1 \wedge p_2) \vee p_3 = (p_1 \vee p_3) \wedge (p_2 \vee p_3) = p_3 \wedge p_3 = p_3$ implies $p_1 \leq p_3$, hence $\leq$ is transitive.

Lemma 4. *For any $p_1, p_2 \in V$, $p_1 \wedge p_2 = p_1$ if and only if $p_1 \vee p_2 = p_2$.*

Proof. Let $p_1 \wedge p_2 = p_1$. Then $p_1 \vee p_2 = (p_1 \wedge p_2) \vee p_2 = p_2$. Similarly, for $p_1 \vee p_2 = p_2$, we have, $p_1 \wedge p_2 = p_1 \wedge (p_1 \vee p_2) = p_1$.

Lemma 5. *For any $p_1, p_2 \in V$, $p_1 \vee p_2 = p_1$ if and only if $p_1 \wedge p_2 = p_2$.*
Proof. Let $p_1 \vee p_2 = p_1$. Then $p_1 \wedge p_2 = (p_1 \vee p_2) \wedge p_2 = p_2$. Similarly, for $p_1 \wedge p_2 = p_2$, we have, $p_1 \vee p_2 = p_1 \vee (p_1 \wedge p_2) = p_1$.

Lemma 6. *For any $p_1, p_2 \in V$, the following holds:*
(1) $(p_1 \vee p_2) \vee p_2 = p_1 \vee p_2$.
(2) $(p_1 \vee p_2) \vee p_1 = p_1 \vee p_2$.
(3) $p_1 \vee (p_1 \vee p_2) = p_1 \vee p_2$.
(4) $p_1 \wedge (p_1 \wedge p_2) = p_1 \wedge p_2$.
(5) $(p_1 \wedge p_2) \wedge p_2 = p_1 \wedge p_2$.
(6) $p_2 \wedge (p_1 \wedge p_2) = p_1 \wedge p_2$.

Proof. (1). By definition 1, we have $(p_1 \vee p_2) \wedge p_2 = p_2$. Now, by Lemma 5, we have $(p_1 \vee p_2) \vee p_2 = p_1 \vee p_2$.
(2). By definition 1, we have $(p_1 \vee p_2) \wedge p_1 = p_1$. Now, by Lemma 5, we have $(p_1 \vee p_2) \vee p_1 = p_1 \vee p_2$.

Lemma 7. *For any $p_1 \in V$, we have*
(I_3) $p_1 \wedge 1 = p_1$.
(I_4) $1 \wedge p_1 = p_1$.
(I_5) $1 \vee p_1 = 1$.

Proof. Let $p_1 \in V$. Then $p_1 \wedge 1 = p_1 \wedge (p_1 \vee 1) = p_1$ and $1 \wedge p_1 = (p_1 \vee 1) \wedge p_1 = p_1$. Similarly, $1 \vee p_1 = 1 \vee (1 \wedge p_1) = 1$ which proves (I_5).

Theorem 1. *For any $p_1, p_2, p_3 \in V$,*
$(RD\wedge)$

$$(p_1 \vee p_2) \wedge p_3 = (p_1 \wedge p_3) \vee (p_2 \wedge p_3).$$

Proof. Let $p_1, p_2, p_3 \in V$. Write $x = (p_1 \vee p_2) \wedge p_3$ and $y = (p_1 \wedge p_3) \vee (p_2 \wedge p_3) = [(p_1 \wedge p_3) \vee p_2] \wedge p_3$. Then, by $(RD\vee)$, $(LD\vee)$, (L_3), (L_5) and (L_6), we have

$$\begin{aligned}
x \vee y &= [(p_1 \vee p_2) \wedge p_3] \vee [(p_1 \wedge p_3) \vee (p_2 \wedge p_3)] \\
&= [(p_1 \vee p_2) \vee [(p_1 \wedge p_3) \vee (p_2 \wedge p_3)]] \wedge p_3 \\
&= [(p_1 \vee p_2) \vee [((p_1 \wedge p_3) \vee p_2) \wedge p_3]] \wedge p_3 \\
&= [(p_1 \vee p_2) \vee [((p_1 \vee p_2) \wedge (p_3 \vee p_2)) \wedge p_3]] \wedge p_3 \\
&= [(p_1 \vee p_2) \wedge [(p_1 \vee p_2) \vee p_3]] \wedge p_3 \\
&= [(p_1 \vee p_2) \wedge p_3] \\
&= x
\end{aligned}$$

Also,

$$\begin{aligned}
x \vee y &= [(p_1 \vee p_2) \wedge p_3] \vee [((p_1 \wedge p_3) \vee p_2) \wedge p_3] \\
&= [((p_1 \vee p_2) \vee ((p_1 \wedge p_3) \vee p_2)) \wedge (p_3 \vee ((p_1 \wedge p_3) \vee p_2))] \wedge p_3 \\
&= [((p_1 \vee p_2) \vee ((p_1 \vee p_2) \wedge (p_3 \vee p_2))) \wedge (p_3 \vee ((p_1 \vee p_2) \wedge (p_3 \vee p_2)))] \wedge p_3 \\
&= [(p_1 \vee p_2) \wedge ((p_3 \vee (p_1 \vee p_2)) \wedge (p_3 \vee p_2))] \wedge p_3 \\
&= [(p_1 \vee p_2) \wedge (p_3 \vee ((p_1 \vee p_2) \wedge p_2))] \wedge p_3 \\
&= [(p_1 \vee p_2) \wedge (p_3 \vee p_2)] \wedge p_3 \\
&= [(p_1 \wedge p_3) \vee p_2] \wedge p_3. \\
&= y
\end{aligned}$$

Therefore, $(p_1 \vee p_2) \wedge p_3 = (p_1 \wedge p_3) \vee (p_2 \wedge p_3)$.

Theorem 2. *For any $p_1, p_2 \in V$, the following are equivalent:*
(1) $(p_1 \wedge p_2) \vee p_1 = p_1$.
(2) $p_1 \wedge (p_2 \vee p_1) = p_1$.
(3) $(p_2 \wedge p_1) \vee p_2 = p_2$.
(4) $p_2 \wedge (p_1 \vee p_2) = p_2$.
(5) $p_1 \wedge p_2 = p_2 \wedge p_1$.
(6) $p_1 \vee p_2 = p_2 \vee p_1$.
(7) $p_1 \wedge p_2 \leq p_1$.
(8) *There exists $a \in V$ such that $a \leq p_1$ and $a \leq p_2$.*
(9) *The g.l.b of p_1 and p_2 exists in V and equals $p_1 \wedge p_2$.*

Proof. Let $p_1, p_2 \in V$.
$(1) \Rightarrow (2)$: Assume (1). Then $p_1 \wedge (p_2 \vee p_1) = (p_1 \vee p_1) \wedge (p_2 \vee p_1) = (p_1 \wedge p_2) \vee p_1 = p_1$.
$(2) \Rightarrow (1)$: Assume (2). Then $(p_1 \wedge p_2) \vee p_1 = (p_1 \vee p_1) \wedge (p_2 \vee p_1) = p_1 \wedge (p_2 \vee p_1) = p_1$.
$(3) \Rightarrow (4)$: Assume (3). Then $p_2 \wedge (p_1 \vee p_2) = (p_2 \vee p_2) \wedge (p_1 \vee p_2) = (p_2 \wedge p_1) \vee p_2 = p_2$
$(4) \Rightarrow (3)$: Assume (4). Then $(p_2 \wedge p_1) \vee p_2 = (p_2 \vee p_2) \wedge (p_1 \vee p_2) = p_2 \wedge (p_1 \vee p_2) = p_2$.
$(1) \Rightarrow (5)$: Assume (1). Then

$$\begin{aligned} p_2 \wedge p_1 &= [(p_1 \wedge p_2) \vee p_2] \wedge [(p_1 \wedge p_2) \vee p_1] \\ &= (p_1 \wedge p_2) \vee (p_2 \wedge p_1) \\ &= [(p_1 \vee (p_2 \wedge p_1)] \wedge [(p_2 \vee (p_2 \wedge p_1)] \\ &= p_1 \wedge p_2. \end{aligned}$$

$(5) \Rightarrow (1)$: Assume (5). Then $(p_1 \wedge p_2) \vee p_2 = (p_2 \wedge p_1) \vee p_2 = p_1$.
$(5) \Rightarrow (3)$: Assume (5). Then $(p_2 \wedge p_1) \vee p_2 = (p_1 \wedge p_2) \vee p_2 = p_2$.
$(3) \Rightarrow (5)$: Assume (3). Then

$$\begin{aligned} p_1 \wedge p_2 &= [(p_2 \wedge p_1) \vee p_1] \wedge [(p_2 \wedge p_1) \vee p_2] \\ &= [p_2 \wedge p_1) \wedge [(p_2 \wedge p_1) \vee p_2]] \vee [p_1 \wedge [(p_2 \wedge p_1) \vee p_2)] \\ &= (p_2 \wedge p_1) \vee (p_1 \wedge p_2) \\ &= [p_2 \vee (p_1 \wedge p_2)] \wedge [p_1 \vee (p_1 \wedge p_2)] \\ &= p_2 \wedge p_1. \end{aligned}$$

$(6) \Rightarrow (3)$: Assume (6). Then $p_1 \wedge (p_2 \vee p_1) = p_1 \wedge (p_1 \vee p_2) = p_1$.
$(3) \Rightarrow (6)$: Assume (3). Then

$$\begin{aligned} p_2 \vee p_1 &= p_2 \vee ((p_1 \vee p_2) \wedge p_1) \\ &= (p_2 \vee (p_1 \vee p_2)) \wedge (p_2 \vee p_1) \\ &= (p_1 \vee p_2) \wedge (p_2 \vee p_1) \\ &= (p_1 \vee p_2) \wedge (p_1 \vee (p_2 \vee p_1)) \\ &= p_1 \vee (p_2 \wedge (p_2 \vee p_1)) \\ &= p_1 \vee p_2. \end{aligned}$$

Now we prove the equivalence of (5), (7), (8) and (9).
$(5) \Rightarrow (7)$: Assume (5). Then $p_1 \wedge p_2 = p_2 \wedge p_1 \leq p_1$. Hence (7) follows.
$(7) \Rightarrow (8)$: Assume (7). Then $p_1 \wedge p_2 \leq p_1$ and let $a = p_1 \wedge p_2$. Therefore $a \leq p_1$ and $a \leq p_2$.
$(8) \Rightarrow (9)$: Assume (8). Then there exists $a \in V$ such that $a \leq p_1$ and $a \leq p_2$.
Now consider

$$\begin{aligned} (p_1 \wedge p_2) \vee p_1 &= (p_1 \vee p_1) \wedge (p_2 \vee p_1) \\ &= p_1 \wedge (p_2 \vee p_1) \\ &= (a \vee p_1) \wedge (p_2 \vee p_1) \\ &= (a \wedge p_2) \vee p_1 \\ &= a \vee p_1 \\ &= p_1 \end{aligned}$$

Therefore $p_1 \wedge p_2$ is the lower bound of p_1 and p_2 in V. Now, for $c \in V$ such that $c \leq p_1$ and $c \leq p_2$, we have $c \vee (p_1 \wedge p_2) = (c \vee p_1) \wedge (c \vee p_2) = p_1 \wedge p_2$, implies $c \leq p_1 \wedge p_2$. Thus

$p_1 \wedge p_2$ is the g.l.b of p_1 and p_2 in V.
(9) $\Rightarrow$ (5): Assume (9). Then $(p_1 \wedge p_2) \vee p_1 = p_1$, since $p_1 \wedge p_2 \leq p_1$ and hence (5) follows by the equivalence of (1) and (5). Thus the conditions (5), (7), (8) and (9) are equivalent.

Theorem 3. *For any $p_1, p_2, p_3 \in V$, $p_1 \vee (p_2 \wedge p_3) = (p_1 \vee p_2) \wedge (p_3 \vee p_1)$ if and only if $p_1 \wedge p_2 = p_2 \wedge p_1$.*

Proof. Let $p_1, p_2, p_3 \in V$ be such that $p_1 \vee (p_2 \wedge p_3) = (p_1 \vee p_2) \wedge (p_3 \vee p_1)$. Then

$$\begin{aligned} p_2 \wedge p_1 &= ((p_1 \wedge p_2) \vee p_2) \wedge ((p_1 \vee (p_1 \wedge p_2)) \\ &= (p_1 \wedge p_2) \vee (p_2 \wedge p_1) \\ &= (p_1 \vee (p_2 \wedge p_1)) \wedge (p_2 \vee (p_2 \wedge p_1)) \\ &= p_1 \wedge p_2. \end{aligned}$$

Converse follows from Theorem 2 and $(LD\vee)$.

Lemma 8. *For any $p_1, p_2, p_3 \in V$, $p_1 \vee (p_2 \wedge p_3) = p_1 \vee (p_3 \wedge p_2)$.*

Proof. Since $p_1 \leq p_1 \vee p_2$ and $p_1 \leq p_1 \vee p_3$, we have $(p_1 \vee p_2) \wedge (p_1 \vee p_3) = (p_1 \vee p_3) \wedge (p_1 \vee p_2)$. Therefore

$$p_1 \vee (p_2 \wedge p_3) = (p_1 \vee p_2) \wedge (p_1 \vee p_3) = (p_1 \vee p_3) \wedge (p_1 \vee p_2) = p_1 \vee (p_3 \wedge p_2).$$

Theorem 4. *The operation $\vee$ is associative in a PDL V.*

Proof. Let $p_1, p_2, p_3 \in V$. Then

$$\begin{aligned} p_1 \vee (p_2 \vee p_3) &= [p_1 \vee (p_3 \wedge p_1)] \vee (p_2 \vee p_3) \\ &= [((p_1 \vee p_2) \wedge p_1) \vee (p_3 \wedge p_1)] \vee (p_2 \vee p_3) \\ &= [[(p_1 \vee p_2) \vee p_3] \wedge p_1] \vee (p_2 \vee p_3) \\ &= [[(p_1 \vee p_2) \vee p_3] \vee (p_2 \vee p_3)] \wedge [p_1 \vee (p_2 \vee p_3)] \\ &= ((p_1 \vee p_2) \vee p_3) \wedge (p_1 \vee (p_2 \vee p_3)) \\ &= ((p_1 \vee p_2) \wedge (p_1 \vee (p_2 \vee p_3))) \vee (p_3 \wedge (p_1 \vee (p_2 \vee p_3))) \\ &= (p_1 \vee p_2) \vee ((p_1 \vee (p_2 \vee p_3)) \wedge p_3) \\ &= (p_1 \vee p_2) \vee [(p_1 \wedge p_3) \vee ((p_2 \vee p_3) \wedge p_3)] \\ &= (p_1 \vee p_2) \vee [(p_1 \wedge p_3) \vee p_3] \\ &= (p_1 \vee p_2) \vee p_3 \end{aligned}$$

Therefore $\vee$ is associative in V.

"Note that, since $\vee$ is associative in a PDL, we can write $(p_1 \vee p_2) \vee p_3$, simply, as $p_1 \vee p_2 \vee p_3$. Hence the notation $(\bigvee_{i=1}^{n} \alpha_i)$ is meaningful, as well".
The operation $\wedge$ is not associative, as shown by the following example:

Example 4. *Let* $V = \{0, 1, 2, 3, 4, 5\}$ *be a set with binary operations* $\vee$ *and* $\wedge$ *given in the following tables:*

$\vee$	0	1	2	3	4	5
0	0	1	4	0	4	0
1	1	1	1	1	1	1
2	2	1	2	2	2	2
3	3	1	2	3	2	3
4	4	1	4	4	4	4
5	5	1	4	5	4	5

$\wedge$	0	1	2	3	4	5
0	0	0	3	3	0	5
1	0	1	2	3	4	5
2	0	2	2	3	4	5
3	0	3	3	3	5	5
4	0	4	2	3	4	5
5	0	5	3	3	5	5

Then $(V; \vee \wedge 1)$ *is a Paradistributive Latticoid. But the operation* $\wedge$ *is not associative, since* $0 \wedge (2 \wedge 4) = 0 \wedge 4 = 0 \neq 5 = 3 \wedge 4 = (0 \wedge 2) \wedge 4$.

Definition 2. *A Paradistributive Latticoid* $(V, \vee, \wedge, 1)$ *is said to be associative if it satisfies the following condition*

$$p_1 \wedge (p_2 \wedge p_3) = (p_1 \wedge p_2) \wedge p_3$$

for all $p_1, p_2, p_3 \in V$.

Example 5. *Let* $V = \{0, 1, 2, 3, 4\}$ *be a set with binary operations* $\vee$ *and* $\wedge$ *given in the following tables:*

$\vee$	0	1	2	3	4
0	0	1	0	3	3
1	1	1	1	1	1
2	2	1	2	4	4
3	3	1	3	3	3
4	4	1	4	4	4

$\wedge$	0	1	2	3	4
0	0	0	2	0	2
1	0	1	2	3	4
2	0	2	2	0	2
3	0	3	2	3	4
4	0	4	2	3	4

Then $(V; \vee \wedge 1)$ *is an associative Paradistributive Latticoid.*

Lemma 9. *For any* $p_1, p_2, p_3 \in V$, $p_1 \vee p_2 \vee p_3 = p_1 \vee p_3 \vee p_2$.

Proof. Since $p_1 \leq p_1 \vee p_2$ and $p_1 \leq p_1 \vee p_3$, we have

$$(p_1 \vee p_2) \vee (p_1 \vee p_3) = (p_1 \vee p_3) \vee (p_1 \vee p_2).$$

Therefore $(p_1 \vee p_2 \vee p_1) \vee p_3 = (p_1 \vee p_3 \vee p_1) \vee p_2$. Hence $p_1 \vee p_2 \vee p_3 = p_1 \vee p_3 \vee p_2$.

Lemma 10. *For any* $p_1, p_2 \in V$, $p_1 \vee p_2 = 1$ *if and only if* $p_2 \vee p_1 = 1$.

Theorem 5. *Let* $(V, \vee, \wedge, 1)$ *be a PDL. Then the following are equivalent:*
(1) $(V, \vee, \wedge, 1)$ *is a distributive lattice.*
(2) *The poset* $(V, \leq)$ *is directed below.*
(3) $(p_1 \wedge p_2) \vee p_1 = p_1$.
(4) *The operation* $\wedge$ *is commutative.*
(5) *The operation* $\vee$ *is commutative.*
(6) *The relation* $\chi = \{(p_1, p_2) \in V \times V \mid p_2 \vee p_1 = p_2\}$ *is antisymmetric.*

Proof. (1) $\Rightarrow$ (2): Let $(V, \vee, \wedge, 1)$ be a distributive lattice. Then, by Theorem 2, V is directed below. The equivalence of $(2), (3), (4), (5)$ also follows from Theorem 2.
(5) $\Rightarrow$ (6): Given $\chi = \{(p_1, p_2) \in V \times V \mid p_2 \vee p_1 = p_2\}$. If $(p_1, p_2) \in \chi$, then $p_2 \vee p_1 = p_1$ and $(p_2, p_1) \in \chi$ and hence $p_1 = p_2$.
(6) $\Rightarrow$ (1): Let $p_1, p_2 \in V$. Then $(p_1 \vee p_2) \vee (p_2 \vee p_1) = p_1 \vee p_2 \vee p_1 = p_1 \vee p_1 \vee p_2 = p_1 \vee p_2$ and hence $(p_1 \vee p_2, p_2 \vee p_1) \in \chi$. Also, $(p_2 \vee p_1) \vee (p_1 \vee p_2) = p_2 \vee p_1 \vee p_2 = p_2 \vee p_2 \vee p_1 = p_2 \vee p_1$ which shows $(p_2 \vee p_1, p_1 \vee p_2) \in \chi$. Since χ is antisymmetric, we have $p_1 \vee p_2 = p_2 \vee p_1$. So that V is a lattice and hence distributive.

Theorem 6. *An algebra* $(V, \vee, \wedge, 1)$ *of type* $(2, 2, 0)$ *is a PDL if and only if it satisfies the following:*
$(LD\vee)$ $p_1 \vee (p_2 \wedge p_3) = (p_1 \vee p_2) \wedge (p_1 \vee p_3)$
$(RD\vee)$ $(p_1 \wedge p_2) \vee p_3 = (p_1 \vee p_3) \wedge (p_2 \vee p_3)$
$(RD\wedge)$ $(p_1 \vee p_2) \wedge p_3 = (p_1 \wedge p_3) \vee (p_2 \wedge p_3)$
(L_1) $(p_1 \vee p_2) \wedge p_2 = p_2$
(L_3) $p_1 \vee (p_1 \wedge p_2) = p_1$
(I_1) $p_1 \vee 1 = 1$
(I_2) $1 \wedge p_1 = p_1$.
for all $p_1, p_2, p_3 \in V$.

Corollary 3. *For all* $a \in V$, V *contains an element* 0 *such that* $0 \vee a = a$ *if and only if* V *is a bounded distributive lattice, and hence for any* $a \in V$, *the set*

$$V_a = \{a \leq p_1 \mid p_1 \in V\}$$

is a bounded distributive lattice under the induced operations $\vee$ *and* $\wedge$ *with* a *as its least element.*

3. Ideals and Filters

In this section, we introduce the notion of an ideal and a filter in a paradistributive latticoid and investigate its important properties.

Definition 4. *A non-empty subset* U *of* V *is said to be an ideal if it satisfies the following:*

$$p_1, p_2 \in U \Rightarrow p_1 \vee p_2 \in U.$$
$$p_1 \in U, a \in V \Rightarrow p_1 \wedge a \in U.$$

It should be noted that every ideal of V is a PDL. Next theorem describes the ideal generated by a non-empty subset S of V.

Theorem 7. *Let* S *be a non-empty subset of* V. *Then*

$$(S] = \{(\bigvee_{i=1}^{n} \alpha_i) \wedge s \mid \alpha_i \in S, s \in V \text{ and } n \text{ is a positive integer } \}$$

is the smallest ideal of V *containing* S.

Proof. Let S be a non-empty subset of V.
Choose

$$T = \{\bigvee_{i=1}^{n} \alpha_i \mid \alpha_i \in S \text{ for } 1 \leq i \leq n \text{ and n is a positive integer } \}$$

Clearly $S \subseteq T \subseteq (S]$. First we prove that

$$(S] = \{p_1 \in V \mid t \vee p_1 = t \text{ for some } t \in T\} = M.$$

Let $x \in (S]$. Then

$$x = (\bigvee_{i=1}^{n} \alpha_i) \wedge s$$

where $\alpha_i \in S$, $s \in V$.
Now

$$\begin{aligned}(\bigvee_{i=1}^{n} \alpha_i) \vee x &= (\bigvee_{i=1}^{n} \alpha_i) \vee ((\bigvee_{i=1}^{n} \alpha_i) \wedge s). \\ &= \bigvee_{i=1}^{n} \alpha_i\end{aligned}$$

which implies $x \in M$. Therefore $(S] \subseteq M$. Conversely, let $s \in M$. Then $t \vee s = t$ for some $t = \bigvee_{i=1}^{n} \alpha_i$, $\alpha_i \in S$. Now $s = t \wedge s = (\bigvee_{i=1}^{n} \alpha_i) \wedge s \in (S]$. Therefore $M \subseteq (S]$. Hence $(S] = \{s \in V \mid t \vee s = t \text{ for some } t \in T\} = M$.
Let $s, l \in (S]$, then there exists $t_1, t_2 \in T$ such that

$$t_1 \vee s = t_1 \text{ and } t_2 \vee l = t_2.$$

Then $(S]$ is an ideal as

$$\begin{aligned}(t_1 \vee t_2) \vee (s \vee l) &= t_1 \vee (t_2 \vee (s \vee l)) \\ &= t_1 \vee (t_2 \vee l \vee s) \\ &= t_1 \vee t_2 \vee s \\ &= t_1 \vee s \vee t_2 \\ &= t_1 \vee t_2\end{aligned}$$

Therefore $s \vee l \in (S]$. Also, for $s \in (S]$ and $u \in V$, we have $t \in T$ such that $t \vee s = t$. Now $t \vee (s \wedge u) = (t \vee s) \wedge (t \vee u) = t \wedge (t \vee u) = t$. Hence $s \wedge u \in (S]$.
Thus $(S]$ is an ideal of V containing S. Now, let U be any ideal of V such that $S \subseteq U$.
Let $s \in (S]$. Then

$$s = (\bigvee_{i=1}^{n} \alpha_i) \wedge p_2$$

where $\alpha_i \in S \subseteq U$ for $1 \leq i \leq n$ and $p_2 \in V$. Since U is an ideal of V, we have

$$s = (\bigvee_{i=1}^{n} \alpha_i) \wedge p_2 \in U.$$

Hence $(S] \subseteq U$. Therefore $(S]$ is the smallest ideal of V containing S.

Note that if $S = \{a\}$, then we write $(S] = (a]$, the principal ideal of V generated by 'a'.

Corollary 5. *$p_1 \in (p_2]$ if and only if $p_2 \wedge p_1 = p_1$, where $p_1, p_2 \in V$.*

Lemma 11. *Let U be an ideal of V. Then, for any $p_1, p_2 \in V$, $p_1 \wedge p_2 \in U$ if and only if $p_2 \wedge p_1 \in U$.*

Corollary 6. *For any $p_1, p_2 \in V$*

$$\begin{aligned} (p_1] \vee (p_2] &= (p_1 \vee p_2] = (p_2 \vee p_1] \\ (p_1] \wedge (p_2] &= (p_1 \wedge p_2] = (p_2 \wedge p_1] \end{aligned}$$

Theorem 8. *The set $P(V)$ of all ideals of V forms a distributive lattice with greatest element under the set inclusion in which the g.l.b and l.u.b for any P and Q are respectively, $P \wedge Q = P \cap Q$ and $P \vee Q = \{p_1 \vee p_2 \mid p_1 \in P, p_2 \in Q\}$.*

Definition 7. *A non-empty subset F of V is said to be a filter if it satisfies the following:*

$$p_1, p_2 \in F \Rightarrow p_1 \wedge p_2 \in F.$$
$$p_1 \in F, a \in V \Rightarrow a \vee p_1 \in F.$$

Theorem 9. *Let S be a non-empty subset of V. Then*

$$[S) = \{p_1 \vee (\bigwedge_{i=1}^{n} s_i) \mid s_i \in S, p_1 \in V, 1 \leq i \leq n \text{ and } n \text{ is a positive integer } \}$$

is the smallest filter of V containing S.

Proof. Let $a, b \in [S)$. Then $a = p_1 \vee (\bigwedge_{i=1}^{n} s_i)$, $b = p_2 \vee (\bigwedge_{i=1}^{m} t_j)$

$$\begin{aligned} a \wedge b &= [p_1 \vee (\bigwedge_{i=1}^{n} s_i)] \wedge [p_2 \vee (\bigwedge_{j=1}^{m} t_j)] \\ &= [p_1 \wedge (p_2 \vee (\bigwedge_{j=1}^{m} t_j))] \vee [(\bigwedge_{i=1}^{n} s_i) \wedge (p_2 \vee (\bigwedge_{j=1}^{m} t_j))] \\ &= [p_1 \wedge (p_2 \vee (\bigwedge_{j=1}^{m} t_j))] \vee [(p_2 \vee (\bigwedge_{j=1}^{m} t_j)) \wedge (\bigwedge_{i=1}^{n} s_i)] \\ &= p_3 \vee (\bigwedge_{j=1}^{m} t_j \wedge \bigwedge_{i=1}^{n} s_i) \end{aligned}$$

(where $p_3 = (p_1 \wedge (p_2 \vee (\bigwedge_{j=1}^{m} t_j))) \vee (p_2 \wedge (\bigwedge_{i=1}^{n} s_i))$ and hence $a \wedge b \in [S)$.
Now, we prove $u \vee a \in [S)$ for $u \in V$.
Consider

$$u \vee a = u \vee (p_1 \vee (\bigwedge_{i=1}^{n} s_i)) = (u \vee p_1) \vee (\bigwedge_{i=1}^{n} s_i) \in [S)$$

Therefore $[S)$ is a filter of V, and clearly it is the smallest filter of V containing S.

Note that if $S = \{a\}$, then we write $[S) = [a)$, the principal filter of V generated by 'a'.

Corollary 8. *$p_1 \in [p_2)$ if and only if $p_1 = p_1 \vee p_2$ for all $p_1, p_2 \in V$.*

Lemma 12. *Let F be a filter of V and $p_1, p_2 \in V$. Then*
(1) $p_1 \vee p_2 \in F$ *if and only if* $p_2 \vee p_1 \in F$.
(2) *For any* $p_1, p_2 \in V$, $[p_1 \vee p_2) = [p_2 \vee p_1)$.
(3) *For any* $p_1, p_2 \in V$, $[p_1 \wedge p_2) = [p_2 \wedge p_1) = [p_1) \vee [p_2)$.

Theorem 10. *Let $p_1, p_2 \in V$. Then the following are equivalent.*
(1) $(p_1] \subseteq (p_2]$.
(2) $p_2 \wedge p_1 = p_1$.
(3) $p_2 \vee p_1 = p_2$.
(4) $[p_2) \subseteq [p_1)$.

Theorem 11. *The collection $F(V)$ of all filters of a PDL V forms a distributive lattice under set inclusion, in which,the glb and lub of any F and G are given respectively by* $F \wedge G = F \cap G$ *and* $F \vee G = \{p_1 \wedge p_2 \mid p_1 \in F \text{ and } p_2 \in G\}$.

Theorem 12. *The class $PU(V)(PF(V))$ of all principal ideals(filters) of V is a sublattice of the distributive lattice $U(V)(F(V))$ of all the ideals(filters) of V. Moreover, the lattice $PU(V)$ is "dually isomorphic" on to the lattice $PF(V)$.*

Proof. Let $PU(V)$ be the set of all principal ideals of the PDL V and $PU(V) = \{(a] \mid a \in V\}$.
First, we prove that $PU(V)$ is a sublattice of $U(V)$. Let $(a], (b] \in PU(V)$. Then $(a] \vee (b] = (a \vee b]$ for $a, b \in V$. Hence $(a] \vee (b] \in PU(V)$. Similarly, $(a] \wedge (b] = (a \wedge b] \in PU(V)$. Therefore $PU(V)$ is a sublattice of $U(V)$.
Finally, we prove that there exists a dual isomorphism from $PU(V)$ to $PF(V)$.
Define $\zeta : PU(V) \to PF(V)$ by $\zeta\{(a]\} = [a)$, $a \in V$.

(i) ζ is a homomorphism:

$$\begin{aligned} \zeta\{(a] \vee (b]\} &= \zeta\{(a \vee b]\} \\ &= [a \vee b) \\ &= [a) \wedge [b) \\ &= \zeta\{(a]\} \wedge \zeta\{(b]\} \end{aligned}$$

also,

$$\begin{aligned} \zeta\{(a] \wedge (b]\} &= \zeta\{(a \wedge b]\} \\ &= [a \wedge b) \\ &= [a) \vee [b) \\ &= \zeta\{(a]\} \vee \zeta\{(b]\} \end{aligned}$$

(ii) ζ is one-one:
Let $a, b \in V$. Then

$$\begin{aligned} \zeta\{(a]\} &= \zeta\{(b]\} \\ \Rightarrow [a) &= [b) \\ \Rightarrow (a] &= (b] \end{aligned}$$

(iii) ζ is onto: Let $b \in PF(V)$. Then $b = [a)$ for some $a \in V$. Hence $(a] \in PU(V)$. Therefore $\zeta((a]) = [a) = b$.
Therefore there exists a dual isomorphism from $PU(V)$ onto $PF(V)$.

4. Subdirectly Irreducible PDLs

For any algebra A, we denote the structure lattice of A, that is the lattice of all congruence relations on A, by $\mathbf{B}$ in which the least element is Δ_A where $\Delta_A = \{(x,y) \in A \times A \mid x = y\}$ and greatest element is $A \times A$. Recall that a non-trivial algebra A is said to be subdirectly irreducible if intersection of any family of nonzero congruences is again non-zero; or equivalently, $\mathbf{B}$ has smallest non-zero congruence. We characterize subdirectly irreducible associative PDLs in this section, and then use Birkhoff's subdirect representation theorem to obtain a subdirect representation for an associative PDL. This subdirect representation of a PDL V is crucial in the theory of PDLs since it simplifies numerous lattice theoretic computations.

Definition 9. *Let V be a PDL, an element $a \in V$ is said to be minimal if for any $u \in V$, $u \leq a \Rightarrow u = a$.*

Lemma 13. *Let V be a PDL. Then for any $a \in V$, the following are equivalent:*
(1). *a is minimal*
(2). *$p_1 \wedge a = a$ for all $p_1 \in V$*
(3). *$p_1 \vee a = p_1$ for all $p_1 \in V$.*

Definition 10. *An equivalence relation θ on a PDL V, is called a congruence relation on V if $(a \wedge c, b \wedge d), (a \vee c, b \vee d) \in \theta$, for all $(a,b), (c,d) \in \theta$.*

Lemma 14. *For any $a \in V$, $\varphi^a = \{(p_1,p_2) \in V \times V \mid p_1 \vee a = p_2 \vee a\}$ is a congruence relation on V. Further, $\varphi^a = \Delta_V$ if and only if a is minimal element of V.*

Proof. Given $\varphi^a = \{(p_1,p_2) \in V \times V \mid p_1 \vee a = p_2 \vee a\}$.
Clearly φ^a is an equivalence relation on V. Let $(p_1,p_2), (p_3,s) \in \varphi^a$. Then $p_1 \vee a = p_2 \vee a$ and $p_3 \vee a = s \vee a$. Hence

$$\begin{aligned} (p_1 \vee p_3) \vee a &= p_1 \vee p_3 \vee a \\ &= p_1 \vee s \vee a \\ &= p_1 \vee a \vee s \\ &= p_2 \vee a \vee s \\ &= p_2 \vee s \vee a \end{aligned}$$

and

$$\begin{aligned} (p_1 \wedge p_3) \vee a &= (p_1 \vee a) \wedge (p_3 \vee a) \\ &= (p_2 \vee a) \wedge (s \vee a) \\ &= (p_2 \wedge s) \vee a. \end{aligned}$$

Therefore $(p_1 \vee p_3, p_2 \vee s), \in \varphi^a$ and $(p_1 \wedge p_3, p_2 \wedge s) \in \varphi^a$.
Therefore φ^a is a congruence relation on V. Assume $\varphi^a = \Delta_V$. Let $p_1 \in V$ be such that $p_1 \leq a$. Then $a \vee a = a = p_1 \vee a$ which implies $(a, p_1) \in \varphi^a = \Delta_V$. Therefore $a = p_1$. Hence a is minimal element. Let $(p_1, p_2) \in \varphi^a$. Then $p_1 \vee a = p_2 \vee a$. Since a is minimal element, we have $p_1 = p_2$. Hence $\varphi^a = \Delta_V$.

Lemma 15. *If V is an associative PDL, then for any $a \in V$,*
$\theta_a = \{(p_1, p_2) \in V \times V \mid p_1 \wedge a = p_2 \wedge a\}$ is a congruence relation on V. Further, $\theta_a = \Delta_V$ if and only if a is unity (greatest) element of V.

Proof. Let $\theta_a = \{(p_1, p_2) \in V \times V \mid p_1 \wedge a = p_2 \wedge a\}$.
Clearly θ_a is an equivalence relation on V.
Let $(p_1, p_2) \in \theta_a$ and $(p_3, t) \in \theta_a$. Then $p_1 \wedge a = p_2 \wedge a$ and $p_3 \wedge a = t \wedge a$. Hence

$$\begin{aligned}(p_1 \wedge p_3) \wedge a &= p_1 \wedge p_3 \wedge a \\ &= p_1 \wedge t \wedge a \\ &= p_1 \wedge a \wedge t \wedge a \\ &= p_2 \wedge a \wedge t \wedge a \\ &= p_2 \wedge t \wedge a \\ &= (p_2 \wedge t) \wedge a\end{aligned}$$

and

$$\begin{aligned}(p_1 \vee p_3) \wedge a &= (p_1 \wedge a) \vee (p_3 \wedge a) \\ &= (p_2 \wedge a) \vee (t \wedge a) \\ &= (p_2 \vee t) \wedge a.\end{aligned}$$

Therefore $(p_1 \wedge p_3, p_2 \wedge t), (p_1 \vee p_3, p_2 \vee t) \in \theta_a$. Hence θ_a is a congruence relation on V. Now assume $\theta_a = \Delta_V$. Then, for any $p_1 \in V$, $a \wedge a = a = (p_1 \vee a) \wedge a$ implies $(a, p_1 \vee a) \in \theta_a = \Delta_V$ which shows that a is unity element of V.
Conversely, suppose that a is unity element of V. Then, for $(p_1, p_2) \in \theta_a$, we have $p_1 \wedge a = p_2 \wedge a$ which implies $p_1 = p_2$. Therefore $\theta_a = \Delta_V$, which concludes the lemma.

Theorem 13. *Let V be a subdirectly irreducible associative PDL. Then every non-unity of V is minimal and V contains atmost two non-unity elements.*

Proof. Let V be an associative PDL. Suppose V is subdirectly irreducible.
Let θ_1 be the smallest non-zero congruence on V. Select $p_1, p_2 \in V$ with $p_1 \neq p_2$ such that $(p_1, p_2) \in \theta_1$. Then atleast one of p_1 and p_2 is minimal. For if, assume that p_1 and p_2 both are not minimal elements of V, then $\varphi^{p_1} \neq \Delta_V \neq \varphi^{p_2}$, implies $(p_1, p_2) \in \varphi^{p_1} \cap \varphi^{p_2}$. Since $p_1 = p_1 \vee p_1 = p_2 \vee p_1$ and $p_2 = p_1 \vee p_2$. Thus $p_1 = p_2$, which is a contradiction. Thus, atleast one of p_1, p_2 is minimal. Without loss of generality, let us assume that p_1 is minimal. Let a be a non-unity element of V. Suppose, a is not minimal, then p_1 being the minimal element we have $a \wedge p_1 = p_1$. Hence $a \vee p_1 = a \vee (a \wedge p_1) = a$, so that $p_1 \vee a$ is a non-unity element of V. Therefore $\theta_{p_1 \vee a} \neq \Delta_V$. Hence $(p_1, p_2) \in \theta_{p_1 \vee a}$. Also, since a is not minimal, we obtain $(p_1, p_2) \in \varphi^a$. Now $p_1 = p_1 \wedge (p_1 \vee a) = p_2 \wedge (p_1 \vee a) = p_2 \wedge (p_2 \vee a) = p_2$ which

is a contradiction. Thus a is minimal. Suppose $a, b, c \in V$ be three distinct non-unity elements of V. Then since a, b, c are minimal elements, we have $\varphi = \Delta_V \cup \{(a, b), (b, a)\}$ and $\psi = \Delta_V \cup \{(b, c), (c, b)\}$ are two non-zero congruences on V such that $\varphi \cap \psi = \Delta_V$ which contradicts the fact that V is subdirectly irreducible. Hence V has atmost two non-unity elements.

Corollary 11. *A subdirectly irreducible distributive lattice V is a two element chain.*

Remark: It is evident that the converse of Theorem 13 is also true. i.e., if V is a PDL containing atmost two non-unity elements and every non-unity element of V is minimal.

Theorem 14. *Let V be a PDL. Then the following are equivalent:*
(1) *V is associative.*
(2) *θ_a is a congruence relation on V for all $a \in V$.*
(3) *V is a subdirect product of PDLs in each of which there are atmost two non-unity elements and every non-unity element is minimal.*

Proof. Let V be a PDL. We have (1) $\Rightarrow$ (2) by Lemma 15. Assume (2). Then, for any congruence η on V, p_1, p_2 and $p_3 \in V$ we have $(p_1 \wedge p_3, p_2 \wedge p_3) \in \eta$ if and only if $(p_1, p_2) \in \theta_{p_3} \wedge \eta$. Hence for any congruence relation η on V, the quotient PDL $V|\eta$ also satisfies the assumption (2). By Birkhoff's theorem and Theorem 13, we have (3). (3) $\Rightarrow$ (1) is clear.

5. Conclusions

In this paper, we have introduced the concept of Paradistributive Latticoid and studied certain properties related to the structure. Further, provided a set of equivalence conditions for the PDL to become a distributive lattice. We have introduced the notions of an ideal and a filter in a PDL and studied their properties. We have obtained subdirect representation of a PDL. In future, our work will focus on parapseudo-complementation on a PDL, Stone PDL, Normal PDL and study their topological properties.

Conflicts of interest or competing interests

The authors declare that they have no conflicts of interest.

Data and code Availability

No data were used to support this study

Supplementary information

Not Applicable

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors

Informed Consent

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