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On Common Fixed Point for Contractive Mappings in p-Pompeiu-Hausdorff Metric Spaces

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Abstract. In this paper we establish the existence of a common fixed point from a pair of set-valued mappings. By utilizing the concept of convergence of set-valued mappings' sequences, both ordinary and pointwise convergence, we establish a common fixed point theorem. This our newly result is a generalization of common fixed point theorem of set-valued mappings on partial metric spaces. Further, we establish newly common fixed point theorem under ϕ-contraction on partial metric spaces.

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1. Introduction

Discussions regarding Banach's principle of contraction often appear in various references. Many generalizations are also given for which a comparative study of these generalizations is given by Rhoades [18]. One of the generalizations of the Banach contraction principle that is also quite widely discussed is in the set-valued mapping. Various results of the generalization of Banach's contraction principle can be found in [8, 13, 14, 17, 20] and reference therein. Further results on the general fixed point of the set-valued mapping of the contractive type may be found in Kubiak [12] and Singh [19]. On the other hand a generalization of the principle of Banach contraction for single-valued mapping on partial metric spaces can be seen in [2, 5, 10, 11] and reference therein. Furthermore, a generalization of the Banach contraction principle for set-valued mappings in partial

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metric spaces can be found in [1, 4, 6]. This generalization builds upon the Banach contraction principle for set-valued mappings, which was initially introduced by Nadler [16]. And further results on the general fixed point of the set-valued mapping on partial metric space be found in Aydi et. al. [7] and Ahmad et. al. [3]. In this paper, we will generalize some results of Aydi et. al.[7] and Ahmad et. al. [3]. Referring to Kubiak [12], we will use the common fixed point existence of a sequence of set-valued mappings to derived on a pair of set-valued mappings so that the existence of a common fixed point is guaranted. Furthermore, referring to Singh [19] we will use some functions that he has defined to give a new generalization of the contraction of Banach's principle for set-valued mappings on partial metric spaces. By using this contraction we obtain the common fixed points of a pair of set-valued mappings.

2. Preliminaries

Let (X, p) be a partial metric spaces. Suppose that $CB^p(X)$ be class of all nonempty, closed and bounded subsets of X. Let mapping $H^p : X \to CB^p(X)$ define

$$H^p(A, B) = \max\{\sup\{p(x, B) : x \in A\}, \sup\{p(y, A) : y \in B\}\},$$

for each $A, B \in X$ and $p(x, B) = \inf\{p(x, y) : y \in B\}$. The mapping H^p is p-Pompeiu-Hausdorff (partial Pompeiu Hausdorff) metric, and the pairs $(CB^p(X), H^p)$ is called p-Pompeiu-Hausdorff metric spaces. (The use of the term Pompeiu-Hausdorf refers to [9]). Some properties of metric H^p can be found in [6, 7, 15].

Definition 1. *[15] Let $(CB^p(X), H^p)$ be a p-Pompeiu-Hausdorff metric spaces. A sequence (F_n) in $CB^p(X)$ converges to set $F \in CB^p(X)$ if*

$$\lim_{n\to\infty} H^p(F_n, F) = H^p(F, F).$$

Definition 2. *[15] Let $(CB^p(X), H^p)$ be a p-Pompeiu-Hausdorff metric spaces. A sequence (F_n) in $CB^p(X)$ is said to a Cauchy sequence if*

$$\lim_{n,m\to\infty} H^p(F_n, F_m)$$

exists and finite.

Sequence (F_n) is Cauchy sequence if the sequence $H^p(F_n, F_m)$ tends to some $\lambda \in \mathbb{R}$ as n, m approach to infinity, that is, $\lim_{n,m\to\infty} H^p(F_n, F_m) = \lambda < \infty$, i.e. for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|H^p(F_n, F_m) - \lambda| < \varepsilon,$$

for all $n, m \geq N$.

Furthermore, lets consider the properties of Cauchy sequence (F_n) in $(CB^p(X), H^p)$.

Theorem 1. *[15] A sequence (F_n) in p-Pompeiu-Hausdorff metric spaces $(CB^p(X), H^p)$ is Cauchy if and only if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$H^p(F_n, F_m) - H^p(F_m, F_m) < \varepsilon,$$

for every $n, m \geq N$.

Definition 3. *[15] A p-Pompeiu-Hausdorff metric spaces $(CB^p(X), H^p)$ is called complete if every Cauchy sequences $F_n \in CB^p(X)$ converges to $F \in CB^p(X)$ and*

$$\lim_{n\to\infty} H^p(F_n, F) = H^p(F, F).$$

One of the relationships between the partial metric space (X, p) and the p-Pompeiu-Hausdorff metric space $(CB^p(X), H^p)$ can be seen in its completeness. This is shown in the following Theorem 2.

Theorem 2. *[15] If $(CB^p(X), H^p)$ be a complete partial metric spaces then $(CB^p(X), H^p)$ is complete.*

For set-valued mapping $F : X \to CB^p(X)$, a point $x \in X$ is called a fixed point of F if $x \in F(x)$. Analogously, for F and G set-valued mappings from X into $CB^p(X)$, a point $x \in X$ is called as a common fixed point of F and G if $x \in F(x)$ and $x \in G(x)$.

3. Main Results

In the following discussion, we assume that (X, p) is a complete partial metric space.

Theorem 3. *Let $(CB^p(X), H^p)$ be a p-Pompeiu-Hausdorff metric spaces. Suppose that $F_n, G_n : X \to CB^p(X), n \in \mathbb{N}$ be sequence of set-valued mappings on $CB^p(X)$, there exists κ where $0 \leq \kappa < 1$ such that*

$$H^p(F_m(x), G_n(y)) \leq \kappa \max\left\{p(x, y), p(x, F_m(x)), p(y, G_n(y)), \frac{1}{2}\left(p(x, G_n(y)) + p(y, F_m(x))\right)\right\},$$

for each $m, n \in \mathbb{N}$ and $x, y \in X$, then (F_n) and (G_n) have a common fixed point, i.e. there exist a point $x \in X$ such that $x \in F_m(x)$ and $x \in G_n(x)$ for each $m, n \in \mathbb{N}$.

Proof. Let we consider that $0 \leq \kappa < 1$. For the first we assume that $\kappa = 0$. Suppose that $x_0 \in X$ and $x_1 \in F_1(x_0)$, then for all $n \in \mathbb{N}$ we have

$$p(x_1, G_n(x_1) \leq H^p(F_1(x_0), G_n(x_1)) = 0.$$

It means $p(x_1, G_n(x_1)) = 0$. Since G_n are closed for each n then $x_1 \in G_n(x_1)$. In the similar way, we can obtain that for $x_0 \in X$ and $x_1 \in G_1(x_0)$, then for all $n \in \mathbb{N}$ we have

$$p(x_1, F_n(x_1)) \leq H^p(G_1(x_0), F_n(x_1)) = 0,$$

i.e., $p(x_1, F_n(x_1)) = 0$, then $x_1 \in F_n(x_1)$. From this result, it can be seen that x_1 is the common fixed point of F_n and G_n.
Next we assume that $\kappa \neq 0$. Suppose that $x_0 \in X$ and $x_1 \in F_1(x_0)$. Furthermore, define the sequence (x_n) where $x_{2n} \in G_n(x_{2n-1})$ and $x_{2n-1} \in F_n(x_{2n-2})$ are such that

$$p(x_{2n-1}, x_{2n}) \leq \frac{1}{\sqrt{\kappa}} H^p(F_n(x_{2n-2}), G_n(x_{2n-1}))$$

$$p(x_{2n}, x_{2n+1}) \leq \frac{1}{\sqrt{\kappa}} H^p(F_n(x_{2n}), G_n(x_{2n-1})),$$

for $n = 1, 2, 3, \ldots$.
Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. For n being even, we have $x_{2n} \in F_{n+1}(x_{2n})$ thus for each $m \in \mathbb{N}$

$$\begin{aligned} p(x_{2n}, G_m(x_{2n})) &\leq H^p(F_{n+1}(x_{2n}), G_m(x_{2n})) \\ &\leq \kappa \max\{p(x_{2n}, x_{2n}), p(x_{2n}, F_{n+1}(x_{2n})), p(x_{2n}, G_m(x_{2n})), \\ &\quad \tfrac{1}{2}(p(x_{2n}, F_{n+1}(x_{2n})) + p(x_{2n}, G_m(x_{2n}))\} \\ &\leq \kappa p(x_{2n}, G_m(x_{2n})). \end{aligned}$$

Since $0 < \kappa < 1$ then $p(x_{2n}, G_m(x_{2n}) = 0$. Therefore, we have $x_{2n} \in G_m(x_{2n})$ for each $m \in \mathbb{N}$. Similarly, for n being odd numbers, we have $x_{2n+1} \in G_{n+1}(x_{2n+1})$, and for every m implies

$$\begin{aligned} p(x_{2n+1}, F_m(x_{2n+1})) &\leq H^p(G_{n+1}(x_{2n+1}), F_m(x_{2n+1})) \\ &\leq \kappa \max\{p(x_{2n+1}, x_{2n+1}), p(x_{2n+1}, G_{n+1}(x_{2n+1})), \\ &\quad p(x_{2n+1}, F_m(x_{2n+1})), \tfrac{1}{2}(p(x_{2n+1}, F_m(x_{2n+1})) \\ &\quad + p(x_{2n+1}, G_{n+1}(x_{2n+1}))\} \\ &\leq \kappa p(x_{2n+1}, F_m(x_{2n+1})). \end{aligned}$$

Analogous to n is even, it can be concluded that $p(x_{2n+1}, F_m(x_{2n+1})) = 0$, it means $x_{2n+1} \in F_m(x_{2n+1})$.
For the next step, we will show that (x_n) is Cauchy sequence in (X, p). Let we consider

$$\begin{aligned} p(x_{2n}, x_{2n+1})) &\leq \tfrac{1}{\sqrt{\kappa}} H^p(F_{n+1}(x_{2n}), G_n(x_{2n-1})) \\ &\leq \tfrac{1}{\sqrt{\kappa}} \kappa \max\{p(x_{2n}, x_{2n-1}), p(x_{2n}, F_{n+1}(x_{2n})), p(x_{2n-1}, G_n(x_{2n-1})), \\ &\quad \tfrac{1}{2}(p(x_{2n}, G_n(x_{2n-1})) + p(x_{2n-1}, F_{n+1}(x_{2n}))\} \\ &\leq \sqrt{\kappa} \max\{p(x_{2n}, x_{2n-1}), p(x_{2n}, x_{2n+1}), p(x_{2n-1}, x_{2n}), \\ &\quad \tfrac{1}{2}(p(x_{2n}, x_{2n+1}) + p(x_{2n-1}, x_{2n}))\} \\ &\leq \sqrt{\kappa} \max\{p(x_{2n-1}, x_{2n}), p(x_{2n}, x_{2n+1})\}, \end{aligned}$$

when $p(x_{2n}, x_{2n+1})$ is the maximum then we have $x_{2n} = x_{2n+1}$. Since $x_n \neq x_{x+1}$ for each n thus we get a contradiction. Therefore, we have the maximum is $p(x_{2n-1}, x_{2n})$. It implies

$$p(x_{2n}, x_{2n+1}) \leq \sqrt{\kappa} p(x_{2n-1}, x_{2n}).$$

In the similar way, we have

$$p(x_{2n+1}, x_{2n+2}) \leq \sqrt{\kappa} p(x_{2n}, x_{2n+1}).$$

Therefore, we obtain

$$\begin{aligned} p(x_{2n}, x_{2n+1})) &\leq \sqrt{\kappa} p(x_{2n-1}, x_{2n}) \\ &\leq \sqrt{\kappa}\sqrt{\kappa} p(x_{2n-2}, x_{2n-1}) = (\sqrt{\kappa})^2 p(x_{2n-2}, x_{2n-1}) \\ &\leq (\sqrt{\kappa})^2 \sqrt{\kappa} p(x_{2n-3}, x_{2n-2}) = (\sqrt{\kappa})^3 p(x_{2n-3}, x_{2n-2}) \\ &\vdots \\ &\leq (\sqrt{\kappa})^{2n} p(x_0, x_1) \\ &= \kappa^n p(x_0, x_1). \end{aligned}$$

And also we have $p(x_{2n+1}, x_{2n+2})) \leq \kappa^n p(x_1, x_2)$.
Let $t(x_0) := \max\{p(x_0, x_1), p(x_1, x_2)\}$, then for $m > n$ w have

$$\begin{aligned} p(x_m, x_n)) &\leq \sum_{i=0}^{m-(n+1)} p(x_{n+i}, x_{n+1+i}) \\ &\leq \sum_{i=0}^{m-(n+1)} h^{n+i} t(x_0) \\ &= t(x_0) \sum_{i=0}^{m-(n+1)} h^{n+i} \\ &= th^n \sum_{i=0}^{m-(n+1)} h^i \\ &\leq \frac{t(x_0)h^n}{1-h} \end{aligned}$$

Since $\frac{t(x_0)h^n}{1-h} \to 0$ as $n \to \infty$, it means we are already shown that (x_n) is a Cauchy sequence in X. Since (X, p) is complete partial metric space then there exists $x \in X$ such that $x_n \to x$ whereas $n \to \infty$. Let we observe the following condition

$$\begin{aligned} p(x_{2n-1}, G_m(x)) &\leq H^p(F_n(x_{2n-2}), G_m(x)) \\ &\leq \kappa \max\{p(x_{2n-2}, x), p(x_{2n-2}, F_n(x_{2n-2})), p(x, G_m(x)), \\ &\quad \tfrac{1}{2}(p(x_{2n-2}, G_m(x)) + p(x, F_n(x_{2n-2})))\} \\ &\leq \kappa \max\{p(x_{2n-2}, x), p(x_{2n-2}, x_{2n-1}), p(x, G_m(x)), \\ &\quad \tfrac{1}{2}(p(x_{2n-2}, G_m(x)) + p(x, x_{2n-1})))\} \end{aligned}$$

by taking $n \to \infty$ we obtain

$$p(x, G_m(x)) \leq \kappa \max\left\{p(x, x), p(x, x), p(x, G_m(x)), \frac{1}{2}(p(x, G_m(x)) + p(x, x))\right\},$$

for each m. Therefore, we have

$$p(x, G_m(x)) \leq \kappa p(x, G_m(x)).$$

Since $0 \leq \kappa < 1$ then $p(x, G_m(x)) = 0$. It implies that $x \in G_m(x)$ because $G_m(x)$ is closed. Similarly, we can show that $x \in F_n(x)$ for each n. So, we obtain $x \in G_m(x)$ and $x \in F_n(x)$ for each m, n. It means x is common fixed point of G_m and F_n for every m and n. This complete the proof.

In Theorem 3 above, we have the principle of contraction on set-valued mapping sequences as follows:

$$H^p(F_n(x), G_n(y)) \leq \kappa \max \left\{ p(x,y), p(x, F_n(x)), p(y, G_n(y)), \frac{1}{2}\left(p(x, G_n(y)) + p(y, F_n(x))\right) \right\},$$

for each $n \in \mathbb{N}$ and $x, y \in X$ and $\kappa \in [0,1)$.

By looking at the sequences F_n and G_n in Theorem 3 respectively as constant sequences, Corollary 1 can be obtained as follows. This Corollary 1 is a generalization of the result [12] on the partial metric space.

Corollary 1. *Let $(CB^p(X), H^p)$ be a p-Pompeiu-Hausdorff metric spaces. Suppose that $F, G : X \to CB^p(X)$ with the following condition*

$$H^p(F(x), G(y)) \leq \kappa \max \left\{ p(x,y), p(x, F(x)), p(y, G(y)), \frac{1}{2}(p(x, G(y)) + p(y, F(x))) \right\}, \tag{1}$$

for each $x, y \in X$ where $0 \leq \kappa < 1$, then F and G have a common fixed point.

By utilizing the concept of pointwise convergence of set-valued sequences, we can also investigate the existence of a common fixed point of set-valued mappings. Let see on the following theorem.

Theorem 4. *Let $(CB^p(X), H^p)$ be a p-Pompeiu-Hausdorff metric spaces. Suppose that $F_n, G_n : X \to CB^p(X)$ sequences in $CB^p(X)$. Sequences F_n, G_n converging pointwise to $F, G : X \to CB^p(X)$ respectively. If the following condition holds*

$$H^p(F_n(x), G_n(y)) \leq \kappa \max \left\{ p(x,y), p(x, F_n(x)), p(y, G_n(y)), \frac{1}{2}\left(p(x, G_n(y)) + p(y, F_n(x))\right) \right\}, \tag{2}$$

for every $x, y \in X$ and $n \in \mathbb{N}$ where $0 \leq \kappa < 1$ then F and G have a common fixed point.

Proof. Take any point $x, y \in X$. Let $u \in F_n(x)$ and $v \in F(x)$, then we have

$$\begin{aligned} p(y,u) &\leq p(y,v) + p(v,u) - p(v,v) \\ &\leq p(y,v) + p(v,u) \\ &\leq p(y, F(x)) + p(u, F(x)). \end{aligned}$$

Consequently $p(y, F_n(x)) \leq p(y, F(x)) + H^p(F_n(x), F(x))$. On the other side we also have the following condition

$$\begin{aligned} p(y,v) &\leq p(y,u) + p(u,v) - p(u,u) \\ &\leq p(y,u) + p(u,v) \\ &\leq p(y, F_n(x)) + p(v, F_n(x)). \end{aligned}$$

It implies $p(y, F(x)) \leq p(y, F_n(x)) + H^p(F(x), F_n(x))$. Therefore, we have

$$|p(y, F_n(x)) - p(y, F(x))| \leq H^p(F_n(x), F(x)). \tag{3}$$

On the similar way we can also show that

$$|p(x, G_n(y)) - p(x, G(y))| \leq H^p(G_n(y), G(y)). \tag{4}$$

Furthermore, by using inequality (3) and (4) and also the continuity of H^p then by taking $n \to \infty$ in inequality (2) we obtain

$$H^p(F(x), G(y)) \leq \kappa \max\left\{p(x, y), p(x, F(x)), p(y, G(y)), \frac{1}{2}\left(p(x, G(y)) + p(y, F(x))\right)\right\},$$

These conditions show that the set-valued mapping F and G satisfies the hypothesis on Corollary 1. Thus, based on corollary 1 it can be concluded that F and G have a common fixed point. This complete the proof.

Let we consider that for any positive real numbers s and t holds

$$\frac{1}{2}(s + t) \leq \max\{s, t\}.$$

It implies for any positive real numbers p, q, r, s and t we have

$$\max\left\{p, q, r, \frac{1}{2}(s + t)\right\} \leq \max\{p, q, r, s, t\}. \tag{5}$$

Therefore, we can derive a generalization of contractions that the theorem uses as well as the corollary on the previous discussion. In corollary 1, which indicates the existence of a common fixed point of set-valued mapping, by utilizing inequality (5) we can obtain a generalization of contractions (1) as follows

$$H^p(F(x), G(y)) \leq \kappa \max\left\{p(x, y), p(x, F(x)), p(y, G(y)), p(x, G(y)), p(y, F(x))\right\}. \tag{6}$$

On the other sides, Singh has given a definition of a function in generalizing the principle of contraction of several references therein (Definition 2.1 in [19]) as follows.

Definition 4. *Suppose that $\phi : [0, \infty) \to [0, \infty)$ a function that satisfy the following conditions:*

(i) ϕ is non-decreasing upper semi-continuous,

(ii) $\phi(2u) < u$ for each $u > 0$.

Using this definition, Singh established the existence of common fixed points of set-valued mappings (Theorem 2.2 in [19]). Referring to these results, we will generalize the theorem to a more general metric space, which is a partial metric space.

Theorem 5. *Let $(CB^p(X), H^p)$ be a p-Pompeiu-Hausdorff metric spaces. Suppose that $F, G : X \to CB^p(X)$ be set-valued mappings that satisfy the following conditions*

$$H^p(F(x), G(y)) \leq \phi(\max\{p(x, y), p(x, F(x)), p(y, G(y)), p(x, G(y)), p(y, F(x))\}), \tag{7}$$

for each $x, y \in X$ where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that ϕ be a non-decreasing upper semi-continuous and $\phi(2u) < u$ for $u > 0$, then set-valued mappings F and G have a unique common fixed point.

Proof. Take any $x_0 \in X$, but fixed. Let $x_0 \notin F(x_0)$ and take $x_1 \in F(x_0)$, then from (7) we obtain

$$\begin{aligned} p(x_1, G(x_1)) &\leq H^p(F(x_0), G(x_1)) \\ &\leq \phi(\max\{p(x_0, x_1), p(x_0, F(x_0)), p(x_1, G(x_1), p(x_0, G(x_1)), p(x_1, F(x_0))\}) \\ &\leq \phi(\max\{p(x_0, x_1), p(x_1, G(x_1))\}) \\ &\leq \phi(p(x_0, x_1) + p(x_1, G(x_1))). \end{aligned}$$

Consider that: if $p(x_0, x_1) < p(x_1, G(x_1))$ then

$$\begin{aligned} p(x_1, G(x_1)) &\leq \phi(p(x_0, x_1) + p(x_1, G(x_1))) \\ &< \phi(p(x_1, G(x_1)) + p(x_1, G(x_1))) \\ &= \phi(2p(x_1, G(x_1))) \\ &< p(x_1, G(x_1). \end{aligned}$$

This condition shows a contradiction, then it must be $p(x_0, x_1) \geq p(x_1, G(x_1))$. Therefore, we have

$$\begin{aligned} p(x_1, G(x_1)) &\leq \phi(p(x_0, x_1) + p(x_0, x_1)) \\ &= \phi(2p(x_0, x_1)) \\ &< p(x_0, x_1). \end{aligned}$$

Furthermore, we can take $x_2 \in G(x_1)$ such that $p(x_1, x_2) \leq p(x_0, x_1)$. Thus, by using inequality (7) we have

$$\begin{aligned} p(x_2, F(x_2)) &\leq H^p(G(x_1), F(x_2)) \\ &\leq \phi(\max\{p(x_1, x_2), p(x_1, G(x_1)), p(x_2, F(x_2), p(x_1, F(x_2)), p(x_2, G(x_1))\}) \\ &\leq \phi(\max\{p(x_1, x_2), p(x_2, F(x_2))\}) \\ &\leq \phi(p(x_1, x_2) + p(x_2, F(x_2))). \end{aligned}$$

Let's observe, when $p(x_1, x_2) < p(x_2, F(x_2))$ then we obtain

$$\begin{aligned} p(x_2, F(x_2)) &\leq \phi(p(x_1, x_2) + p(x_2, F(x_2))) \\ &< \phi(p(x_2, F(x_2)) + p(x_2, F(x_2))) \\ &= \phi(2p(x_2, F(x_2))) \\ &< p(x_2, F(x_2). \end{aligned}$$

Thus, we found a contradiction. It must holds $p(x_1, x_2) \geq p(x_2, F(x_2))$. Therefore, we obtain

$$\begin{aligned} p(x_2, F(x_2)) &\leq \phi(p(x_1, x_2) + p(x_1, x_2)) \\ &= \phi(2p(x_1, x_2)) \\ &< p(x_1, x_2). \end{aligned}$$

In the similar line, we can choose $x_3 \in F(x_2)$, then we will have $p(x_2, x_3) \leq p(x_1, x_2)$. If this process is continued then a sequence (x_n) in X is obtained with the form as follows

$$x_{2n+1} \in F(x_{2n}),$$

and

$$x_{2n+2} \in G(x_{2n+1}),$$

and also

$$p(x_n, x_{n+1}) \leq p(x_{n-1}, x_n). \tag{8}$$

Furthermore, we defined $p_n = p(x_n, x_{n+1})$. From inequality (8) then we obtain

$$p_n \leq p_{n+1}.$$

This means that p_n is a non-decreasing sequences of real numbers and is bounded below by zero. Therefore p_n is a convergent sequences. Suppose that

$$lim_{n\to\infty} p_n = q.$$

Let $q > 0$, consider that $p(x_n, x_{n+1}) \leq \phi(2p(x_{n-1}, x_n)) < p(x_{n-1}, x_n)$, thus

$$p(x_n) \leq \phi(2p_{n-1}) < p_{n-1}. \tag{9}$$

Take $n \to \infty$ on inequality (9) then we obtain

$$q \leq \phi(2q) < q.$$

Therefore, we have a contradiction. Hence, $q = 0$, i.e.,

$$lim_{n\to\infty} p_n = lim_{n\to\infty} p(x_n, x_{n+1}) = 0.$$

Furthermore, we will show that (x_n) is a Cauchy sequences. Based on the construction of sequence (x_n), in showing that sequence (x_n) is a Cauchy can be done by showing that (x_{2n}) is a Cauchy sequence. As for the proof using contradiction, that is, if (x_{2n}) is not a Cauchy sequence then there exist $\varepsilon > 0$ such that for every positive integer $2t$ there is sequence $(2m_t)$ and $(2n_t)$ where $t < n_t < m_t$ and we have

$$p(x_{2n_t}, x_{2m_t}) > \varepsilon, t = 1, 2, 3, \ldots \tag{10}$$

Suppose that $2m_t$ is the smallest integer that greater than $2n_t$ and satisfies the inequality (10) then we have

$$p(x_{2n_t}, x_{2m_t-2}) \leq \varepsilon.$$

Hence

$$\begin{aligned}
\varepsilon &\leq p(x_{2n_t}, x_{2m_t}) \\
&\leq p(x_{2n_t}, x_{2m_t-2}) + p(x_{2m_t-2}, x_{2m_t}) - p(x_{2m_t-2}, x_{2m_t-2}) \\
&\leq p(x_{2n_t}, x_{2m_t-2}) + p(x_{2m_t-2}, x_{2m_t}) \\
&\leq \varepsilon + p(x_{2m_t-2}, x_{2m_t}).
\end{aligned}$$

Let we consider that $p(x_{2m_t-2}, x_{2m_t}) \to 0$ as $m_t \to \infty$, then we have $\varepsilon \le p(x_{2n_t}, x_{2m_t}) \le \varepsilon$. Consequently,

$$lim_{n_t,m_t\to\infty} p(x_{2n_t}, x_{2m_t}) = \varepsilon.$$

It is noted that

$$\begin{aligned}
p(x_{2n_t+1}, x_{2m_t}) &\le H^p(F(x_{2n_t}), G(x_{2m_t-1}))\\
&\le \phi(\max\{p(x_{2n_t}, x_{2m_t-1}), p(x_{2n_t}, F(x_{2n_t})), p(x_{2m_t-1}, G(x_{2m_t-1})),\\
&\quad p(x_{2n_t}, G(x_{2m_t-1})), p(x_{2m_t-1}, F(x_2 n_t))\})\\
&\le \phi(\max\{p(x_{2n_t}, x_{2m_t-1}), p(x_{2m_t-1}, G(x_{2m_t-1}))\})\\
&\le \phi(p(x_{2n_t}, x_{2m_t-1}) + p(x_{2m_t-1}, G(x_{2m_t-1})))\\
&\le \phi(p(x_{2n_t}, x_{2m_t-1}) + p(x_{2m_t-1}, x_{2n_t}) + p(x_{2n_t}, G(x_{2m_t-1}))\\
&\quad -p(x_{2n_t}, x_{2n_t}))\\
&\le \phi(p(x_{2n_t}, x_{2m_t-1}) + p(x_{2m_t-1}, x_{2n_t}) + p(x_{2n_t}, G(x_{2m_t-1}))\\
&\le \phi(p(x_{2n_t}, x_{2m_t-1}) + p(x_{2m_t-1}, x_{2n_t})\\
&\le \phi(2p(x_{2n_t}, x_{2m_t-1}))\\
&\le \phi(2\varepsilon)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
p(x_{2n_t}, x_{2m_t}) &\le p(x_{2n_t}, x_{2n_t+1}) + p(x_{2n_t+1}, x_{2m_t})\\
&\le p(x_{2n_t}, x_{2n_t+1}) + H^p(F(x_{2n_t}), G(x_{2m_t-1}))\\
&\le p(x_{2n_t}, x_{2n_t+1}) + \phi(2\varepsilon)
\end{aligned}$$

Thus for $n_t, m_t \to \infty$ we have $\varepsilon \le \phi(2\varepsilon)$. Since $\phi(2\varepsilon) < \varepsilon$ then we have a contradiction. Therefore, it can be concluded that (x_n) is Cauchy sequences in X. Since (X, p) is complete partial metric space, then there exist $x \in X$ such that $lim_{n\to\infty} x_n = x$. Furthermore, we will establish that x is common fixed point of F and G.
Let $p(x, F(x)) > 0$. Let we consider that

$$\begin{aligned}
p(x_{2n}, F(x)) &\le H^p(F(x), G(x_{2n-1}))\\
&\le \phi(\max\{p(x, x_{2n-1}), p(x, F(x)), p(x_{2n-1}, G(x_{2n-1})),\\
&\quad p(x, G(x_{2n-1})), p(x_{2n-1}, F(x))\})
\end{aligned} \tag{11}$$

Taking $n \to \infty$ on the inequality (11) above, we obtain

$$\begin{aligned}
p(x, F(x)) &\le \phi(\max\{p(x, F(x)), p(x, F(x))\})\\
&\le \phi(p(x, F(x)) + p(x, F(x)))\\
&= \phi(2p(x, F(x)))\\
&< p(x, F(x)).
\end{aligned}$$

Then we have a contradiction. Hence, $p(x, F(x)) = 0$, i.e., $x \in F(x)$. In the similar way it can be shown that $p(x, G(x)) = 0$, i.e., $x \in G(x)$. It means, x is a common fixed point of set-valued mapping F and G. Furthermore, we will show the uniqueness of this common fixed points.

Suppose that v is another common fixed point of set-valued mappings F and G such that $v \in F(v)$ and $v \in G(v)$. Let $p(x,v) > 0$ then

$$\begin{aligned} H^p(F(x,G(v)) &\leq \phi(\max\{p(x,v),p(x,F(x)),p(v,G(v)),p(x,G(v)),p(v,F(x))\}) \\ &\leq \phi(p(x,v),p(x,G(v)),p(v,F(x))) \\ &\leq \phi(p(x,v),p(v,F(x))) \\ &\leq \phi(p(x,v),p(v,x)) \\ &\leq \phi(2p(x,v)) \end{aligned}$$

Since $p(x,v) \leq H^p(F(x,G(v)) \leq \phi(2p(x,v)) < p(x,v)$, thus we have a contradiction. Hence $p(x,v) = 0$, i.e., $x = v$. Therefore, we can conclude that common fixed point x is unique. This complete the proof.

Further, we have

Corollary 2. *Let $(CB^p(X), H^p)$ be a p-Pompeiu-Hausdorff metric spaces. Suppose that $F : X \to CB^p(X)$ be set-valued mappings which satisfy*

$$H^p(F(x),F(y)) \leq \phi(\max\{p(x,y),p(x,F(x)),p(y,F(y)),p(x,F(y)),p(y,F(x))\}), \quad (12)$$

for each $x,y \in X$ and ϕ as defined in Theorem 5, then set-valued mappings F has a unique fixed point.

The existence of a common fixed point of set-valued mapping that satisfies the contraction as in inequality (6) is the consequence of Theorem 5. For $\phi(\beta u) = \beta u$ where $\beta \in [0, \frac{1}{2})$ in Theorem 5 then we have Corollary 3 below.

Corollary 3. *Let $(CB^p(X), H^p)$ be a p-Pompeiu-Hausdorff metric spaces. Suppose that $F, G : X \to CB^p(X)$ be set-valued mappings which satisfy the contraction as in inequality (6),*

$$H^p(F(x),G(y)) \leq \kappa \max\{p(x,y),p(x,F(x)),p(y,G(y)),p(x,G(y)),p(y,F(x))\},$$

for each $x,y \in X$ and $\kappa \in [0,1)$, then set-valued mappings F and G have a unique common fixed point.

4. Conclusion

In this manuscript, we have established several theorems concerning common fixed points for set-valued mappings. These theorems introduce novel forms of contraction, which extend the Banach contraction principle to set-valued mappings. Among them are contractions for sequences of set-valued mappings, indicating the existence of a common fixed point for the sequence. This common fixed point is then utilized to infer shared fixed points of the set-valued mappings through sequence convergence. Furthermore, we present a new, more general contraction principle. This principle employs a non-decreasing upper semi-continuous ϕ function to construct a contraction mapping which is then used to ensure the existence of common points for the set-valued mappings.

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