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On the McShane-Dunford-Stieltjes Integral and McShane-Pettis-Stieltjes Integral

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Abstract. This paper combines the McShane-Stieltjes integral and Pettis approaches by utilizing Pettis' definition, which coincides with the Dunford integral rather than the version applicable to weakly measurable functions with Lebesgue integrable images. In this way, another integration process without measure theoretic standpoint will be introduced. To this end, we will define the McShane-Dunford-Stieltjes integral and McShane-Pettis-Stieltjes integral in Banach Space and provide its simple properties such as the uniqueness, linearity property of both the integrand and integrator, additivity and formulate the Cauchy criterion of these integrals. In addition, the existence theorem of McShane-Dunford Stieltjes integral will also be presented.

2020 Mathematics Subject Classifications: 40A99, 46BXX, 49Q15, 26A42, 26A45

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1. Introduction

The Kurzweil-Henstock integral, also known as the generalized Riemann integral, in most cases, is an expansion of the Lebesgue integral in the context of Riemann. It is well-known that English mathematician Ralph Henstock and Czech mathematician Jaroslav Kurzweil provided a slight but clever alteration to the conventional Riemann integral to obtain this definition [1]. Unlike the Lebesgue integral, which requires a foundation in measure theory for its definition making it a difficult one, the Kurzweil-Henstock integral is accessible through a more straightforward gauge-based approach.

Now, for real-valued functions the generalization of Riemann integral yields to an integral referred to as the McShane integral, that is, in most cases, equivalent to the Lebesgue integral [4]. Meanwhile, in the late 1960's, McShane [9] proved that the Lebesgue integral is indeed equivalent to a modified version of the Kurzweil-Henstock integral [6]. Note that the Kurzweil-Henstock and McShane integrals differ in how the tagged intervals

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are constructed. In the Kurzweil-Henstock integral, the tags must be within the interval while for the McShane integral the tags can be located outside the interval.

On the other hand, another integration process called Pettis integral was discovered being an extension of the Lebesgue integral to a Banach-valued functions on a measure space. The Pettis integral has two definitions: one for Dunford integrable functions, where the integral coincides with the Dunford integral; the other for weakly measurable functions with Lebesgue integrable images, ensuring the existence of an element satisfying a certain condition for every measurable set [10]. It is also important to note that the convergence theorems of the Dunford integral wherein in a sequence of Dunford integrable function is uniform convergent or weakly convergent, weakly monotone and its limit exists [11]. But Gouju and Schwabik (2002) mentioned that the relation between the Pettis integral and the McShane integral for arbitrary spaces is unknown [5]. Meanwhile, Benitez and Flores [3] introduced the McShane-Stieltjes integral in Banach Space and provided some of its simple properties. Furthermore, the applicability of this study spans diverse areas, including the theory of curve integrals, probability, hysteresis, and functional-differential and generalized differential equations.

2. Preliminaries

This section will be discussing essential notions that are necessary in formulating the McShane-Pettis-Stieltjes integral in a Banach Space. Throughout the paper, we let X to be a Banach Space, $\mathbb{R}^n$ denotes the n-Euclidean space, $\mathbb{R}^+$ is the set of positive real numbers,

$$[\boldsymbol{a},\boldsymbol{b}] = \prod_{i=1}^{n} [a_i, b_i],$$

where $-\infty < a_i < b_i < \infty$ for $i = 1, \cdots, n$ to be a **compact interval** in $\mathbb{R}^n$, $\mathcal{I}_n([\boldsymbol{a},\boldsymbol{b}])$ is the collection of all compact subintervals of $[\boldsymbol{a},\boldsymbol{b}]$ and $V([\boldsymbol{u},\boldsymbol{v}])$ is the collection of all vertices of $[\boldsymbol{u},\boldsymbol{v}]$. Moreover, $\mathbb{R}^n$ is equipped with the maximum norm $\|\cdot\|_{\mathbb{R}^n}$. So, for each $\boldsymbol{x} \in \mathbb{R}^n$, we define $\|\cdot\|$ the **maximum norm** of $\boldsymbol{x}$ by

$$\|\boldsymbol{x}\|_{\mathbb{R}^n} = \max\{|x_i| : i = 1, \cdots, n\},$$

where $\boldsymbol{x} = (x_1, \cdots, x_n)$ and given $r > 0$, we set $B(\boldsymbol{x}, r) = \{\boldsymbol{y} \in \mathbb{R}^n : \|\boldsymbol{x} - \boldsymbol{y}\|_{\mathbb{R}^n} < r\}$, where $\boldsymbol{x} - \boldsymbol{y} = (x_1 - y_2, x_2 - y_2, \cdots, x_n - y_n)$.

Definition 1. [8] *Two compact intervals $[\boldsymbol{q},\boldsymbol{r}], [\boldsymbol{s},\boldsymbol{t}] \in \mathbb{R}^n$ are said to be **non-overlapping** if*

$$\prod_{i=1}^{n}(q_i, r_i) \bigcap \prod_{i=1}^{n}(s_i, t_i) = \varnothing,$$

where $\boldsymbol{q} = (q_1, q_2, \cdots, q_n), \boldsymbol{r} = (r_1, r_2, \cdots, r_n), \boldsymbol{s} = (s_1, s_2, \cdots, s_n)$ and $\boldsymbol{t} = (t_1, t_2, \cdots, t_n)$.

Definition 2. [8] *A function $\delta : [\boldsymbol{a},\boldsymbol{b}] \to \mathbb{R}^+$ is called a **gauge** on $[\boldsymbol{a},\boldsymbol{b}]$.*

Definition 3. [10] *A pair* $(\boldsymbol{t}, [\boldsymbol{u}, \boldsymbol{v}])$ *of a point* $\boldsymbol{t} \in \mathbb{R}^n$ *and a compact interval* $[\boldsymbol{u}, \boldsymbol{v}]$ *of* $\mathbb{R}^n$ *is called a* ***tagged interval****. Here,* $\boldsymbol{t}$ *is known as the tag of* $[\boldsymbol{u}, \boldsymbol{v}]$*.*

Definition 4. [10] *A finite collection* $\{(\boldsymbol{t}_k, [\boldsymbol{u}_k, \boldsymbol{v}_k]), k = 1, 2, \cdots, p\}$ *of pairwise non-overlapping tagged intervals is called an* ***M-system*** *in* $[\boldsymbol{a}, \boldsymbol{b}]$ *if* $[\boldsymbol{u}_k, \boldsymbol{v}_k] \subseteq [\boldsymbol{a}, \boldsymbol{b}]$ *for* $k = 1, 2, \cdots, p$*.*

Definition 5. [10] *An* $\boldsymbol{M}$*-system* $\{(\boldsymbol{t}_k, [\boldsymbol{u}_k, \boldsymbol{v}_k]), k = 1, 2, \cdots, p\}$ *in* $[\boldsymbol{a}, \boldsymbol{b}]$ *is called an* ***M-partition*** *of the interval if* $\bigcup_{k=1}^{p} [\boldsymbol{u}_k, \boldsymbol{v}_k] = [\boldsymbol{a}, \boldsymbol{b}]$*.*

Definition 6. [10] *Given a gauge* δ *defined on* $\{\boldsymbol{t}_1, \cdots, \boldsymbol{t}_p\}$*, a tagged interval* $(\boldsymbol{t}, [\boldsymbol{u}, \boldsymbol{v}])$ *is said to be* ***δ-fine*** *if* $[\boldsymbol{u}, \boldsymbol{v}] \subseteq B(\boldsymbol{t}, \delta(\boldsymbol{t}))$*, where* $B(\boldsymbol{t}, \delta(\boldsymbol{t}))$ *is the open ball in* $\mathbb{R}^n$ *centered at* $\boldsymbol{t}$ *with radius* $\delta(\boldsymbol{t})$*. Here,* $\boldsymbol{M}$*-systems or* $\boldsymbol{M}$*-partitions are called* δ*-fine if all the tagged intervals* $(\boldsymbol{t}_k, [\boldsymbol{u}_k, \boldsymbol{v}_k])$*, where* $k = 1, 2, \cdots, p$*, are* δ*-fine with respect to the gauge* δ*.*

For simplicity, we denote $\{(\boldsymbol{t}_k, [\boldsymbol{u}_k, \boldsymbol{v}_k]), k = 1, 2, \cdots, p\}$ by $\{(\boldsymbol{t}, [\boldsymbol{u}, \boldsymbol{v}])\}$.

Definition 7. *[12]* Let δ be a gauge defined on $\{\boldsymbol{t}\}$. An M-system $P = \{(\boldsymbol{t}, [\boldsymbol{u}, \boldsymbol{v}])\}$ is said to be a **δ-fine division** of $[\boldsymbol{a}, \boldsymbol{b}]$ if $\boldsymbol{t} \in [\boldsymbol{u}, \boldsymbol{v}] \subseteq B(\boldsymbol{t}, \delta(\boldsymbol{t}))$.

Lemma 1. [8] ***(Cousin's Lemma)*** *If* δ *is a gauge on* $[\boldsymbol{a}, \boldsymbol{b}]$*, then there exists a* δ*-fine M-Partition of* $[\boldsymbol{a}, \boldsymbol{b}]$*.*

Definition 8. [8] Let $g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$. The **total variation of g** over $[\boldsymbol{a}, \boldsymbol{b}]$ is given by

$$Var(g, [\boldsymbol{a}, \boldsymbol{b}]) = sup\left\{ \sum_{[\boldsymbol{u}, \boldsymbol{v}] \in \mathcal{D}} |\Delta_g([\boldsymbol{u}, \boldsymbol{v}])| : \mathcal{D} \text{ is a partition of } [\boldsymbol{a}, \boldsymbol{b}] \right\}$$

where

$$\Delta_g([\boldsymbol{u}, \boldsymbol{v}]) = \sum_{\boldsymbol{t} \in V([\boldsymbol{u}, \boldsymbol{v}])} \left(g(\boldsymbol{t}) \prod_{k=1}^{n} (-1)^{\chi_{\{u_k\}}(t_k)} \right) \qquad (\star)$$

and $[\boldsymbol{u}, \boldsymbol{v}] \in \mathcal{I}_n\big([\boldsymbol{a}, \boldsymbol{b}]\big)$.

Definition 9. [8] *A function* $g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ *is said to be of* ***bounded variation*** *on* $[\boldsymbol{a}, \boldsymbol{b}]$ *if* $Var(g, [\boldsymbol{a}, \boldsymbol{b}])$ *is finite.*

Definition 10. [2] *Let* f *be Banach-valued function defined on* $[\boldsymbol{a}, \boldsymbol{b}]$ *and* g *be a real-valued function defined* $[\boldsymbol{a}, \boldsymbol{b}]$*. A function* f *is said to be* ***McShane-Stieltjes integrable****, or simply* ***MS-integrable****, with respect to* g *on* $[\boldsymbol{a}, \boldsymbol{b}]$ *if there exists* $J \in X$ *with the following property: for each* $\varepsilon > 0$ *there is a gauge* δ *such that for every* δ*-fine M-partition* $P = \{(\boldsymbol{t}, [\boldsymbol{u}, \boldsymbol{v}])\}$ *of* $[\boldsymbol{a}, \boldsymbol{b}]$ *the inequality*

$$\left\| \sum_{(\boldsymbol{t}, [\boldsymbol{u}, \boldsymbol{v}]) \in P} f(\boldsymbol{t}) \Delta_g([\boldsymbol{u}, \boldsymbol{v}]) - J \right\|_X < \varepsilon$$

holds. In this case, the McShane-Stieltjes integral is $J = (MS)\int_{[\boldsymbol{a},\boldsymbol{b}]} f dg$. *For brevity, denote* $S(f, g, P) = \sum_{(\boldsymbol{t},[\boldsymbol{u},\boldsymbol{v}])\in P} f(\boldsymbol{t})\Delta_g([\boldsymbol{u},\boldsymbol{v}])$ *and* $\boldsymbol{MS}([\boldsymbol{a},\boldsymbol{b}], g)$ *be the collection of* $\boldsymbol{MS}$-*integrable function with respect to* g *on* $[\boldsymbol{a},\boldsymbol{b}]$.

Definition 11. [3] *Let* $f : [\boldsymbol{a},\boldsymbol{b}] \to X$ *and* $g : [\boldsymbol{a},\boldsymbol{b}] \to \mathbb{R}$ *be a function. We say that* f *is PUL-**Stieltjes integrable** to* $A \in X$ *with respect to* g *on* $[\boldsymbol{a},\boldsymbol{b}]$ *if for every* $\varepsilon > 0$, *there exists a gauge* δ *on* $[\boldsymbol{a},\boldsymbol{b}]$ *such that for every* δ-*fine division* D *of* $[\boldsymbol{a},\boldsymbol{b}]$, *we have* $\|S(f, g, D) - A)\| < \varepsilon$. *In this case,* A *is the PUL-Stieltjes integral of* f *with respect to* g *and we write* $A = (P)\int_{[\boldsymbol{a},\boldsymbol{b}]} f dg$.

Theorem 1. [2] *A function* $f : [\boldsymbol{a},\boldsymbol{b}] \to X$ *is PUL-Stieltjes integrable with respect to* $g : [\boldsymbol{a},\boldsymbol{b}] \to \mathbb{R}$ *if and only if* f *is McShane-Stieltjes integrable with respect to* g *on* $[\boldsymbol{a},\boldsymbol{b}]$. *Moreover,*

$$(P)\int_{[\boldsymbol{a},\boldsymbol{b}]} f dg = (MS)\int_{[\boldsymbol{a},\boldsymbol{b}]} f dg.$$

Definition 12. [10] *If* $f : [\boldsymbol{a},\boldsymbol{b}] \to X$ *is weakly measurable such that the function* $x^*(f) : [\boldsymbol{a},\boldsymbol{b}] \to \mathbb{R}$ *is McShane integrable for each* $x^* \in X^*$ *then* f *is called **Dunford integrable**. The Dunford integral* $(\boldsymbol{D})\int_{\boldsymbol{E}} f$ *of* f *over a measurable set* $\boldsymbol{E} \subseteq [\boldsymbol{a},\boldsymbol{b}]$ *is defined by the element* $x^{**}_{\boldsymbol{E}} \in X^{**}$, *that is,*

$$(D)\int_{\boldsymbol{E}} f = x^{**}_{\boldsymbol{E}} \in X^{**},$$

where $x^{**}_{\boldsymbol{E}}(x^*) = (\boldsymbol{D})\int_{\boldsymbol{E}} x^*(f)$ *for all* $x^* \in X^*$. *Here, denote by* $\boldsymbol{D}[\boldsymbol{a},\boldsymbol{b}]$ *the set of all Dunford integrable functions on* $[\boldsymbol{a},\boldsymbol{b}]$.

Definition 13. [10] *If* $f : [\boldsymbol{a},\boldsymbol{b}] \to X$ *is Dunford integrable where* $(\boldsymbol{D})\int_{\boldsymbol{E}}(f)$ *in* X *(or more precisely* $(\boldsymbol{D})\int_{\boldsymbol{E}} f \in \mathfrak{e}(X) \subseteq X^{**}$, *where* $\mathfrak{e}$ *is the canonical embedding of* X *into* X^{**}*) for every measurable* $\boldsymbol{E} \subseteq [\boldsymbol{a},\boldsymbol{b}]$, *then* f *is called **Pettis integrable** and*

$$(\boldsymbol{P})\int_{\boldsymbol{E}} f = (\boldsymbol{D})\int_{\boldsymbol{E}} f$$

is called the Pettis integral of f *over the set* $\boldsymbol{E}$.

Definition 14. [7] *A **linear functional*** f *is a linear operator with domain in a vector space* X *and range in the scalar field* K *of* X, *that is,* $f : X \to K$, *where* $K = \mathbb{R}$ *or* $K = \mathbb{C}$.

Definition 15. [2] *Let $f : [\boldsymbol{a}, \boldsymbol{b}] \to X$ be any Banach-valued function. We say that f is continuous at a point $\boldsymbol{y} \in [\boldsymbol{a}, \boldsymbol{b}]$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $\boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}]$ with $\|\boldsymbol{x} - \boldsymbol{y}\|_{\mathbb{R}^n} < \delta$, we have $|f(\boldsymbol{x}) - f(\boldsymbol{y})| < \varepsilon$.*

Definition 16. [7] *Let X be a vector space and $x \in X$. Define $X^* = \{f : X \to K \mid f \text{ is a linear functional }\}$ and the following operations in X^*:*

(i) $(f_1 + f_2)(x) = f_1(x) + f_2(x)$; *and*

(ii) $(\alpha f)(x) = \alpha(f(x))$.
Then $(X^*, +, \cdot)$ is a vector space called the ***algebraic dual space****.*

Definition 17. [7] *Let X be a vector space and fix $x \in X$. Define $X^{**} = \{g : X^* \to K \mid g(f) = g_x(f) = f(x) \quad \forall\ f \in X^* \text{ is a linear functional }\}$ and the following operations in X^{**}:*

(i) $(g_1 + g_2)(f) = g_1(f) + g_2(f)$; *and*

(ii) $(\alpha g)(f) = \alpha(g(f))$.
Then $(X^{**}, +, \cdot)$ is a vector space called the ***second algebraic dual space****.*

3. McShane-Dunford-Stieltjes Integral and McShane-Pettis-Stieltjes Integral in Banach Space

Before going through the paper an important proposition is provided.

Proposition 1. *Let $f : [\boldsymbol{a}, \boldsymbol{b}] \to X$ and $g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ be functions. If f is McShane-Stieltjes integrable with $(MS)\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg \in X$ with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$. Then for all $x^* \in X^*$ the real function $x^*(f) : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ is McShane-Stieltjes integrable and*

$$(MS)\int_{[\boldsymbol{a},\boldsymbol{b}]} x^*(f) dg = x^*\big((MS)\int_{[\boldsymbol{a},\boldsymbol{b}]} f dg\big).$$

Proof. Suppose that f is MS-integrable to $A = (MS)\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg$ with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$. Let $x^* \in X^*$ and let $\varepsilon > 0$. Then there exists a gauge δ such that

$$\left\| \sum_{(\boldsymbol{t},[\boldsymbol{u},\boldsymbol{v}])\in \boldsymbol{P}} f(\boldsymbol{t})\Delta_g([\boldsymbol{u}, \boldsymbol{v}]) - A \right\|_X < \frac{\varepsilon}{\|x^*\|_{X^*} + 1}.$$

for every δ-fine M-partition $\boldsymbol{P} = \{(\boldsymbol{t}, [\boldsymbol{u}, \boldsymbol{v}])\}$ of $[\boldsymbol{a}, \boldsymbol{b}]$. Observe that

$$\left\| \sum_{(\boldsymbol{t},[\boldsymbol{u},\boldsymbol{v}])\in \boldsymbol{P}} (x^*f)(\boldsymbol{t})\Delta_g([\boldsymbol{u}, \boldsymbol{v}]) - x^*\Big((MS)\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg\Big) \right\|_{X^*}$$

$$= \left\| x^*\Big(\sum_{(\boldsymbol{t},[\boldsymbol{u},\boldsymbol{v}])\in \boldsymbol{P}} f(\boldsymbol{t})\Delta_g([\boldsymbol{u},\boldsymbol{v}])\Big) - x^*\Big((MS)\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg\Big)\right\|_{X^*}$$

$$= \left\| x^*\Big(\sum_{(\boldsymbol{t},[\boldsymbol{u},\boldsymbol{v}])\in \boldsymbol{P}} f(\boldsymbol{t})\Delta_g([\boldsymbol{u},\boldsymbol{v}]) - (MS)\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg\Big)\right\|_{X^*}$$

$$\leq \|x^*\|_{X^*}\left\| \sum_{(\boldsymbol{t},[\boldsymbol{u},\boldsymbol{v}])\in \boldsymbol{P}} f(\boldsymbol{t})\Delta_g([\boldsymbol{u},\boldsymbol{v}]) - (MS)\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg\right\|_{X^*}$$

$$< \|x^*\|_{X^*}\frac{\varepsilon}{\|x^*\|_{X^*}+1} = \varepsilon.$$

Since ε is arbitrarily chosen, we have $(MS)\int_{[\boldsymbol{a},\boldsymbol{b}]} x^*(f)dg = x^*\big((MS)\int_{[\boldsymbol{a},\boldsymbol{b}]} fdg\big)$; which means that $x^*(f)$ is MS-integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$ integrable for all $x^* \in X^*$. □

In this part, the McShane-Dunford-Stieltjes integral is established as it is a necessary concept in introducing the McShane-Pettis-Stieltjes integral.

Definition 18. *Let $f : [\boldsymbol{a},\boldsymbol{b}] \to X$ and $g : [\boldsymbol{a},\boldsymbol{b}] \to \mathbb{R}$ be functions. If the function $x^*(f) : [\boldsymbol{a},\boldsymbol{b}] \to \mathbb{R}$ is McShane-Stieltjes integrable for all $x^* \in X^*$ with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$ and if for every interval $\boldsymbol{J} \subseteq [\boldsymbol{a},\boldsymbol{b}]$, there exists an element $x^{**}_{\boldsymbol{J}} \in X^{**}$ such that $x^{**}_{\boldsymbol{J}}(x^*) = (MS)\int_{\boldsymbol{J}} x^*(f)\ dg$ for all $x^* \in X^*$, then f is called* ***McShane-Dunford-Stieltjes integrable*** *(or simply MDS-integrable) with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$. For an interval $\boldsymbol{J} \subseteq [\boldsymbol{a},\boldsymbol{b}]$, we write the MDS integral of f with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$ by*

$$(MDS)\int_{\boldsymbol{J}} f\ dg = x^{**}_{\boldsymbol{J}} \in X^{**}.$$

Denote by $MDS([\boldsymbol{a},\boldsymbol{b}],g)$ the set of all McShane-Dunford-Stieltjes integrable functions $f : [\boldsymbol{a},\boldsymbol{b}] \to X$ with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$.

Theorem 2. *There is at most one value satisfying Definition 18.*

Proof. Assume that f is MDS-integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$. By Definition 18, $x^*(f)$ is MS- integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$ for all $x^* \in X^*$. For each interval $\boldsymbol{J} \subseteq [\boldsymbol{a},\boldsymbol{b}]$, there is an element $x^{**}_{\boldsymbol{J}} \in X^{**}$ such that

$$x^{**}_{\boldsymbol{J}}(x^*) = (MS)\int_{\boldsymbol{J}} x^*(f)\ dg \quad \forall x^* \in X^*.$$

Now, for an interval $\boldsymbol{J} \subseteq [\boldsymbol{a},\boldsymbol{b}]$. Suppose that $x^{**}_{\boldsymbol{J}}, y^{**}_{\boldsymbol{J}} \in X^{**}$ are the values of MDS integral of f with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$. Let $x^* \in X^*$. Then $x^*(f)$ is McShane-Stieltjes integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$. But

$$x^{**}_{\boldsymbol{J}}(x^*) = (MS)\int_{\boldsymbol{J}} x^*(f)\ dg = y^{**}_{\boldsymbol{J}}(x^*).$$

Hence, $x_{\boldsymbol{J}}^{**} = (MDS)\int_{\boldsymbol{J}} f\ dg = y_{\boldsymbol{J}}^{**}$. This means that $x_{\boldsymbol{J}}^{**} = y_{\boldsymbol{J}}^{**}$. Thus, $(MDS)\int_{\boldsymbol{J}} f\ dg$ is unique. □

Theorem 3. ***(Linearity of MDS-integral over the Integrand)***

Let $f_1, f_2 : [\boldsymbol{a}, \boldsymbol{b}] \to X$ and $g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ be functions. If $f_1, f_2 \in MDS([\boldsymbol{a}, \boldsymbol{b}], g)$, then for every $\alpha, \beta \in \mathbb{R}$, $\alpha f_1 + \beta f_2 \in MDS([\boldsymbol{a}, \boldsymbol{b}], g)$ and for all $\boldsymbol{J} \subseteq [\boldsymbol{a}, \boldsymbol{b}]$

$$(MDS)\int_{\boldsymbol{J}} (\alpha f_1 + \beta f_2)\ dg = \alpha \cdot (MDS)\int_{\boldsymbol{J}} f_1\ dg + \beta \cdot (MDS)\int_{\boldsymbol{J}} f_2\ dg.$$

Proof. Fix $x^* \in X^*$. Assume that f_1, f_2 is MDS-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$. Then $x^*(f_1)$ and $x^*(f_2)$ is MS-integrable. This further means that $x^*(f_1) + x^*(f_2)$ is MS-integrable. Now, let $\alpha, \beta \in \mathbb{R}$. Notice that

$$\begin{aligned} x^*(\alpha f_1 + \beta f_2) &= x^*(\alpha f_1) + x^*(\beta f_2) \\ &= \alpha \cdot x^*(f_1) + \beta \cdot x^*(f_2). \end{aligned}$$

And so, $\alpha \cdot x^*(f_1) + \beta \cdot x^*(f_2)$ is also MS-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$. By the linearity property of the MS-integral and for an interval $\boldsymbol{J} \subseteq [\boldsymbol{a}, \boldsymbol{b}]$,

$$(MS)\int_{\boldsymbol{J}} [\alpha \cdot x^*(f_1) + \beta \cdot x^*(f_2)]\ dg = \alpha \cdot (MS)\int_{\boldsymbol{J}} x^*(f_1) dg + \beta \cdot (MS)\int_{\boldsymbol{J}} x^*(f_2) dg.$$

Fix an interval $\boldsymbol{J} \subseteq [\boldsymbol{a}, \boldsymbol{b}]$. Since $f_1, f_2 \in MDS([\boldsymbol{a}, \boldsymbol{b}], g)$, it follows that we can pick operators $x_{\boldsymbol{J}}^{**}, y_{\boldsymbol{J}}^{**} \in X^{**}$ such that

$$x_{\boldsymbol{J}}^{**}(x^*) = (MS)\int_{\boldsymbol{J}} x^*(f_1)\ dg \quad \text{and} \quad y_{\boldsymbol{J}}^{**}(x^*) = (MS)\int_{\boldsymbol{J}} x^*(f_2)\ dg \quad \forall x^* \in X^*.$$

Now, $\alpha x_{\boldsymbol{J}}^{**} + \beta y_{\boldsymbol{J}}^{**} \in X^{**}$. Take $w_{\boldsymbol{J}}^{**} = \alpha x_{\boldsymbol{J}}^{**} + \beta y_{\boldsymbol{J}}^{**}$. Fix $x^* \in X^*$. Then

$$\begin{aligned} w_{\boldsymbol{J}}^{**}(x^*) &= (\alpha x_{\boldsymbol{J}}^{**} + \beta y_{\boldsymbol{J}}^{**})(x^*) \\ &= \alpha x_{\boldsymbol{J}}^{**}(x^*) + \beta y_{\boldsymbol{J}}^{**}(x^*) \\ &= \alpha (MS)\int_{\boldsymbol{J}} x^*(f_1)\ dg + \beta (MS)\int_{\boldsymbol{J}} x^*(f_2)\ dg \\ &= (MS)\int_{\boldsymbol{J}} \alpha \cdot x^*(f_1)\ dg + (MS)\int_{\boldsymbol{J}} \beta \cdot x^*(f_2)\ dg \\ &= (MS)\int_{\boldsymbol{J}} [\alpha \cdot x^*(f_1) + \beta \cdot x^*(f_2)]\ dg \\ &= (MS)\int_{\boldsymbol{J}} x^*(\alpha f_1 + \beta f_2)\ dg. \end{aligned}$$

Consequently, $\alpha f_1 + \beta f_2$ is MDS-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$. Therefore,

$$\begin{aligned} (MDS)\int_{\boldsymbol{J}} (\alpha f_1 + \beta f_2)\ dg &= w_{\boldsymbol{J}}^{**} = \alpha x_{\boldsymbol{J}}^{**} + \beta y_{\boldsymbol{J}}^{**} \\ &= \alpha \cdot (MDS)\int_{\boldsymbol{J}} f_1\ dg + \beta \cdot (MDS)\int_{\boldsymbol{J}} f_2\ dg. \end{aligned}$$

□

Theorem 4. ***(Linearity of MDS-integral over the Integrator)***

Let $f : [\boldsymbol{a}, \boldsymbol{b}] \to X$ and $g_1, g_2 : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ be functions. If f is MDS-integrable with respect to g_1 and g_2 on $[\boldsymbol{a}, \boldsymbol{b}]$, then $f \in MDS([\boldsymbol{a}, \boldsymbol{b}], \alpha g_1 + \beta g_2)$ and for every $\boldsymbol{J} \subseteq [\boldsymbol{a}, \boldsymbol{b}]$

$$(MDS)\int_{\boldsymbol{J}} f \ d[\alpha g_1 + \beta g_2] = \alpha \cdot (MDS)\int_{\boldsymbol{J}} f \ dg_1 + \beta \cdot (MDS)\int_{\boldsymbol{J}} f \ dg_2.$$

Proof. Fix $x^* \in X^*$. Assume that f is McShane-Dunford-Stieltjes integrable with respect to g_1 and g_2 on $[\boldsymbol{a}, \boldsymbol{b}]$. Then $x^*(f)$ is MS-integrable with respect to g_1 and g_2 on $[\boldsymbol{a}, \boldsymbol{b}]$. Utilizing the linearity property of MS-integral over an integrator and for an interval $J \subseteq [\boldsymbol{a}, \boldsymbol{b}]$,

$$(MS)\int_{[\boldsymbol{a},\boldsymbol{b}]} x^*(f) d[\alpha g_1 + \beta g_2] = \alpha (MS)\int_{[\boldsymbol{a},\boldsymbol{b}]} x^*(f) dg_1 + \beta (MS)\int_{[\boldsymbol{a},\boldsymbol{b}]} x^*(f) dg_2.$$

Let an interval $\boldsymbol{J} \subseteq [\boldsymbol{a}, \boldsymbol{b}]$. Since f is MDS-integrable with respect to g_1 and g_2 on $[\boldsymbol{a}, \boldsymbol{b}]$, it implies that we can choose operators $x^{**}_{\boldsymbol{J}}, y^{**}_{\boldsymbol{J}} \in X^{**}$ such that

$$x^{**}_{\boldsymbol{J}}(x^*) = (MS)\int_{\boldsymbol{J}} x^*(f) \ dg_1 \quad \text{and} \quad y^{**}_{\boldsymbol{J}}(x^*) = (MS)\int_{\boldsymbol{J}} x^*(f) \ dg_2 \quad \forall x^* \in X^*.$$

Note that $\alpha x^{**}_{\boldsymbol{J}} + \beta y^{**}_{\boldsymbol{J}} \in X^{**}$. Then write $w^{**}_{\boldsymbol{J}} = \alpha x^{**}_{\boldsymbol{J}} + \beta y^{**}_{\boldsymbol{J}}$. Fix $x^* \in X^*$. Now,

$$\begin{aligned} w^{**}_{\boldsymbol{J}}(x^*) &= (\alpha x^{**}_{\boldsymbol{J}} + \beta y^{**}_{\boldsymbol{J}})(x^*) \\ &= \alpha x^{**}_{\boldsymbol{J}}(x^*) + \beta y^{**}_{\boldsymbol{J}}(x^*) \\ &= \alpha (MS)\int_{\boldsymbol{J}} x^*(f) \ dg_1 + \beta (MS)\int_{\boldsymbol{J}} x^*(f) \ dg_2 \\ &= (MS)\int_{\boldsymbol{J}} x^*(f) d[\alpha g_1 + \beta g_2]. \end{aligned}$$

And so, we see that f is MDS-integrable with respect to $\alpha g_1 + \beta g_2$. Thus,

$$\begin{aligned} (MDS)\int_{\boldsymbol{J}} f \ d[\alpha g_1 + \beta g_2] &= w^{**}_{\boldsymbol{J}} = \alpha x^{**}_{\boldsymbol{J}} + \beta y^{**}_{\boldsymbol{J}} \\ &= \alpha \cdot (MDS)\int_{\boldsymbol{J}} f \ dg_1 + \beta \cdot (MDS)\int_{\boldsymbol{J}} f \ dg_2. \end{aligned}$$

□

Theorem 5. *Let $g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ be a function. A function $f : [\boldsymbol{a}, \boldsymbol{b}] \to X$ is McShane-Dunford-Stieltjes integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$ if and only if $x^*(f) : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ is McShane-Stieltjes integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$ for all $x^* \in X^*$.*

Proof. Suppose that f is MDS-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$. By Definition 18, $x^*(f)$ is McShane-Stieltjes integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$ and so we are done. Conversely, assume that $x^*(f) : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ is McShane-Stieltjes integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$ for all $x^* \in X^*$. Let an interval $\boldsymbol{J} \subseteq [\boldsymbol{a}, \boldsymbol{b}]$. Then $x^*(f)$ is McShane-Stieltjes integrable with respect to g on $\boldsymbol{J}$ implying that $(MS)\int_{\boldsymbol{J}} x^*(f) dg \in \mathbb{R}$. Now, let

$x^{**} : X^* \to \mathbb{R}$ by setting $x^{**}_{\boldsymbol{J}}(x^*) = (MS)\displaystyle\int_{\boldsymbol{J}} x^*(f)dg$ for all $x^* \in X^*$. This means that $x^{**}_{\boldsymbol{J}} \in X^{**}$. Hence, f is MDS-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$. □

Theorem 6. ***(Cauchy Criterion of the MDS-integral)***

Let $f : [\boldsymbol{a}, \boldsymbol{b}] \to X$ and $g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ be functions. A function f is McShane-Dunford-Stieltjes integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$ if and only if for every $\varepsilon > 0$, there exists a gauge $\delta : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}^+$ such that if N_1 and N_2 are two δ-fine M-partitions, then

$$\|S(f, g, N_1) - S(f, g, N_2)\|_X < \varepsilon.$$

Proof. Suppose that f is McShane-Dunford-Stieltjes integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$. By Theorem 5, $x^*(f) : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ is McShane-Stieltjes integrable with respect to to g on $[\boldsymbol{a}, \boldsymbol{b}]$ for all $x^* \in X^*$. If $x^* \in X^*$ is the zero map so that $x^*(x) = 0$ for all $x \in X$. Then we can choose $\delta : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}^+$ such that for any two δ-fine M-partitions N_1 and N_2 of $[\boldsymbol{a}, \boldsymbol{b}]$,

$$\|S(x^*(f), g, N_1) - S(x^*(f), g, N_2)\|_{X^*} = \|x^*S(f, g, N_1) - x^*S(f, g, N_2)\|_{X^*} = \|0 - 0\| < \varepsilon.$$

If $x^* \in X^*$ is not the zero map. Then we can pick $\delta : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}^+$ such that for every two δ-fine M-partitions N_1 and N_2 of $[\boldsymbol{a}, \boldsymbol{b}]$, we have

$$\|x^*[S(f, g, N_1) - S(f, g, N_2)]\|_{X^*} = \|S(x^*(f), g, N_1) - S(x^*(f), g, N_2)\|_{X^*} < \|x^*\|_{X^*} \cdot \varepsilon.$$

This implies that

$$\|x^*\|_{X^*}\|S(f, g, N_1) - S(f, g, N_2)\|_{X^*} < \|x^*\|_{X^*} \cdot \varepsilon.$$

In other words,

$$\|S(f, g, N_1) - S(f, g, N_2)\|_X < \varepsilon.$$

Conversely, let $x^* \in X^*$. Suppose that for every $\varepsilon > 0$, there exists a gauge $\delta : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}^+$ such that if N_1 and N_2 are two δ-fine M-partitions, then

$$\|S(f, g, N_1) - S(f, g, N_2)\|_X < \frac{\varepsilon}{\|x^*\|_{X^*} + 1} \tag{1}$$

By (1), we obtain

$$\begin{aligned}
\|S(x^*(f), g, N_1) - S(x^*(f), g, N_2)\|_{X^*} &= \|x^*[S(f, g, N_1) - S(f, g, N_2)]\|_X \\
&\le \|x^*\|_{X^*}\|S(f, g, N_1) - S(f, g, N_2)\|_X \\
&< \|x^*\|_{X^*} \cdot \frac{\varepsilon}{\|x^*\|_{X^*} + 1} \\
&< \varepsilon.
\end{aligned}$$

This implies that $x^*(f) : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ is MS-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$. Fix $\boldsymbol{J} \subseteq [\boldsymbol{a}, \boldsymbol{b}]$ be a compact interval. Define $x^{**} : X^* \to \mathbb{R}$ such that

$$x^{**}_{\boldsymbol{J}}(x^*) = (MS)\int_{\boldsymbol{J}} x^*(f)dg.$$

Now, by definition of X^{**}, $x^{**}_{\boldsymbol{J}} \in X^{**}$ which means that $x^{**}_{\boldsymbol{J}}$ is linear. Consequently, $x^{**}_{\boldsymbol{J}} = (MDS)\int_{\boldsymbol{J}} fdg$. Thus, f is MDS-integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$. □

Theorem 7. ***(Additivity of the MDS-integral)***
Let $f:[\boldsymbol{a},\boldsymbol{b}]\to X$, $g:[\boldsymbol{a},\boldsymbol{b}]\to\mathbb{R}$ and $\boldsymbol{I},\boldsymbol{J}\in\mathcal{I}_n\big([\boldsymbol{a},\boldsymbol{b}]\big)$ that forms an M-partition of $[\boldsymbol{a},\boldsymbol{b}]$. Assume that $f\in MDS(\boldsymbol{I},g)\cap MDS(\boldsymbol{J},g)$, then $f\in MDS([\boldsymbol{a},\boldsymbol{b}],g)$ and

$$(MDS)\int_{[\boldsymbol{a},\boldsymbol{b}]} fdg = (MDS)\int_{\boldsymbol{I}} fdg + (MDS)\int_{\boldsymbol{J}} fdg.$$

Theorem 8. *Let $f:[\boldsymbol{a},\boldsymbol{b}]\to X$ and $g:[\boldsymbol{a},\boldsymbol{b}]\to\mathbb{R}$ be functions. If f is MDS-integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}$, then for each $\boldsymbol{J}\subseteq\mathcal{I}_n\big([\boldsymbol{a},\boldsymbol{b}]\big)$,*

$$(MDS)\int_{\boldsymbol{J}} fdg = (MDS)\int_{[\boldsymbol{a},\boldsymbol{b}]} f\cdot\chi_{\boldsymbol{J}}dg.$$

Proof. Let $f,g:[\boldsymbol{a},\boldsymbol{b}]\to X$ be functions and $\boldsymbol{J}\in\mathcal{I}_n\big([\boldsymbol{a},\boldsymbol{b}]\big)$. Assume that f is MDS-integrable with respect to g over $[\boldsymbol{a},\boldsymbol{b}]$. If $\boldsymbol{J}=[\boldsymbol{a},\boldsymbol{b}]$, then

$$(MDS)\int_{\boldsymbol{J}} f\cdot\chi_{\boldsymbol{J}}\ dg = (MDS)\int_{[\boldsymbol{a},\boldsymbol{b}]} f\cdot\chi_{[\boldsymbol{a},\boldsymbol{b}]}\ dg = (MDS)\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg = (MDS)\int_{\boldsymbol{J}} f\ dg.$$

If $\boldsymbol{I}\subset[\boldsymbol{a},\boldsymbol{b}]$, then f is $\boldsymbol{MDS}$-integrable with respect to g on $\boldsymbol{J}$. Notice that

$$[\boldsymbol{a},\boldsymbol{b}] = ([\boldsymbol{a},\boldsymbol{b}]\setminus\boldsymbol{J})\ \cup\ \boldsymbol{J}.$$

Now,

$$\begin{aligned}(MDS)\int_{[\boldsymbol{a},\boldsymbol{b}]} f\cdot\chi_{\boldsymbol{J}}\ dg &= (MDS)\int_{[\boldsymbol{a},\boldsymbol{b}]\setminus\boldsymbol{J}} f\cdot\chi_{\boldsymbol{J}}\ dg + (MDS)\int_{\boldsymbol{J}} f\cdot\chi_{\boldsymbol{J}}\ dg\\ &= (MDS)\int_{[\boldsymbol{a},\boldsymbol{b}]\setminus\boldsymbol{J}} f\cdot(0)\ dg + (MDS)\int_{\boldsymbol{J}} f\cdot(1)\ dg\\ &= (MDS)\int_{\boldsymbol{J}} f\ dg.\end{aligned}$$

Thus,

$$(MDS)\int_{[\boldsymbol{a},\boldsymbol{b}]} f\cdot\chi_{\boldsymbol{J}}\ dg = (MDS)\int_{\boldsymbol{J}} f\ dg.$$ □

.

Proposition 2. *If $f:[\boldsymbol{a},\boldsymbol{b}]\to X$ is continuous on $[\boldsymbol{a},\boldsymbol{b}]$, then for each $x^*\in X^*$, $x^*(f):[\boldsymbol{a},\boldsymbol{b}]\to\mathbb{R}$ is also continuous on $[\boldsymbol{a},\boldsymbol{b}]$.*

Proof. Let $\varepsilon>0$ and $x^*\in X^*$. Suppose that $f:[\boldsymbol{a},\boldsymbol{b}]\to X$ is continuous on $[\boldsymbol{a},\boldsymbol{b}]$. Then for every $\varepsilon>0$, there exists a $\delta>0$ such that for every $\boldsymbol{u},\boldsymbol{v}\in[\boldsymbol{a},\boldsymbol{b}]$ with $\|\boldsymbol{u}-\boldsymbol{v}\|_{\mathbb{R}^n}<\delta$, we have

$$|f(\boldsymbol{u})-f(\boldsymbol{v})| < \frac{\varepsilon}{\|x^*\|_{X^*}+1}.$$

Now,

$$\begin{aligned}\|x^*(f(\boldsymbol{u})) - x^*(f(\boldsymbol{v}))\|_{X^*} &= \|x^*[f(\boldsymbol{u}) - f(\boldsymbol{v})]\|_{X^*} \\ &\leq \|x^*\|_{X^*}\|f(\boldsymbol{u}) - f(\boldsymbol{v})\| \\ &\leq \|x^*\|_{X^*}|f(\boldsymbol{u}) - f(\boldsymbol{v})| \\ &< \|x^*\|_{X^*}\frac{\varepsilon}{\|x^*\|_{X^*}+1} = \varepsilon.\end{aligned}$$

And so, $x^*(f)$ is continuous. □

Theorem 9. ***(Existence Theorem of** MDS**-integral)***

Let f be continuous on $[\boldsymbol{a}, \boldsymbol{b}]$ and g be a function of bounded variation. Then f is MDS-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$.

Proof. Fix $x^* \in X^*$. Suppose that f is continuous on $[\boldsymbol{a}, \boldsymbol{b}]$ and g be a function of bounded variation. Applying Proposition 2, $x^*(f)$ is continuous. By the Existence Theorem of the PUL-Stieltjes integral and Theorem 1, $x^*(f)$ is continuous and g is a function of bounded variation means that the McShane-Stieltjes integral of $x^*(f)$ with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$ exists. And so, $x^*(f)$ is MS-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$. Let $\boldsymbol{J}$ be a compact subinterval of $[\boldsymbol{a}, \boldsymbol{b}]$. Then $x^*(f)$ is also MS-integrable with respect to g on $\boldsymbol{J}$. Take $x^{**}_{\boldsymbol{J}}(x^*) = (MS)\int_{\boldsymbol{J}} x^*(f)dg$ implying that $x^{**}_{\boldsymbol{J}} \in X^{**}$. This means that $x^{**}_{\boldsymbol{J}} = (MDS)\int_{\boldsymbol{J}} fdg$. Therefore, f is MDS-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$. □

Definition 19. *Let $f : [\boldsymbol{a}, \boldsymbol{b}] \to X$ and $g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ be functions. If f is MDS-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$ where $(MDS)\int_{\boldsymbol{J}} fdg \in X$ for every interval $\boldsymbol{J} \subseteq [\boldsymbol{a}, \boldsymbol{b}]$ (more precisely $(MDS)\int_{\boldsymbol{J}} fdg \in e(X) \subseteq X^{**}$, where e is the canonical embedding of X into X^{**}), then f is called **McShane-Pettis-Stieltjes integrable** (or simply MPS-integrable) with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$ and*

$$(MDS)\int_{\boldsymbol{J}} f\ dg = (MPS)\int_{\boldsymbol{J}} f\ dg \in X$$

*is the **McShane-Pettis-Stieltjes integral** of f with respect to g on $\boldsymbol{J} \subseteq [\boldsymbol{a}, \boldsymbol{b}]$. Here, we write $\boldsymbol{MPS}([\boldsymbol{a}, \boldsymbol{b}], g)$ the set of all McShane-Pettis-Stieltjes integrable functions $f : [\boldsymbol{a}, \boldsymbol{b}] \to X$ with respect to g on $\boldsymbol{J} \subseteq [\boldsymbol{a}, \boldsymbol{b}]$.*

Theorem 10. *There is at most one value satisfying Definition 19*

Proof. Suppose that $(MPS)\int_{\boldsymbol{J}} f\ dg$ exists $\forall\ \boldsymbol{J} \subseteq [\boldsymbol{a}, \boldsymbol{b}]$. By Definition 19,

$$(MDS)\int_{\boldsymbol{J}} f\ dg = (MPS)\int_{\boldsymbol{J}} f\ dg.$$

Now, fix $\boldsymbol{J} \subseteq [\boldsymbol{a}, \boldsymbol{b}]$. From Theorem 2, $(MDS)\int_{\boldsymbol{J}} f\ dg$ is unique, then $(MPS)\int_{\boldsymbol{J}} f\ dg$ is also unique. □

Theorem 11. ***(Linearity of Integrand of the MPS-integral)***
Let $f_1, f_2 : [\boldsymbol{a}, \boldsymbol{b}] \to X$ and $g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ be functions. If $f_1, f_2 \in MPS([\boldsymbol{a}, \boldsymbol{b}], g)$, then for every $\alpha, \beta \in \mathbb{R}$, $\alpha f_1 + \beta f_2 \in MPS([\boldsymbol{a}, \boldsymbol{b}], g)$ and for all $\boldsymbol{J} \subseteq [\boldsymbol{a}, \boldsymbol{b}]$

$$(MPS)\int_{\boldsymbol{J}} (\alpha f_1 + \beta f_2)\ dg = \alpha \cdot (MPS)\int_{\boldsymbol{J}} f_1\ dg + \beta \cdot (MPS)\int_{\boldsymbol{J}} f_2\ dg.$$

Proof. Suppose that f_1, f_2 are MPS-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$. By Definition 19, f_1, f_2 are MDS-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$. Now, let $\alpha, \beta \in \mathbb{R}$ and $\boldsymbol{J} \subseteq [\boldsymbol{a}, \boldsymbol{b}]$. By linearity property of functions, $\alpha \cdot f_1 + \beta \cdot f_2$ is also MDS-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$ and

$$(MDS)\int_{\boldsymbol{J}} (\alpha \cdot f_1 + \beta \cdot f_2)\ dg = \alpha \cdot (MDS)\int_{\boldsymbol{J}} f_1\ dg + \beta \cdot (MDS)\int_{\boldsymbol{J}} f_2\ dg. \tag{2}$$

By the definition of MPS-integral,

$$(MDS)\int_{\boldsymbol{J}} f_1\ dg, (MDS)\int_{\boldsymbol{J}} f_2\ dg \quad \in e(X).$$

Set $m_1, m_2 \in X$ so that

$$(MDS)\int_{\boldsymbol{J}} f_1\ dg = m_1 \text{ and } (MDS)\int_{\boldsymbol{J}} f_2\ dg = m_2.$$

So we have $\alpha \cdot m_1 + \beta \cdot m_2 \in X$. By equation (2),

$$\begin{aligned}(MDS)\int_{\boldsymbol{J}} (\alpha \cdot f_1 + \beta \cdot f_2)\ dg &= \alpha \cdot (MDS)\int_{\boldsymbol{J}} f_1\ dg + \beta \cdot (MDS)\int_{\boldsymbol{J}} f_2\ dg \\ &= \alpha \cdot e(m_1) + \beta \cdot e(m_2) \\ &= e(\alpha m_1 + \beta m_2) \in e(X).\end{aligned}$$

This implies that $\alpha \cdot f_1 + \beta \cdot f_2$ is MPS-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$ and

$$\begin{aligned}(MPS)\int_{\boldsymbol{J}} (\alpha \cdot f_1 + \beta \cdot f_2)\ dg &= (MDS)\int_{\boldsymbol{J}} (\alpha \cdot f_1 + \beta \cdot f_2)\ dg \\ &= \alpha \cdot (MDS)\int_{\boldsymbol{J}} f_1\ dg + \beta \cdot (MDS)\int_{\boldsymbol{J}} f_2\ dg \\ &= \alpha \cdot (MPS)\int_{\boldsymbol{J}} f_1\ dg + \beta \cdot (MPS)\int_{\boldsymbol{J}} f_2\ dg.\end{aligned}$$

Hence, the proof. □

Theorem 12. ***(Linearity of Integrator of the MPS-integral)***
Let $f : [\boldsymbol{a}, \boldsymbol{b}] \to X$ and $g_1, g_2 : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ be functions. If f is MPS-integrable with respect to g_1 and g_2 on $[\boldsymbol{a}, \boldsymbol{b}]$, then for any $\alpha, \beta \in \mathbb{R}$, $f \in MPS([\boldsymbol{a}, \boldsymbol{b}], \alpha g_1 + \beta g_2)$ and

$$(MPS) \int_{\boldsymbol{J}} f \; d[\alpha g_1 + \beta g_2] = \alpha \cdot (MPS) \int_{\boldsymbol{J}} f \; dg_1 + \beta \cdot (MPS) \int_{\boldsymbol{J}} f \; dg_2.$$

for every $\boldsymbol{J} \subseteq [\boldsymbol{a}, \boldsymbol{b}]$.

Proof. The proof is analogous to Theorem (11). □

Theorem 13. ***(Cauchy Criterion of the MPS-integral)***
Let $f : [\boldsymbol{a}, \boldsymbol{b}] \to X$ and $g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ be functions. Then f is McShane-Pettis-Stieltjes integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$ if and only if for every $\varepsilon > 0$, there exists a gauge $\delta : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}^+$ such that if M_1 and M_2 are two δ-fine M-partitions, then

$$\|S(f, g, M_1) - S(f, g, M_2)\|_X < \varepsilon.$$

Proof. Suppose that f is MPS-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$. By Definition 19, f is MDS-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$. Using the Cauchy Criterion of the MDS-integral, there exists a gauge
$\delta : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}^+$ such that if M_1 and M_2 are two δ-fine M-partitions, then

$$\|S(f, g, M_1) - S(f, g, M_2)\|_X < \varepsilon.$$

Conversely, assume that for every $\varepsilon > 0$, there exists a gauge $\delta : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}^+$ such that if M_1 and M_2 are two δ-fine M-partitions, then

$$\|S(f, g, M_1) - S(f, g, M_2)\|_{e(X)} < \varepsilon.$$

By Cauchy Criterion on MDS-integral, $x^*(f) : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ is MS-integrable. Let an interval $\boldsymbol{J} \subseteq [\boldsymbol{a}, \boldsymbol{b}]$. Define $x^{**} : X^* \to \mathbb{R}$ such that $x^{**}_{\boldsymbol{J}}(x^*) = (MS) \int_{[\boldsymbol{a}, \boldsymbol{b}]} x^*(f) dg$. By definition of X^{**}, $x^{**}_{\boldsymbol{J}} \in X^{**}$. This means that $x^{**}_{\boldsymbol{J}} = (MDS) \int_{[\boldsymbol{a}, \boldsymbol{b}]} f dg \in e(X)$. Hence, $(MDS) \int_{[\boldsymbol{a}, \boldsymbol{b}]} f dg = (MPS) \int_{[\boldsymbol{a}, \boldsymbol{b}]} f dg$. Therefore, f is MPS-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$. □

Theorem 14. ***(Additivity of the MPS-integral)***
Let $f : [\boldsymbol{a}, \boldsymbol{b}] \to X$ and $g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ be functions and $\boldsymbol{I}, \boldsymbol{J} \in \mathcal{I}_n\big([\boldsymbol{a}, \boldsymbol{b}]\big)$ that forms an M-partition of $[\boldsymbol{a}, \boldsymbol{b}]$. If $f \in MPS(\boldsymbol{I}, g) \cap MPS(\boldsymbol{J}, g)$, then $f \in MPS([\boldsymbol{a}, \boldsymbol{b}], g)$ and

$$(MPS) \int_{[\boldsymbol{a}, \boldsymbol{b}]} f dg = (MPS) \int_{\boldsymbol{I}} f dg + (MPS) \int_{\boldsymbol{J}} f dg.$$

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