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A Novel Approach for Studying Pythagorean Triples Suitable for Students at all Educational Levels

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Abstract. This paper seeks to showcase how a new approach can breathe new life into research within the traditional domain of Pythagorean triples, introducing innovative applications to invigorate the field. This serves not only as an exemplar but also as a wellspring of inspiration for students at both school and university levels. The demonstrations will underscore that, with fundamental mathematical concepts and unencumbered by intricate calculations, one can unveil novel results and applications with ease. The new results and applications, along with those found in the Preliminary Results section, show how the field of Pythagorean triples is still interesting and stimulating to study, despite the centuries that have elapsed.

2020 Mathematics Subject Classifications: 11D61, 97F99, 97G99, 97H99, 97I99

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1. Introduction

Let x, y, and z be positive integers satisfying

$$x^2 + y^2 = z^2.$$

Such a triple (x, y, z) is called a Pythagorean triple. In particular, if x, y, and z are coprime, the triple is termed a primitive Pythagorean triple.

Pythagorean triples owe their name to the Greek mathematician Pythagoras, who lived in the 6th century B.C. Pythagoras was the founder of the philosophical school known as Pythagoreanism, and Pythagorean triples are often associated with his discoveries and teachings.

According to legend, Pythagoras and his followers became interested in Pythagorean triples while studying numbers and musical proportions. It is said that they noticed certain combinations of lengths of musical strings produced harmonic sounds, and these combinations corresponded to Pythagorean triples. However, it's important to note that Pythagorean triples were not discovered or introduced by Pythagoras himself. Ancient

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Babylonian mathematicians were already aware of some Pythagorean triples long before Pythagoreanism became famous. The earliest known record of the theorem is found in an ancient Babylonian manuscript called Plimpton 322, dating back about 1,800 years before Pythagoras. This text provides a list of Pythagorean triples and demonstrates the Babylonians' knowledge of Pythagorean formulas and how to use them. Nevertheless, it was Pythagoras and his school that placed a particular emphasis on these triples and discovered some of their interesting properties. Pythagorean triples were extensively studied by the Pythagoreans and the subsequent Pythagorean school. They sought to find all possible Pythagorean triples and developed methods to generate new triples. The theory of Pythagorean triples was further developed by mathematicians from various cultures throughout history. For instance, ancient Indians, Chinese, Arabs, and Europeans contributed to further developments in the theory of Pythagorean triples. Over time, numerous Pythagorean triples with increasingly large values for their components have been discovered.

The most famous discovery attributed to Pythagoras regarding Pythagorean triples is the Pythagorean theorem. This theorem is fundamental in geometry and has many practical applications. However, Pythagoras himself may not have provided a general proof of the theorem, and many of the proofs attributed to him may have been developed later by his followers. Pythagorean triples, due to their connection with the Pythagorean theorem, are a fundamental concept with multiple practical applications in different fields of mathematics and applied sciences.

Pythagorean triples continued to be studied and explored by numerous later mathematicians, such as Euclid and Diophantus. Over the centuries, many interesting properties and relationships regarding Pythagorean triples have been discovered, and their importance has extended to various mathematical fields, including number theory, modular arithmetic, geometry, and graph theory.

Pythagorean triples and the Pythagorean theorem have had a lasting impact on mathematics and geometry and have been extensively studied and applied over the centuries, influencing disciplines ranging from trigonometry to mathematical analysis.

Today, Pythagorean triples continue to be a subject of study and appreciation for their unique properties. They are used in various mathematical contexts and practical applications, such as cryptography, random number generation, algorithm design, solving problems involving right-angled triangles, number theory, and many other fields such as physics, engineering, and computer science [11], [12].

Beyond mathematics, Pythagorean triples have been of interest in the history of art and culture, appearing in various forms in architecture, music, and symbolism.

Pythagorean triples have also played a significant role in number theory, with mathematicians like Fermat and Euclid studying their properties. For example, a common formula to generate Pythagorean triples is the following:

$$x = m^2 - n^2, \quad y = 2mn, \quad z = m^2 + n^2$$

where m and n are arbitrary positive integers with $m > n$, and $m, n \in \mathbb{N}$ [10]. The above formula mentioned earlier, involving the numbers m and n, was developed by the Euclid, and the majority of results on Pythagorean triples are due to this formula [9], [8], [1].

In addition to the scientific aspects of Pythagorean triples, we also underscore several valuable didactic perspectives that enrich the learning experience:

Pythagorean triples serve as a tangible gateway to the realm of number theory. By actively engaging with these triples, students can develop a hands-on understanding of fundamental concepts in number theory, such as divisibility, prime factorization, and the properties of integers.

Delving into Pythagorean triples provides an opportunity to reinforce the geometric interpretation of the Pythagorean theorem. Through visualizing right-angled triangles and their associated triples, students gain insights into the geometric relationships embedded in the theorem, fostering a deeper comprehension of its principles.

The exploration of Pythagorean triples naturally encourages students to recognize patterns within numerical relationships. Analyzing these triples prompts discussions about the symmetry inherent in certain configurations, the distinctive roles of odd and even numbers, and the impact of scaling factors on the generation of triples. This process cultivates analytical thinking and the ability to discern mathematical patterns in different contexts.

Working with Pythagorean triples presents students with a variety of mathematical scenarios that require creative problem-solving approaches. As they investigate unique cases and consider different parameterizations, students enhance their problem-solving skills and develop a robust toolkit for addressing mathematical challenges.

Exploring the origins and historical significance of Pythagorean triples provides a broader context for their study. Students can appreciate the cultural contributions of ancient mathematicians like Pythagoras and recognize the enduring legacy of these triples in various mathematical and scientific disciplines.

By embracing these didactic perspectives, the study of Pythagorean triples transcends mere mathematical abstraction, offering a rich and interconnected learning experience that extends beyond the confines of a single theorem. This multifaceted approach not only deepens students' understanding of mathematical concepts but also nurtures a broader appreciation for the historical, cultural, and problem-solving dimensions of mathematics.

Already in 1981, as a student, I had studied how to generate Pythagorean triples, achieving a preliminary result [2]. After many years, returning to study the topic, I found a new and comprehensive result [3] that is suitable for obtaining new results and applications in fields such as geometry, trigonometry, linear algebra, and number theory.

This paper seeks to showcase how a new approach can breathe new life into research within the traditional domain of Pythagorean triples, introducing innovative applications to invigorate the field. This serves not only as an exemplar but also as a wellspring of inspiration for students at both school and university levels. The demonstration will underscore that, with fundamental mathematical concepts and unencumbered by intricate calculations, one can unveil novel results and applications with ease.

2. Prelimunary Results

Let us review recent results concerning some relations among Pythagorean triples that have already been established. The primary tool utilized in those works was the fundamental characterization of Pythagorean triples through a cathetus. This reads as follows.

Theorem 1. *[3] The triple (x, y, z) is a Pythagorean triple if and only if there exists $d \in C(x)$ such that*

$$x = x, \qquad y = \frac{x^2}{2d} - \frac{d}{2}, \qquad z = \frac{x^2}{2d} + \frac{d}{2} \tag{1}$$

with x positive integer, and where

$$C(x) = \begin{cases} D(x), & \text{if } x \text{ is odd,} \\ \\ D(x) \cap P(x), & \text{if } x \text{ is even,} \end{cases}$$

with

$$D(x) = \{d \in \mathbb{N} : d \leq x \text{ with } d \text{ divisor of } x^2\},$$

and if x is even with $x = 2^n k$, $n \in \mathbb{N}$ and $k \geq 1$ odd fixed, with

$$P(x) = \{d \in \mathbb{N} : d = 2^s l \text{ with } l \text{ divisor of } x^2 \text{ and } s \in \{1, 2, ..., 2n-1\}\}.$$

In theorem (1) x is a predetermined integer, which means finding all right triangles whose sides have integer measures and one cathetus is predetermined. Theorem (1) has also one geometrical interpretation. Moreover in [3], based on Theorem (1), we have proved the following theorem.

Theorem 2. *[3] Each $x \in \mathbb{N}$ can be found as cathetus in at least one Pythagorean triple. Every $x \in \mathbb{N}$ can be represented in the form $x = \sqrt{z^2 - y^2}$ with y, $z \in \mathbb{N}$.*

Moreover in [6], an analytic result was found that characterizes primitive Pythagorean triples through a cathetus. This method, which differs from Euler's formulas, offers the advantage of easily identifying all primitive Pythagorean triples $x, y, z \in \mathbb{N}$, where x is a predetermined integer. This reads as follows.

Theorem 3. *[6] Let (x, y, z) be all the Pythagorean triples generated by any predetermined positive integer $x \geq 1$ using (1), $d \in C(x)$, then (x, y, z) is a primitive Pythagorean triple if and only if following both conditions are verified*

$$\text{if } x \text{ is odd then } \begin{cases} d \text{ is a square} \\ \\ \dfrac{x^2}{d} \text{ with } d \text{ are coprime positive odd integers} \end{cases}$$

$$\textit{if } x \textit{ is even then } \begin{cases} \dfrac{d}{2} \textit{ is a square} \\ \\ \dfrac{x^2}{2d} \textit{ with } \dfrac{d}{2} \textit{ are coprime positive integers of different parities.} \end{cases}$$

We remember that the Euclid's formulas do not give all Pythagorean triples that involves a predetermined positive integer x, for example the triples (12, 9, 15), (33, 180, 183) and (33, 44, 55). Moreover it can be laborious to find m and n such that $x = m^2 - n^2$, while using Theorem (1), it is enough to find all the $d \in C(x)$ to obtain all Pythagorean triples.

In particular, if we need to find all primitive Pythagorean triples that involves a predetermined positive integer x, now we can use only the $d \in C(x)$ that satisfy the conditions of the Theorem (3).

In [4], relations were established between the primitive Pythagorean triple (x, y, z) generated by any predetermined positive odd integer x and the primitive Pythagorean triple generated by x^m with $m \in$ N and $m \geq 2$, rispectively, using formulas (1).

Subsequently, additional relations among Pythagorean triples were established in [5]. The primary tool that serves as the foundation of our analysis is Theorem (1) in [3], enabling the determination of relationships between two Pythagorean triples with assigned catheti a and b, and the Pythagorean triple with cathetus a $\cdot$ b. This reads as follows.

Theorem 4. *[5] Let (a, b, c), (d, e, f), (a·d, y, z) be the Pythagorean triples generated by a, d, and a·d, respectively using* (1) *with* $c - b = d_1 \in$ *C(a),* $f - e = d_2 \in$ *C(b), and* $z - y = d_3 \in$ *C(a·d). Then*

$$y = ce + bf, \; z = ce + bf + d_1 d_2,$$

and also

$$y = be + cf - d_1 d_2, \; z = be + cf$$

with $d_3 = d_1 d_2 \in$ *C(a·d).*

Above theorem introduces one suitable binary operation in the set of Pythagorean triples.

In [7], thanks to Theorem (4), we found suitable binary operations on the set of Pythagorean triples, obtaining two commutative infinite groups, one with elements in $\mathbb{Q}$ and the other with elements in $\mathbb{Z}$. Additionally, we obtained a commutative infinite monoid with elements in $\mathbb{N}$ or in $\mathbb{Z}$. In particular, on the set of primitive Pythagorean triples, we established two commutative infinite groups, one with elements in $\mathbb{Q}$ and the other with elements in $\mathbb{Z}$.

All previous results were obtained without advanced techniques and this can be a virtue to reach a wider audience of readers, including students in schools and universities.

3. Applications and Results

In this section, we want to study some applications and results in fields such as geometry, trigonometry, linear algebra and number theory. We will obtain new relations taking into account results seen in Preliminar Results section, and often using only a predeterminatus x and d.

Let's begin to notice that, the formulas of Theorem (1) satisfy the ralation $x^2+y^2=z^2$ for every $x, d \in \mathbb{R}$. It is easy to see that, if $x, d \in \mathbb{R}$ then we obtain all Pythagorean triples in $\mathbb{R}$, that is, also also $y, z \in \mathbb{R}$. Moreover, if $x, d \in \mathbb{R}$ are positive, with $d \leq x$, then also $y, z \in \mathbb{R}$ are positive.

If x is a positive integer and $d \in C(x)$, we want to obtain directly area A, perimeter p and inradius r of all right-angled triangle having only a predeterminatus positive integer cathetus x. We have the following remark.

Remark 1. *In a right-angled triangle, with a predeterminatus cathetus $x \in \mathbb{N}$, we have, regard to area A, perimeter p and inradius r, the following fundamental formulas*

$$A = \frac{x(x^2-d^2)}{4d}, \tag{2}$$

$$p = x + \frac{x^2}{d}, \tag{3}$$

$$r = \frac{x-d}{2}. \tag{4}$$

with $d \in C(x)$.

Formulas (2) and (3) follow directly from (1). To find formula (4), it suffices that we consider the known formula $r = \dfrac{2A}{p}$, obtaining

$$r = \frac{2A}{p} = \frac{\dfrac{2x(x^2-d^2)}{4d}}{x+\dfrac{x^2}{d}} = \frac{x(x^2-d^2)}{2(xd+x^2)} = \frac{x(x-d)(x+d)}{2x(x+d)} = \frac{x-d}{2}.$$

Moreover, if x is a positive integer then $d \in C(x)$,and since x and d have the same parity, we obtain also the known result that r is an integer.

If we have a positive $x \in \mathbb{R}$, (2), (3) and (4) hold, with $d = z - y$. For example, we can find the sides of a right-angled triangle, knowing one cathetus x and the perimeter p. From (3), we have $d = \dfrac{x^2}{p-x}$, that substituted into the formulas of Theorem (1), provides the values of y and z. Other example, we can find the sides of a right-angled triangle, knowing one cathetus x and the inradius r. From (4) we have $d = x - 2r$, that substituted into the formulas of Theorem (1), provides the values of y and z. For both examples, this avoids forming relationships and solving systems of equations using the classical method.

Let the pythagorean triangle ABC be depicted in Figure (1). We consider lines $\overline{OA}, \overline{OB}, \overline{OC}$ from the incentre to the vertices, and $x = \overline{AB}$. The following theorem holds.

Theorem 5. *In a right-angled triangle, with a predeterminatus cathetus $x \in \mathbb{R}$, we have the following relation among lines $\overline{OA}, \overline{OB}, \overline{OC}$ from the incentre to the vertices*

$$\overline{OA} \cdot \overline{OB} = d \cdot \overline{OC} \tag{5}$$

with $\overline{OA}, \overline{OB} < \overline{OC}$, and $d = z - y$.

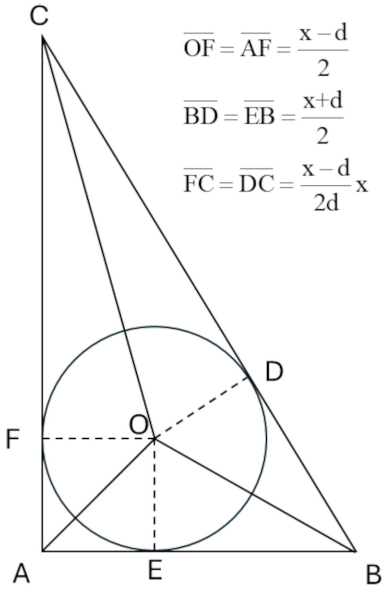

Figure 1:

Proof. Let's begin to notice that $\overline{EB} = \dfrac{x+d}{2}$, while from Theorem 2.1 and (4) we have

$$\overline{FC} = \overline{AC} - r = \frac{x^2 - d^2}{2d} - \frac{x-d}{2} = \frac{x-d}{2d} \cdot x.$$

Applying the Pythagorean theorem, we obtain

$$\overline{OA} = \frac{x-d}{2} \cdot \sqrt{2}\,, \quad \overline{OB} = \frac{\sqrt{x^2+d^2}}{\sqrt{2}}\,, \quad \overline{OC} = \frac{x-d}{2d} \cdot \sqrt{x^2+d^2}$$

from wich (5) easily follows, and consequently, Theorem (5) is proved.

Obviously if x is a positive integer we have $d \in C(x)$.

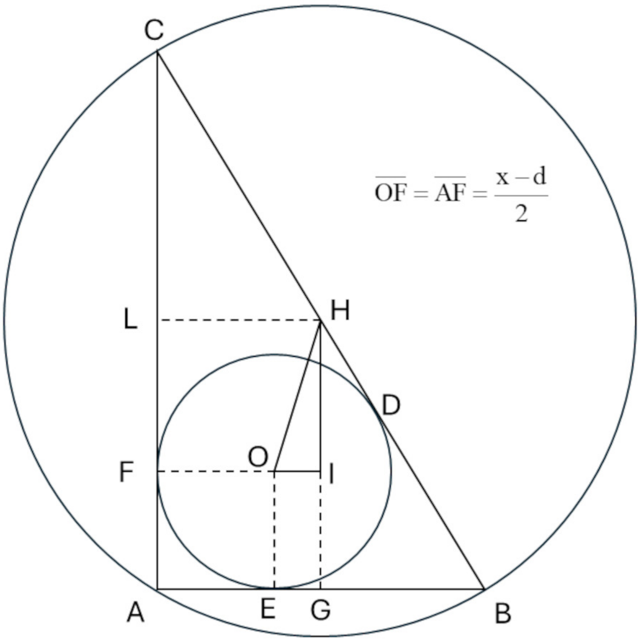

Figure 2:

Let $x, y, z \in \mathbb{R}$ be positive satisfying $x^2 + y^2 = z^2$. From Theorem (1), this triple is generated by x with $d = z - y$ and y with $d' = z - x$ respectively. We want to study the relation betwen d and d' used to obtain the same triple. From formulas (1), we obtain

$$d' = \frac{x^2 + d^2}{2d} - x = \frac{(x-d)^2}{2d}. \tag{6}$$

that is the the relation betwen d and d'.

Let the pythagorean triangle ABC be depicted in Figure (2), and $\overline{OH}$ the line from centre of incircle to centre of circumcircle and $x = \overline{AB}$. The following theorem holds.

Theorem 6. *In a right-angled triangle, with a predeterminatus cathetus $x \in \mathbb{R}$ the line from centre of incircle to centre of circumcircle, is given from*

$$\overline{OH} = \sqrt{\frac{(x-d)^4 + 4d^4}{(4d)^4}} \tag{7}$$

and also

$$\overline{OH} = \sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{d'}{2}\right)^2} \tag{8}$$

with $d = z - y$ and $d' = z - x$.

Proof. Let's begin to notice that

$$\overline{OI} \;=\; \frac{x}{2} \;-\; \frac{x-d}{2} \;=\; \frac{d}{2}$$

while

$$\overline{IH} \;=\; \frac{1}{2}\left(\frac{x^2}{2d}-\frac{d}{2}\right) \;-\; \frac{x-d}{2} \;=\; \frac{(x-d)^2}{4d}\;.$$

Applying the Pythagorean theorem, we obtain (7), and for (6) also (8). Consequently, Theorem (6) is proved.

Obviously if x is a positive integer we have $d \in C(x)$ and $d' \in C(y)$.

Let the pythagorean triangle ABC be depicted in Figure (2). To obtain trigonometric formulas, using only a predeterminatus cathetus $x \in \mathbb{N}$, with $d \in C(x)$, we have the following remark.

Remark 2. *In a right-angled triangle, with a predeterminatus cathetus $x \in \mathbb{N}$ for the angle apposite to x and the acute angle adjacent to x, we have the following trigonometric formulas*

$$sin(\hat{C}) = \cos(\hat{B}) = \frac{2xd}{x^2+d^2}, \quad \cos(\hat{C}) = sin(\hat{B}) = \frac{x^2-d^2}{x^2+d^2}\;, \tag{9}$$

$$sin\left(\frac{\hat{C}}{2}\right) = \frac{d}{\sqrt{x^2+d^2}}, \quad \cos\left(\frac{\hat{C}}{2}\right) = \frac{x}{\sqrt{x^2+d^2}}, \quad \tan\left(\frac{\hat{C}}{2}\right) = \frac{d}{x}\;, \tag{10}$$

$$sin\left(\frac{\hat{B}}{2}\right) = \frac{x-d}{\sqrt{2}\;\cdot\sqrt{x^2+d^2}}, \quad \cos\left(\frac{\hat{B}}{2}\right) = \frac{x+d}{\sqrt{2}\;\cdot\sqrt{x^2+d^2}}, \quad \tan\left(\frac{\hat{B}}{2}\right) = \frac{x-d}{x+d} \tag{11}$$

with $d \in C(x)$.

Let's begin to notice that from formulas of Theorem (1) we have (9). Applying the half-angle formulas, we obtain (10) and (11).
Moreover, if we have a positive $x \in \mathbb{R}$, (9), (10) and (11) hold, with $d = z - y$.

We consider a rectangle with sides and diagonal given by x, y, and $z \in \mathbb{N}$, respectively. Let the parallelepiped have edges and spatial diagonal given by x, y, w, and $t \in \mathbb{N}$, respectively. We want obtain all parallelepipeds with x, y, z, w, and $t \in \mathbb{N}$, where x is a predetermined integer. The following theorem holds.

Theorem 7. *Let a rectangle have sides x, $y \in \mathbb{N}$ and diagonal given by $z \in \mathbb{N}$, and let a parallelepiped have edges x, y, $w \in \mathbb{N}$ and spatial diagonal given by $t \in \mathbb{N}$, respectively, where x is a predetermined integer. The quadruple (x, y, w, t) and z satisfy the conditions*

$$x^2+y^2=z^2 \quad \textit{and} \quad x^2+y^2+w^2=t^2 \tag{12}$$

if and only if

$$x = x, \quad y = \frac{x^2 - d^2}{2d}, \quad w = \frac{\left(\frac{x^2+d^2}{2d}\right)^2 - (d^*)^2}{2d^*}, \quad t = \frac{\left(\frac{x^2+d^2}{2d}\right)^2 + (d^*)^2}{2d^*} \tag{13}$$

with $d \in C(x)$ *and* $d^* \in C(z)$.

Proof. Let's begin to notice that for a predeteminatus $x \in \mathbb{N}$, $f$or Theorem (2) there exists almost a pair $y, z \in \mathbb{N}$ such that $x^2+y^2 = z^2$ with $z-y = d \in C(x)$. Also, for Theorem (2) there exists almost a pair $w, t \in \mathbb{N}$ such that $z^2 + w^2 = t^2$ with $t - w = d^* \in C(z)$, and then there exist x, y, w, and $t \in \mathbb{N}$ such that $x^2 + y^2 + w^2 = t^2$ with $z \in \mathbb{N}$. To obtain formulas (13), using formulas (1) to a predeteminatus $x \in \mathbb{N}$ we obtain y and z such that $x^2 + y^2 = z^2$ with $d \in C(x)$, and re-applying same formulas to $z = \frac{x^2 + d^2}{2d}$, considering it as prededeterminatus positive integer, with $d^* \in C(z)$, we obtain also w and t that togheter with y satisfy (12). To prove that (13) gives every $x^2 + y^2 + w^2 = t^2$ with $z \in \mathbb{N}$, it suffices apply Theorem (1) to the triples $x^2 + y^2 = z^2$ and $z^2 + w^2 = t^2$. Consequently, Theorem (7) is proved.

We observe that if $C(x) = \{1, x\}$ and $C(z) = \{1, z\}$, that is x and z are prime numbers, then we have one unique (x, y, w, t) that satisfies (12), remembering that for $d = x$ and $d^* = z$ we obtain trivial triples. Consequently, if x or z are not prime number, then we obtain more (y, w, t) that togheter x satisfy (12).

More in general, there exists the following theorem.

Theorem 8. *Let* a_1 *be a predetermined integer. There exist at least one Pythagorean n-uple of integers* $(a_1, a_2, ..., a_n)$ *such that*

$$a_1^2 + \sum_{i=2}^{n-1} a_i^2 = a_n^2 \tag{14}$$

Proof. From Theorem (7), we know that for a predetermined integer a_1 there exist $a_2, a_3, b \in \mathbb{N}$ such that $a_1^2 + a_2^2 + a_3^2 = b^2$. From Theorem (2), there exists almost a couple (a_4, c), $a_4, c \in \mathbb{N}$, such that $b^2 = c^2 - a_4^2$ and then $a_1^2 + a_2^2 + a_3^2 + a_4^2 = c^2$. Iterating the procedure $n - 5$ times, then we obtain (14). Consequently, Theorem (8) is proved.

It is interesting to note that we can iterate the previous procedure infinitely. For this reason we have the following corollary.

Corollary 1. *For every predetermined integer* a_1, *there exists at least one* $b \in \mathbb{N}$ *and one infinite set of integers* $a_i, i = 1, 2, ..., \infty$, *such that*

$$a_1^2 + \sum_{i=2}^{\infty} a_i^2 = b^2.$$

Let G be one set of 2x2 symmetric commuting matrices with non-zero determinant, defined in $\mathbb{N}$, in the following form:

$$G = \left\{ \begin{pmatrix} c\, b \\ b\, c \end{pmatrix} \; such that \; c^2 - b^2 = a^2, \;\; with \; a, b, c \in \mathbb{N}, \; a \neq 0 \right\}.$$

Let (a, b, c), (d, e, f), (a·d, y, z) be the Pythagorean triples generated by a, d, a·d, respectively using (1). Taking into account Theorem (4), we have that $y = bf + ce$ and $z = be + cf$. The following theorem holds.

Theorem 9. *Let (a, b, c), (d, e, f), and (ad, bf+ce, be+cf) be Pythagorean triples. We consider* $A = \begin{pmatrix} c\, b \\ b\, c \end{pmatrix}, B = \begin{pmatrix} f\;\, e \\ e\;\, f \end{pmatrix}, C = \begin{pmatrix} be + cf & bf + ce \\ bf + ce & be + cf \end{pmatrix} \in G$. *In the set of Pythagorean triples, the binary operation*

$$(a, b, c) \cdot (d, e, f) = (ad, bf + ce, be + cf)$$

is equivalent, in the set G, to the matrix multiplication A·B=B·A=C and det(A)·det(B)=det(C), and the identity matrix corresponds to the the identity element (1, 0, 1).

Proof. It suffices to apply the properties of the product between matrices and, to obtain the identity matrix, use the trivial Pythagorean triple (1, 0, 1). Consequently, Theorem (9) is proved.

We observe that if a, b, c $\in \mathbb{Q}$, being $det(A) = c^2 - b^2 = a^2 \neq 0$, then there exists the inverse matrix of A in G. Consequently it is easy to obtain that the set of Pythagorean triples is a commutative group in $\mathbb{Q}$, as already seen in [7] but with different approach. If a, b, c $\in \mathbb{Z}$ it is needed to deal with and introduce particular endomorphisms to obtain results also in $\mathbb{Z}$. In particular to study the set of primitive Pythagorean triples to obtain a commutative group with elements in $\mathbb{Q}$ or in $\mathbb{Z}$., Therefore, from the beginning, to have results we preferred in [7] to give an approach that was simpler and suitable for a wider audience.

In order to investigate a possible addition operation among Pythagorean triples, we have the following theorem.

Theorem 10. *Let* $\left(x, \; \frac{x^2-1}{2}, \; \frac{x^2+1}{2}\right)$ *and* $\left(y, \; \frac{y^2-1}{2}, \; \frac{y^2+1}{2}\right)$ *be two Pythagorean triples genetated by odd integers x and y, rispectively, using (1) with* $d = 1 \in$ *C(x) and C(y). One addition operation between the two Pythagorean triples is given by the Pythagorean triple*

$$\left(x + y, \; \frac{x^2-1}{2} + \frac{y^2-1}{2} - \left(\frac{x-y}{2}\right)^2, \; \frac{x^2+1}{2} + \frac{y^2+1}{2} - \left(\frac{x-y}{2}\right)^2 \right). \tag{15}$$

Proof. We note that $x+y$ is even and considering the Pythagorean triple generated by $x+y$, using (1) with $d=2 \in C(x+y)$, we obtain

$$\left(x+y,\ \frac{(x+y)^2-4}{4},\ \frac{(x+y)^2+4}{4}\right).$$

To verify that (15) is a Pythagorean triple, for Theorem (1), then it suffices to prove

$$\frac{(x+y)^2-4}{4} = \frac{x^2-1}{2} + \frac{y^2-1}{2} - \left(\frac{x-y}{2}\right)^2$$

and

$$\frac{(x+y)^2+4}{4} = \frac{x^2+1}{2} + \frac{y^2+1}{2} - \left(\frac{x-y}{2}\right)^2.$$

It is easy to see that above equations are identities and consequently, Theorem (10) is proved.

We observe that (15) is applicable only to every pair of Pythagorean triples generated by odd integers x and y, respectively, using (1) with $d=1$, obtaining one Pythagorean triple generated by the even integer $x+y$ with $d=2$. Currently, the addition operation is not iterable, meaning that the obtained triple cannot be summed with another Pythagorean triple to result in another Pythagorean triple. Additionally, there is a lack of identity and opposite elements. Anyway, it's a start to explore addition operation in the set of Pythagorean triples.

As a consequence of Theorem (1) and (3) let us state the following theorem.

Theorem 11. *Every prime number is present as cathetus in only one Pythagorean triple, and this is a primitive Pythagorean triple.*

Proof. From Theorem (1), we have that $C(x)=\{1,x\}$, for $d=x$ the triple is trivial, and there exists a unique Pythagorean triple generated by x for $d=1$. From Theorem (3), we have that this triple is primitive, and consequently, Theorem (11) is proved.

At last, let $f:]0,\infty[\to \mathbb{R}$ be such that $f(x)=2x$, $\forall\ y,z \in]0,\infty[$, as depicted in Figure (3). We have the following remark.

Remark 3. *Let* $f:]0,\infty[\to \mathbb{R}$ *defined as* $f(x)=2x$, $\forall\ y,z \in]0,\infty[$*, we have*

$$\int_y^z 2x\,dx = d^2 + 2yd \tag{16}$$

with $d=z-y$.

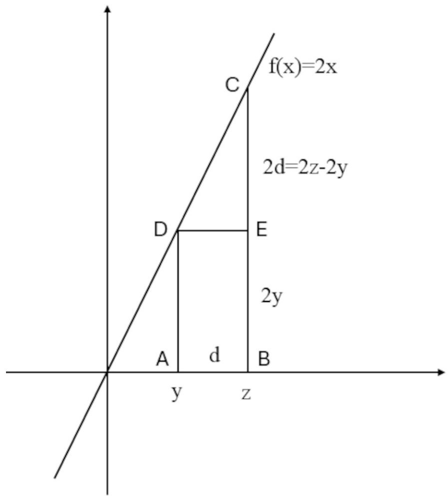

Figure 3:

The above integral is the area of right-angled trapezoid ABCD given by the sum of the areas of the rectangle ABED and the triangle ECD. Consequently, we have (16). Observing that the result of this integral is also in the form $z^2 - y^2 = x^2$, then we get $x^2 = d^2 + 2yd$. From last equation, and taking into account that $z = y + d$. we have (1). In this way, we find another geometric interpretation of (1), different from that found in Theorem (1), and it holds $\forall\ x, y, z \in\ ]0, \infty[$. Obviously, if $x, y, z \in N$ then $d \in C(x)$.

4. Conclusion and Remarks

The discovery of the parametrization and relations among Pythagorean triples that we have found shows the fundamental role of $d \in C(x)$ and characterizing the results.

This approach could be employed to study further relationships among Pythagorean triples. For instance, it might be used to find a suitable addition operation between Pythagorean triples which could allow us, in turn, to define a vector space of Pythagorean triples. This way could be used to study other problems, some of which are still open. One of the next steps could be to study the parametrizations of Pythagorean quadruples, looking for a representation similar to (1) and finding all Pythagorean quadruples. Mainly it will be interesting to find other parametrizations, relations and characterizations regard to the Pythagorean n-uples dependent by $d \in C(x)$.

The new results and applications, along with those found in the Preliminary Results section, show how the field of Pythagorean triples is still interesting and stimulating to study, despite the centuries that have elapsed. In every case, this is a new approach to study the Pythagorean triples for students in schools and universities.

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