EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 3, 2024, 1659-1673
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

The Wiener Index of Prime Graph $PG(\mathbb{Z}_n)$

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Abstract. Let $G = (V, E)$ be a simple graph and $(\mathcal{R}, +, \cdot)$ be a ring with zero element $0_\mathcal{R}$. The Wiener index of G, denoted by $W(G)$, is defined as the sum of distances of every vertex u and v, or half of the sum of all entries of its distance matrix. The prime graph of $\mathcal{R}$, denoted by $PG(\mathcal{R})$, is defined as a graph with $V(PG(\mathcal{R})) = \mathcal{R}$ such that $uv \in E(PG(\mathcal{R}))$ if and only if $u\mathcal{R}v = \{0_\mathcal{R}\}$ or $v\mathcal{R}u = \{0_\mathcal{R}\}$. In this article, we determine the Wiener index of $PG(\mathbb{Z}_n)$ in some cases n by constructing its distance matrix. We partition the set $\mathbb{Z}_n$ into three types of sets, namely zero sets, nontrivial zero divisor sets, and unit sets. There are two objectives to be achieved. Firstly, we revise the Wiener index formula of $PG(\mathbb{Z}_n)$ for $n = p^2$ and $n = p^3$ for prime number p and we compare this results with the results carried out by previous researchers. Secondly, we determine the Wiener index formula of $PG(\mathbb{Z}_n)$ for $n = pq, n = p^2q, n = p^2q^2$, and $n = pqr$ for distinct prime numbers p, q, and r.

2020 Mathematics Subject Classifications: 05C09, 05C25.

Key Words and Phrases: Distance matrix, prime graph, ring, Wiener index

1. Introduction

A graph G is a system consisting of a finite non-empty vertex set and a finite edge set such that every element is identified with a pair of vertices [14]. The development of graph theory and its applications has been carried out by many researchers. One of the developments is the construction of graphs be related with algebraic structures. The study of graph theory for a commutative ring was began in 1988, when Beck in [11] introduced the notion of zero divisor of the graph. Other construction of graphs related to algebraic structures is zero divisor graphs ([4], [5], and [20]). Further development was carried out by Anderson and Badawi in 2008 by defining and discussing the properties of the total

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DOI: https://doi.org/10.29020/nybg.ejpam.v17i3.5166

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graph of the ring [6]. In 2010 Subhakar discussed about associate ring graph [37]. Ashrafi et al [7] investigated the basic properties of unit graph and given some characterization results regarding connectedness, chromatic, index, diameter, girth, and planarity of total graph. The development of prime graph was studied by Bhavanari et al in 2010 [12]. They presented several examples of prime graphs on rings $\mathbb{Z}_n$ for n prime numbers. They also proved the relationship between a prime ring and a prime graph of the ring. Furthermore, Kalita and Patra in 2014 determined the chromatic number of prime graphs of a ring $\mathbb{Z}_n$ [26]. Pawar and Joshi introduced prime graphs $PG_1(\mathcal{R})$ in 2017 [27] and $PG_2(\mathcal{R})$ in 2019 [19].

The application of graph theory has been carried out in many fields such as networks, cryptography, transportation, coding theory, chemistry, crystallography, and information systems (see [1], [2], [15], [21], [22], [23], [24], [29], [30], [31], [35], and [39]). The application of graph theory to the field of chemistry was first introduced by Wiener in 1947 to predict the boiling point of the paraffin molecular structure [40]. The value of that prediction is then known as Wiener index which is defined as the sum of the distances between vertices in a chemical graph representing non-hydrogen atoms in the molecule [13]. The highly anticipated physicochemical features of all sorts of alkanes are found using the distance-based topological indices known as the Wiener index [25]. The mathematical representation of the Wiener index is given by Hosoya as the sum of the distances of each pair of vertices in the graph [17].

Many researchers have developed the concept of the Wiener index for graphs of rings. Ramane et al [32] obtain the Wiener index of line graphs and some class of graphs. Suthar and Prakash provided the Wiener index formula for total graphs over the ring $\mathbb{Z}_n$ [38]. The Wiener index formula for zero divisor graph of ring was given in ([3], [8], [28], [33], [34], and [36]). Asir et al [10] gave the Wiener index formula for unit graph of ring R. Furthermore, Joshi and Pawar [18] introduced the formula of Wiener index of the prime graph of the ring $\mathbb{Z}_n$, for $n = p, n = p^2$ and $n = p^3$ where p is a prime number. All Wiener index formulas that have been constructed for ring graphs (zero divisor graph, unit graph, total graph, and prime graph) can be seen in the survey results arranged by Asir et al [9].

In this article, we determine the Wiener index of $PG(\mathbb{Z}_n)$ for some cases $n = p, p^2, p^3, pq$, p^2q, pqr, with distinct prime p, q, and r, using distance matrix method. We partition the set $\mathbb{Z}_n$ into three types of sets, namely zero sets, nontrivial zero divisor sets, and unit sets. Wiener index is obtained from a half of the sum of all entries of the distance matrix. The discussion is divided into four sections. In the second section, we give a literature review to understand the theory which is used in this article. In the third section, we compare the Wiener index formula of $PG(\mathbb{Z}_n)$ where $n = p, p^2, p^3$ for any prime number p with the results carried out by Joshi and Pawar [18] and we determine the Wiener index formula of $PG(\mathbb{Z}_n)$ where $n = pq, p^2q, p^2q^2, pqr$ for distinct prime numbers p, q, and r. The last section concludes the contents of this article.

2. Preliminaries

The graph referred to in this article is a simple graph, that is, undirected graphs that have no loops and multiple edges. More contents on graph theory can be studied in [14], while topics on algebraic structure can be studied in [16].

A graph $G = (V, E)$ is a system consisting of a finite non-empty vertex set $V(G)$ and a finite edge set $E(G)$, which is a subset of $V(G) \times V(G)$. The order and size of G, are the cardinality of $V(G)$ and $E(G)$, respectively. Generally, the edge $(u, v) \in E(G)$ is written as uv or vu. Two vertex u and v are called adjacent if uv is an edge. The open neighborhood set of $u \in V(G)$, denoted by $N(u)$, is the set of vertices that are adjacent to u, and the cardinality of $N(u)$ is called the degree of u, denoted by $\deg(u)$. A graph G is said to be connected if any two of its vertices are connected. The distance between vertices u and v in a connected graph, denoted as $d(u, v)$, is defined as the size of the shortest path subgraph between vertices u and v. The distance matrix of a graph G, denoted $D(G)$, is the matrix $[d_{ij}]$ defined as $d_{ij} = d(v_i, v_j)$ for $i \neq j$ and $d_{ii} = 0$. The Wiener index of G is the sum of the distances of all pairs of vertices in G. The definition of the Wiener index can be related to the concept of the distance matrix as follows:

Definition 1. *[17] Let G be a graph, and $D(G)$ be the distance matrix of G. The Wiener index of G, denoted $W(G)$, is defined as*

$$W(G) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}.$$

The definition and some properties of prime graphs of ring $\mathcal{R}$ as follows:

Definition 2. *[12] Let $(\mathcal{R}, +, \cdot)$ be a ring. The prime graph of ring $\mathcal{R}$, denoted by $PG(\mathcal{R})$, is a graph with $V(PG(\mathcal{R})) = \mathcal{R}$ and*

$$E(PG(\mathcal{R})) = \{uv | u\mathcal{R}v = \{0_{\mathcal{R}}\} \text{ or } v\mathcal{R}u = \{0_{\mathcal{R}}\} \text{ and } u \neq v\}.$$

The following Theorem 1 shows a property of a prime grphs of a ring. While other properties can be seen in [12].

Theorem 1. [12] *Let $(\mathcal{R}, +, \cdot)$ be a ring and $PG(\mathcal{R})$ be its prime graph.*

(i) Every non-zero vertex $v \in \mathcal{R}$ is adjacent to $0_{\mathcal{R}}$.

(ii) $d(u, v) = 2$ if and only if the vertices u and v are not adjacent.

(iii) If $\mathcal{R}$ is commutative ring unity $1_{\mathcal{R}}$, then the vertices u and v are adjacent if and only if $uv = 0_{\mathcal{R}}$.

(iv) If $\mathcal{R}$ is a commutative ring with unity $1_{\mathcal{R}}$, then u is only adjacent to $0_{\mathcal{R}}$.

(v) If $\mathcal{R} = \mathbb{Z}_p$ for p primes or $p = 4$, then $PG(\mathbb{Z}_p) \cong K_{1,p-1}$.

The Wiener index of the prime graph of the ring $\mathbb{Z}_n$ for $n = p, n = p^2, n = p^3$ with p a prime number, as obtained by Joshi and Pawar [18] as follows:

Theorem 2. [18] *If p is a prime number, then*

$$W(PG(\mathbb{Z}_p)) = (p-1)^2.$$

Theorem 3. [18] *If p is a prime number, then*

$$W(PG(\mathbb{Z}_{p^2})) = \frac{p(p-1)(2p^2-2p+1)}{2}.$$

Theorem 4. [18] *If p is a prime number, then*

$$W(PG(\mathbb{Z}_{p^3})) = \frac{p(p-1)(2p^4+2p^3-2p-3)}{2}.$$

3. Main Result

In this section we will determine the Wiener index formula of $PG(\mathbb{Z}_n)$ in several cases of n, especially for $n = pq, n = p^2q, n = p^2q^2, n = pqr$, where p, q, r are prime numbers. However, firstly we determined the Wiener index formula for the case $n = p, n = p^2, n = p^3$, although the formula for this case has been determined by Joshi and Pawar [18].

Theorem 5. *If p is a prime, then*

$$W(PG(\mathbb{Z}_p)) = (p-1)^2.$$

Proof. For prime p, it holds $d(u, v) = 2$ if u and v are not adjacent and $d(u, v) = 1$ if u and v are adjacent. Next we partition $\mathbb{Z}_p$ into two sets, namely the zero set $O = \{\overline{0}\}$, the unit set

$$U = \{\overline{x} \in \mathbb{Z}_p | \overline{x} \text{ is unit in ring } \mathbb{Z}_p\}.$$

Based on Theorem 1, $PG(\mathbb{Z}_p) \cong K_{1,p-1}$ and the distance matrix of $PG(\mathbb{Z}_p)$ is given as

$$D(PG(\mathbb{Z}_p)) = \begin{array}{c} \\ O \\ U \end{array} \begin{array}{c} \begin{array}{cc} O & U \end{array} \\ \begin{pmatrix} 0 & \mathbf{1}_{1\times(p-1)} \\ \mathbf{1}_{(p-1)\times 1} & 2(J_{p-1} - I_{p-1}) \end{pmatrix} \end{array},$$

where $\mathbf{1}_{m\times n}$ is a matrix of order $m \times n$ with all entries 1 and $J_n = \mathbf{1}_{n\times n}$. We obtain

$$W(PG(\mathbb{Z}_p)) = \frac{1}{2}\left(2(p-1) + 2(p-1)^2 - 2(p-1)\right) = (p-1)^2.$$

Thus, $W(PG(\mathbb{Z}_p)) = (p-1)^2$.

We see that the Wiener index formula of $PG(\mathbb{Z}_p)$ in Theorem 5 is equal to the formula in Theorem 2. Therefore, the result in Theorem 5 strengthen Theorem 2, considering that Theorem 2 does not provide analytical proof.

The Wiener index formula of $PG(\mathbb{Z}_{p^2})$ and $PG(\mathbb{Z}_{p^3})$ with p prime numbers as follows:

Theorem 6. *If p is a prime number, then*

$$W(PG(\mathbb{Z}_{p^2})) = \frac{p(p-1)(2p^2+2p-3)}{2}.$$

Proof. For every prime p, the ring $\mathbb{Z}_{p^2}$ has $p^2 - p$ units and $p-1$ nontrivial zero divisors. Next we partition $\mathbb{Z}_{p^2}$ into three sets, namely the zero set $O = \{\overline{0}\}$, the unit set $U = \{\overline{x} \in \mathbb{Z}_{p^2} | \overline{x} \text{ is unit in ring } \mathbb{Z}_{p^2}\}$, and the set of nontrivial zero divisors $\mathbb{Z}_{p^2}$ are $Z = \{\overline{p}, \overline{2p}, \ldots, \overline{(p-1)p}\} = \langle \overline{p} \rangle \setminus \{\overline{0}\}$. Based on Theorem 1, we obtain

(i) Every vertex in O is adjacent to every vertex in Z and U.

(ii) Every vertex in Z is adjacent to every vertex in O and Z.

(iii) Every vertex in U is only adjcacent to every vertex in O.

Thus, the distance matrix of $PG(\mathbb{Z}_{p^2})$ is

$$D(PG(\mathbb{Z}_{p^2})) = \begin{matrix} & O & Z & U \\ O & 0 & \mathbf{1}_{1\times(p-1)} & \mathbf{1}_{1\times p(p-1)} \\ Z & \mathbf{1}_{(p-1)\times 1} & J_{p-1} - I_{p-1} & \mathbf{2}_{(p-1)\times p(p-1)} \\ U & \mathbf{1}_{p(p-1)\times 1} & \mathbf{2}_{p(p-1)\times(p-1)} & 2(J_{p(p-1)} - I_{p(p-1)}) \end{matrix}.$$

where $\mathbf{1}_{m\times n}$ is a matrix of order $m \times n$ with all entries 1, $\mathbf{2}_{m\times n}$ is a matrix of order $m \times n$ with all entries 2, and $J_n = \mathbf{1}_{n\times n}$. A half of the sum of all entries in the matrix $D(PG(\mathbb{Z}_{p^2}))$ is

$$\begin{aligned} W(PG(\mathbb{Z}_{p^2})) &= \frac{1}{2}(2p^4 - 2p^2 - 2(p^2-1) - (p-1)^2 + (p-1)) \\ &= \frac{2p^4 - 5p^2 + 3p}{2} \\ &= \frac{p(p-1)(2p^2+2p-3)}{2}. \end{aligned}$$

Hence,

$$W(PG(\mathbb{Z}_{p^2})) = \frac{p(p-1)(2p^2+2p-3)}{2}.$$

Theorem 7. *If p is prime, then*

$$W(PG(\mathbb{Z}_{p^3})) = \frac{p(p-1)(2p^4+2p^3+2p^2-4p-1)}{2}.$$

Proof. For every prime p, the ring $\mathbb{Z}_{p^3}$ has $p^3 - p^2$ units and $p^2 - 1$ nontrivial zero divisors. The nontrivial zero divisors in $\mathbb{Z}_{p^3}$ are

$$Z = \{\overline{p}, \overline{2p}, \ldots, \overline{(p-1)p}, \overline{p^2}, \overline{(p+1)p}, \ldots, \overline{2(p-1)p}, \overline{2p^2}, \ldots, \overline{(p^2-1)p}\}.$$

Next, we partition $\mathbb{Z}_{p^3}$ into three sets, namely the zero set $O = \{\overline{0}\}$, the unit set $U = \{\overline{x} \in \mathbb{Z}_{p^3} | \overline{x}$ is unit in ring $\mathbb{Z}_{p^3}\}$, and the set of nontrivial zero divisors $\mathbb{Z}_{p^3}$ are

$$\begin{aligned} A &= \{\overline{p^2}, \overline{2p^2}, \ldots, \overline{(p-1)p^2}\} = \langle \overline{p^2} \rangle \setminus \{\overline{0}\}, & |A| &= p-1, \\ B &= \langle \overline{p} \rangle \setminus \langle \overline{p^2} \rangle, & |B| &= p^2 - p. \end{aligned}$$

Based on Theorem 1, the adjacency of every vertex in $PG(\mathbb{Z}_{p^3})$ is as follows:

(i) Every vertex in O is adjacent to every vertex in A, B, and U.

(ii) Every vertex in A is adjacent to every vertex in O, A and B.

(iii) Every vertex in B is adjacent to every vertex in O and A.

(iv) Every vertex in U is only adjcacent to every vertex in O.

So, we obtain the distance matrix of $PG(\mathbb{Z}_{p^3})$ is

$$D(PG(\mathbb{Z}_{p^3})) = \begin{array}{c} \\ O \\ A \\ B \\ U \end{array} \begin{array}{c} \begin{array}{cccc} O & A & B & U \end{array} \\ \begin{pmatrix} 0 & \mathbf{1}_{1\times(p^2-p)} & \mathbf{1}_{1\times(p-1)} & \mathbf{1}_{1\times(p^3-p^2)} \\ \mathbf{1}_{(p^2-p)\times 1} & 2(J_{p^2-p} - I_{p^2-p}) & \mathbf{1}_{(p^2-p)\times(p-1)} & \mathbf{2}_{(p^2-p)\times(p^3-p^2)} \\ \mathbf{1}_{(p-1)\times 1} & \mathbf{1}_{(p-1)\times(p^2-p)} & J_{p-1} - I_{p-1} & \mathbf{2}_{(p-1)\times(p^3-p^2)} \\ \mathbf{1}_{(p^3-p^2)\times 1} & \mathbf{2}_{(p^3-p^2)\times(p^2-p)} & \mathbf{2}_{(p^3-p^2)\times(p-1)} & 2(J_{p^3-p^2} - I_{p^3-p^2}) \end{pmatrix} \end{array},$$

where $\mathbf{1}_{m\times n}$ is a matrix of order $m \times n$ with all entries 1, $\mathbf{2}_{m\times n}$ is a matrix of order $m \times n$ with all entries 2, and $J_n = \mathbf{1}_{n\times n}$. A half of the sum of all entries in the matrix $D(PG(\mathbb{Z}_{p^3}))$ is

$$\begin{aligned} W(PG(\mathbb{Z}_{p^3})) &= \frac{1}{2}(2p^6 - 2p^3 - 2(p^3-1) - 2(p^2-p)(p-1) - (p-1)^2 + (p-1)) \\ &= \frac{2p^6 - 6p^3 + 3p^2 + p}{2} \\ &= \frac{p(p-1)(2p^4 + 2p^3 + 2p^2 - 4p - 1)}{2}. \end{aligned}$$

Thus, we obtain

$$W(PG(\mathbb{Z}_{p^3})) = \frac{p(p-1)(2p^4 + 2p^3 + 2p^2 - 4p - 1)}{2}.$$

The Wiener index formulas of $PG(\mathbb{Z}_n)$ in Theorem 6 and Theorem 7 are different from the formula in Theorem 3 and Theorem 4, respectively. We claim that Theorem 6 and Theorem 7 are correct. We give a simple example for $p = 2$. The prime graphs of $PG(\mathbb{Z}_4)$ and $PG(\mathbb{Z}_8)$ are given in Figure 1.

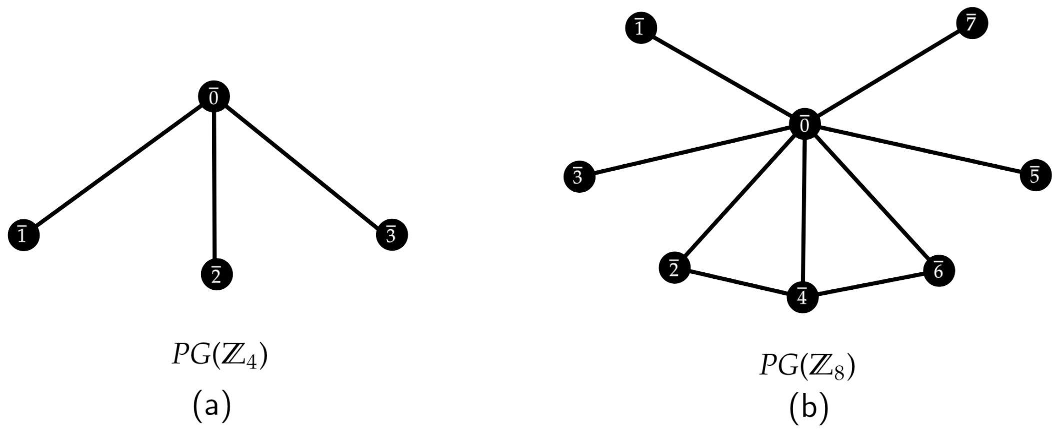

Figure 1: (a). Prime graph of ring $\mathbb{Z}_4$, and (b). Prime graph of ring $\mathbb{Z}_8$

We compare the Wiener index formula of $PG(\mathbb{Z}_4)$ resulting from Theorem 6 and Theorem 3 with Definition 1. Based on Theorem 3, we get $W(PG(\mathbb{Z}_4)) = 5$ and based on Theorem 6, we get $W(PG(\mathbb{Z}_4)) = 9$. However, by Definition 1, since $d(0,1) = d(0,2) = d(0,3) = 1, d(1,2) = d(1,3) = d(2,3) = 2$, the Wiener index of $PG(\mathbb{Z}_4)$ is $W(PG(\mathbb{Z}_4)) = 9$.

Now, we compare the Wiener index formula of $PG(\mathbb{Z}_8)$ resulting from Theorem 7 and Theorem 4 with Definition 1. Based on Theorem 4, we get $W(PG(\mathbb{Z}_8)) = 41$ and based on Theorem 7, we get $W(PG(\mathbb{Z}_8)) = 47$. However, by Definition 1, the Wiener index of $PG(\mathbb{Z}_8)$ is $W(PG(\mathbb{Z}_8)) = 47$.

Next, we investigate the Wiener index formula for other cases of n, that is, for $n = pq, n = pqr, n = p^2q$, and $n = p^2q^2$ where p, q, and r are different primes. The Wiener index formula for $PG(\mathbb{Z}_{pq})$ is as follows:

Theorem 8. *If p, q are two distinct prime numbers, then*

$$W(PG(\mathbb{Z}_{pq})) = p^2q^2 - 3pq + p + q.$$

Proof. For every prime number p, q, the ring $\mathbb{Z}_{pq}$ has $u = (p-1)(q-1)$ units and $p + q - 2$ nontrivial zero divisors. Next, we partition $\mathbb{Z}_{pq}$ into three sets, namely the zero set $O = \{\bar{0}\}$, the unit set

$$U = \{\bar{x} \in \mathbb{Z}_{pq} | \bar{x} \text{ is unit in ring } \mathbb{Z}_{pq}\},$$

and the set of nontrivial zero divisors $\mathbb{Z}_{pq}$ are

$$A = \{\bar{q}, \overline{2q}, \ldots, \overline{(p-1)q}\} = \langle \bar{q} \rangle \setminus \{\bar{0}\}, \qquad |A| = p - 1,$$
$$B = \{\bar{p}, \overline{2p}, \ldots, \overline{(q-1)p}\} = \langle \bar{p} \rangle \setminus \{\bar{0}\}, \qquad |B| = q - 1.$$

Based on Theorem 1, the adjacency of every vertex in $PG(\mathbb{Z}_{pq})$ is as follows:

(i) Every vertex in O is adjacent to every vertex in A, B, and U.

(ii) Every vertex in A is adjacent to every vertex in O and B.

(iii) Every vertex in B is adjacent to every vertex in O and B.

(iv) Every vertex in U is only adjacent to every vertex in O.

Then, the distance matrix of $PG(\mathbb{Z}_{pq})$ is

$$D(PG(\mathbb{Z}_{pq})) = \begin{array}{c} \\ O \\ A \\ B \\ U \end{array}\begin{array}{c} \begin{array}{cccc} O & A & B & U \end{array} \\ \begin{pmatrix} 0 & \mathbf{1}_{1\times(p-1)} & \mathbf{1}_{1\times(q-1)} & \mathbf{1}_{1\times u} \\ \mathbf{1}_{(p-1)\times 1} & 2(J_{p-1} - I_{p-1}) & \mathbf{1}_{(p-1)\times(q-1)} & \mathbf{2}_{(p-1)\times u} \\ \mathbf{1}_{(q-1)\times 1} & \mathbf{1}_{(q-1)\times(p-1)} & 2(J_{q-1} - I_{q-1}) & \mathbf{2}_{(q-1)\times u} \\ \mathbf{1}_{u\times 1} & \mathbf{2}_{u\times(p-1)} & \mathbf{2}_{u\times(q-1)} & 2(J_u - I_u) \end{pmatrix} \end{array},$$

where $\mathbf{1}_{m\times n}$ is a matrix of order $m \times n$ with all entries 1, $\mathbf{2}_{m\times n}$ is a matrix of order $m \times n$ with all entries 2, and $J_n = \mathbf{1}_{n\times n}$. A half of the sum of all entries in the matrix $D(PG(\mathbb{Z}_{pq}))$ is

$$\begin{aligned} W(PG(\mathbb{Z}_{pq})) =& \frac{1}{2}(2p^2q^2 - 2pq - 2(pq-1) - 2(p-1)(q-1)) \\ =& p^2q^2 - 3pq + p + q. \end{aligned}$$

Thus, we obtain $W(PG(\mathbb{Z}_{pq})) = p^2q^2 - 3pq + p + q$.

Theorem 9. *If p, q, and r are distinct prime numbers, then*

$$W(PG(\mathbb{Z}_{pqr})) = p^2q^2r^2 - 5pqr + 2(pq + qr + pr) - (p + q + r) + 1.$$

Proof. For every three distinct prime numbers p, q, and r, the ring $\mathbb{Z}_{pqr}$ has $u = (p-1)(q-1)(r-1) = pqr - pq - qr - pr + p + q + r - 1$ units and $pq + qr + pr - p - q - r$ nontrivial zero divisors. The nontrivial zero divisor of $\mathbb{Z}_{pqr}$ is

$$Z = (\langle\overline{p}\rangle \cup \langle\overline{q}\rangle \cup \langle\overline{r}\rangle) \setminus \{\overline{0}\}.$$

Next, we partition $\mathbb{Z}_{pqr}$ into three sets, namely the zero set $O = \{\overline{0}\}$, the unit set $U = \{\overline{x} \in \mathbb{Z}_{pqr} | \overline{x}$ is unit in ring $\mathbb{Z}_{pqr}\}$, and the set of nontrivial zero divisors $\mathbb{Z}_{pqr}$ are

$$\begin{aligned} A &= \langle\overline{qr}\rangle \setminus \{\overline{0}\}, & |A| &= p - 1 = a, \\ B &= \langle\overline{pr}\rangle \setminus \{\overline{0}\}, & |B| &= q - 1 = b, \\ C &= \langle\overline{pq}\rangle \setminus \{\overline{0}\}, & |C| &= r - 1 = c, \\ D &= \langle\overline{p}\rangle \setminus (\langle\overline{pq}\rangle \cup \langle\overline{pr}\rangle), & |D| &= qr - (q + r - 1) = d, \\ E &= \langle\overline{q}\rangle \setminus (\langle\overline{pq}\rangle \cup \langle\overline{qr}\rangle), & |E| &= pr - (p + r - 1) = e, \\ F &= \langle\overline{r}\rangle \setminus (\langle\overline{pr}\rangle \cup \langle\overline{qr}\rangle), & |F| &= pq - (p + q - 1) = f. \end{aligned}$$

Based on Theorem 1, the adjacency of every vertex in $PG(\mathbb{Z}_{pqr})$ is as follows:

(i) Every vertex in O is adjacent to every vertex in A, B, C, D, E, F, and U.

(ii) Every vertex in A is adjacent to every vertex in O, B, C, and D.

(iii) Every vertex in B is adjacent to every vertex in O, A, C, and E.

(iv) Every vertex in C is adjacent to every vertex in O, A, B, and F.

(v) Every vertex in D is adjacent to every vertex in O and A.

(vi) Every vertex in E is adjacent to every vertex in O and B.

(vii) Every vertex in F is adjacent to every vertex in O and C.

(viii) Every vertex in U is only adjacent to every vertex in O.

The distance matrix $D(PG(\mathbb{Z}_{pqr}))$ can then be written as

$$\begin{array}{c} \\ O \\ A \\ B \\ C \\ D \\ E \\ F \\ U \end{array}\begin{array}{c} \begin{array}{cccccccc} O & A & B & C & D & E & F & U \end{array} \\ \begin{pmatrix} 0 & \mathbf{1}_{1\times a} & \mathbf{1}_{1\times b} & \mathbf{1}_{1\times c} & \mathbf{1}_{1\times d} & \mathbf{1}_{1\times e} & \mathbf{1}_{1\times f} & \mathbf{1}_{1\times u} \\ \mathbf{1}_{a\times 1} & 2(J_a-I_a) & \mathbf{1}_{a\times b} & \mathbf{1}_{a\times c} & \mathbf{1}_{a\times d} & \mathbf{2}_{a\times e} & \mathbf{2}_{a\times f} & \mathbf{2}_{a\times u} \\ \mathbf{1}_{b\times 1} & \mathbf{1}_{b\times a} & 2(J_b-I_b) & \mathbf{1}_{b\times c} & \mathbf{2}_{b\times d} & \mathbf{1}_{b\times e} & \mathbf{2}_{b\times f} & \mathbf{2}_{b\times u} \\ \mathbf{1}_{c\times 1} & \mathbf{1}_{c\times a} & \mathbf{1}_{c\times b} & 2(J_c-I_c) & \mathbf{2}_{c\times d} & \mathbf{2}_{c\times e} & \mathbf{1}_{c\times f} & \mathbf{2}_{c\times u} \\ \mathbf{1}_{d\times 1} & \mathbf{1}_{d\times a} & \mathbf{2}_{d\times b} & \mathbf{2}_{d\times c} & 2(J_d-I_d) & \mathbf{2}_{d\times e} & \mathbf{2}_{d\times f} & \mathbf{2}_{d\times u} \\ \mathbf{1}_{e\times 1} & \mathbf{2}_{e\times a} & \mathbf{1}_{e\times b} & \mathbf{2}_{e\times c} & \mathbf{2}_{e\times d} & 2(J_e-I_e) & \mathbf{2}_{e\times f} & \mathbf{2}_{e\times u} \\ \mathbf{1}_{f\times 1} & \mathbf{2}_{f\times a} & \mathbf{2}_{f\times b} & \mathbf{1}_{f\times c} & \mathbf{2}_{f\times d} & \mathbf{2}_{f\times e} & 2(J_f-I_f) & \mathbf{2}_{f\times u} \\ \mathbf{1}_{u\times 1} & \mathbf{2}_{u\times a} & \mathbf{2}_{u\times b} & \mathbf{2}_{u\times c} & \mathbf{2}_{u\times d} & \mathbf{2}_{u\times e} & \mathbf{2}_{u\times f} & 2(J_u-I_u) \end{pmatrix} \end{array},$$

where $\mathbf{1}_{m\times n}$ is a matrix of order $m \times n$ with all entries 1, $\mathbf{2}_{m\times n}$ is a matrix of order $m \times n$ with all entries 2, and $J_n = \mathbf{1}_{n\times n}$. A half of the sum of all entries in the matrix $D(PG(\mathbb{Z}_{pqr}))$ is

$$\begin{aligned} W(PG(\mathbb{Z}_{pqr})) =& \frac{1}{2}(2p^2q^2r^2 - 2pqr - 2(pqr-1) - 2(p-1)(qr-1) - 2(q-1)(pr-p)) \\ & - 2(r-1)(pq-p-q+1) \\ =& p^2q^2r^2 - 5pqr + 2(pq+qr+pr) - (p+q+r) + 1. \end{aligned}$$

Thus, we obtain

$$W(PG(\mathbb{Z}_{pqr})) = p^2q^2r^2 - 5pqr + 2(pq+qr+pr) - (p+q+r) + 1.$$

Theorem 10. *If p, q are two distinct prime numbers, then*

$$W(PG(\mathbb{Z}_{p^2q})) = \frac{2p^4q^2 - 8p^2q + 3p^2 + 4pq - p}{2}.$$

Proof. For every two distinct prime numbers p and q, the ring $\mathbb{Z}_{p^2q}$ has $u = p(p-1)(q-1) = p^2q - pq - p^2 + p$ units and $pq + p^2 - p - 1$ nontrivial zero divisors. Next, we partition $\mathbb{Z}_{p^2q}$ into three sets, namely the zero set $O = \{\overline{0}\}$, the unit set $U = \{\overline{x} \in \mathbb{Z}_{p^2q} | \overline{x}$ is unit in ring $\mathbb{Z}_{p^2q}\}$, and the set of nontrivial zero divisors $\mathbb{Z}_{p^2q}$ are

$$\begin{aligned}
&A = \{\overline{pq}, \overline{2pq}, \ldots, \overline{(p-1)pq}\} = \langle \overline{pq} \rangle \setminus \{\overline{0}\}, && |A| = p-1,\\
&B = \{\overline{p^2}, \overline{2p^2}, \ldots, \overline{(q-1)p^2}\} = \langle \overline{p^2} \rangle \setminus \{\overline{0}\}, && |B| = q-1,\\
&C = \langle \overline{q} \rangle \setminus \langle \overline{pq} \rangle, && |C| = (p^2-1) - (p-1) = p^2 - p,\\
&D = \langle \overline{p} \rangle \setminus (\langle \overline{pq} \rangle \cup \langle \overline{p^2} \rangle), && |D| = (pq-1) - (p+q-2) = pq - p - q + 1 = d.
\end{aligned}$$

Based on Theorem 1, the adjacency of every vertex in $PG(\mathbb{Z}_{p^2q})$ is as follows:

(i) Every vertex in O is adjacent to every vertex in A, B, C, D, and U.

(ii) Every vertex in A is adjacent to every vertex in O, A, B, and D.

(iii) Every vertex in B is adjacent to every vertex in O, A and C.

(iv) Every vertex in C is adjacent to every vertex in O and B.

(v) Every vertex in D is adjacent to every vertex in O and A.

(vi) Every vertex in U is only adjacent to every vertex in O.

The distance matrix $D(PG(\mathbb{Z}_{p^2q}))$ can then be written as

$$\begin{array}{c} \\ \overline{0} \\ A \\ B \\ C \\ D \\ U \end{array}
\begin{array}{c}
\begin{array}{cccccc} \overline{0} & A & B & C & D & U \end{array}\\
\begin{pmatrix}
0 & \mathbf{1}_{1\times(p-1)} & \mathbf{1}_{1\times(q-1)} & \mathbf{1}_{1\times(p^2-p)} & \mathbf{1}_{1\times d} & \mathbf{1}_{1\times u}\\
\mathbf{1}_{(p-1)\times 1} & J_{p-1} - I_{p-1} & \mathbf{1}_{(p-1)\times(q-1)} & \mathbf{2}_{(p-1)\times(p^2-p)} & \mathbf{1}_{(p-1)\times d} & \mathbf{2}_{(p-1)\times u}\\
\mathbf{1}_{(q-1)\times 1} & \mathbf{1}_{(q-1)\times(p-1)} & 2(J_{q-1} - I_{q-1}) & \mathbf{1}_{(q-1)\times(p^2-p)} & \mathbf{2}_{(q-1)\times d} & \mathbf{2}_{(q-1)\times u}\\
\mathbf{1}_{(p^2-p)\times 1} & \mathbf{2}_{(p^2-p)\times(p-1)} & \mathbf{1}_{(p^2-p)\times(q-1)} & 2(J_{p^2-p} - I_{p^2-p}) & \mathbf{2}_{(p^2-p)\times d} & \mathbf{2}_{(p^2-p)\times u}\\
\mathbf{1}_{d\times 1} & \mathbf{1}_{d\times(p-1)} & \mathbf{2}_{d\times(q-1)} & \mathbf{2}_{d\times(p^2-p)} & 2(J_d - I_d) & \mathbf{2}_{d\times u}\\
\mathbf{1}_{u\times 1} & \mathbf{2}_{u\times(p-1)} & \mathbf{2}_{u\times(q-1)} & \mathbf{2}_{u\times(p^2-p)} & \mathbf{2}_{u\times d} & 2(J_u - I_u)
\end{pmatrix}
\end{array},$$

where $\mathbf{1}_{m\times n}$ is a matrix of order $m \times n$ with all entries 1, $\mathbf{2}_{m\times n}$ is a matrix of order $m \times n$ with all entries 2, and $J_n = \mathbf{1}_{n\times n}$. A half of the sum of all entries in the matrix $D(PG(\mathbb{Z}_{p^2q}))$ is

$$\begin{aligned}
W(PG(\mathbb{Z}_{p^2q})) =& \frac{1}{2}(2p^4q^2 - 2p^2q - 2(p^2q-1) - (p-1)^2 + (p-1) - 2(p-1)(pq-p)\\
&- 2(p^2-p)(q-1))\\
=& \frac{2p^4q^2 - 8p^2q + 3p^2 + 4pq - p}{2}.
\end{aligned}$$

Thus, we obtain

$$W(PG(\mathbb{Z}_{p^2q})) = \frac{2p^4q^2 - 8p^2q + 3p^2 + 4pq - p}{2}.$$

Theorem 11. *If p, q are two distinct prime numbers, then*

$$W(PG(\mathbb{Z}_{p^2q^2})) = \frac{2p^4q^4 - 11p^2q^2 + 6p^2q + 6pq^2 - 3pq}{2}.$$

Proof. For every two distinct prime numbers p and q, the ring $\mathbb{Z}_{p^2q^2}$ has $u = p(p-1)q(q-1) = p^2q^2 - pq^2 - p^2q + pq$ units and $pq^2 + p^2q - pq - 1$ nontrivial zero divisors. The nontrivial zero divisor of $\mathbb{Z}_{p^2q^2}$ is

$$Z = (\langle \overline{p} \rangle \cup \langle \overline{q} \rangle) \setminus \langle 0 \rangle.$$

Next, we partition we partition $\mathbb{Z}_{p^2q^2}$ into three sets, namely the zero set $O = \{\overline{0}\}$, the unit set

$$U = \{\overline{x} \in \mathbb{Z}_{p^2q^2} | \overline{x} \text{ is unit in ring } \mathbb{Z}_{p^2q^2}\},$$

and the set of nontrivial zero divisors $\mathbb{Z}_{p^2q^2}$ are

$$\begin{aligned}
A &= \langle \overline{pq^2} \rangle \setminus \{\overline{0}\}, & |A| &= p - 1 = a, \\
B &= \langle \overline{p^2q} \rangle \setminus \{\overline{0}\}, & |B| &= q - 1 = b, \\
C &= \langle \overline{q^2} \rangle \setminus \langle \overline{pq^2} \rangle, & |C| &= p^2 - p = c, \\
D &= \langle \overline{p^2} \rangle \setminus \langle \overline{p^2q} \rangle, & |D| &= q^2 - q = d, \\
E &= \langle \overline{pq} \rangle \setminus (\langle \overline{pq^2} \rangle \cup \langle \overline{p^2q} \rangle), & |E| &= pq - p - q + 1 = e, \\
F &= \langle \overline{p} \rangle \setminus (\langle \overline{pq} \rangle \cup \langle \overline{p^2} \rangle), & |F| &= pq^2 - q^2 - pq + q = f, \\
G &= \langle \overline{q} \rangle \setminus (\langle \overline{pq} \rangle \cup \langle \overline{q^2} \rangle), & |G| &= p^2q - p^2 - pq + p = g.
\end{aligned}$$

Based on Theorem 1, the adjacency of every vertex in $PG(\mathbb{Z}_{p^2q^2})$ is as follows:

(i) Every vertex in O is adjacent to every vertex in A, B, C, D, E, F, G, and U.

(ii) Every vertex in A is adjacent to every vertex in O, A, B, D, E, and F.

(iii) Every vertex in B is adjacent to every vertex in O, A, B, C, E, and G.

(iv) Every vertex in C is adjacent to every vertex in O, B and D.

(v) Every vertex in D is adjacent to every vertex in O, A and C.

(vi) Every vertex in E is adjacent to every vertex in O, A, B, and E.

(vii) Every vertex in F is adjacent to every vertex in O and A.

(viii) Every vertex in G is adjacent to every vertex in O and B.

(ix) Every vertex in U is only adjacent to every vertex in O.

The distance matrix of $D(PG(\mathbb{Z}_{p^2q^2}))$ can then be written as

$$\begin{array}{c} \\ \overline{0} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \\ U \end{array}\begin{array}{c} \begin{array}{ccccccccc} \overline{0} & A & B & C & D & E & F & G & U \end{array} \\ \begin{pmatrix} 0 & \mathbf{1}_{1\times a} & \mathbf{1}_{1\times b} & \mathbf{1}_{1\times c} & \mathbf{1}_{1\times d} & \mathbf{1}_{1\times e} & \mathbf{1}_{1\times f} & \mathbf{1}_{1\times g} & \mathbf{1}_{1\times u} \\ \mathbf{1}_{a\times 1} & J_a - I_a & \mathbf{1}_{a\times b} & \mathbf{2}_{a\times c} & \mathbf{1}_{a\times d} & \mathbf{1}_{a\times e} & \mathbf{1}_{a\times f} & \mathbf{2}_{a\times g} & \mathbf{2}_{a\times u} \\ \mathbf{1}_{b\times 1} & \mathbf{1}_{b\times a} & J_b - I_b & \mathbf{1}_{b\times c} & \mathbf{2}_{b\times d} & \mathbf{1}_{b\times e} & \mathbf{2}_{b\times f} & \mathbf{1}_{b\times g} & \mathbf{2}_{b\times u} \\ \mathbf{1}_{c\times 1} & \mathbf{2}_{c\times a} & \mathbf{1}_{c\times b} & 2(J_c - I_c) & \mathbf{1}_{c\times d} & \mathbf{2}_{c\times e} & \mathbf{2}_{c\times f} & \mathbf{2}_{c\times g} & \mathbf{2}_{c\times u} \\ \mathbf{1}_{d\times 1} & \mathbf{1}_{d\times a} & \mathbf{2}_{d\times b} & \mathbf{1}_{d\times c} & 2(J_d - I_d) & \mathbf{2}_{d\times e} & \mathbf{2}_{d\times f} & \mathbf{2}_{d\times g} & \mathbf{2}_{d\times u} \\ \mathbf{1}_{e\times 1} & \mathbf{1}_{e\times a} & \mathbf{1}_{e\times b} & \mathbf{2}_{e\times c} & \mathbf{2}_{e\times d} & J_e - I_e & \mathbf{2}_{e\times f} & \mathbf{2}_{e\times g} & \mathbf{2}_{e\times u} \\ \mathbf{1}_{f\times 1} & \mathbf{1}_{f\times a} & \mathbf{2}_{f\times b} & \mathbf{2}_{f\times c} & \mathbf{2}_{f\times d} & \mathbf{2}_{f\times e} & 2(J_f - I_f) & \mathbf{2}_{f\times g} & \mathbf{2}_{f\times u} \\ \mathbf{1}_{g\times 1} & \mathbf{2}_{g\times a} & \mathbf{1}_{g\times b} & \mathbf{2}_{g\times c} & \mathbf{2}_{g\times d} & \mathbf{2}_{g\times e} & \mathbf{2}_{g\times f} & 2(J_g - I_g) & \mathbf{2}_{g\times u} \\ \mathbf{1}_{u\times 1} & \mathbf{2}_{u\times a} & \mathbf{2}_{u\times b} & \mathbf{2}_{u\times c} & \mathbf{2}_{u\times d} & \mathbf{2}_{u\times e} & \mathbf{2}_{u\times f} & \mathbf{2}_{u\times g} & 2(J_u - I_u) \end{pmatrix} \end{array},$$

where $\mathbf{1}_{m\times n}$ is a matrix of order $m \times n$ with all entries 1, $\mathbf{2}_{m\times n}$ is a matrix of order $m \times n$ with all entries 2, and $J_n = \mathbf{1}_{n\times n}$. A half of the sum of all entries in the matrix $D(PG(\mathbb{Z}_{p^2q^2}))$ is

$$\begin{aligned} W(PG(\mathbb{Z}_{p^2q^2})) =& \frac{1}{2}(2p^4q^4 - 2p^2q^2 - 2(p^2q^2 - 1) - (p-1)^2 + (p-1) - 2(p-1)(pq^2 - p) \\ & - (q-1)^2 + (q-1) - 2(q-1)(p^2q - p - q + 1) \\ & - 2(p^2 - p)(q^2 - q) - (pq - p - q + 1)^2 + (pq - p - q + 1)) \\ =& \frac{2p^4q^4 - 11p^2q^2 + 6p^2q + 6pq^2 - 3pq}{2}. \end{aligned}$$

Thus, we obtain

$$W(PG(\mathbb{Z}_{p^2q^2})) = \frac{2p^4q^4 - 11p^2q^2 + 6p^2q + 6pq^2 - 3pq}{2}.$$

4. Conclusion

Based on the results and discussion above, we obtain the Wiener index formulas for prime graphs of the ring $\mathbb{Z}_n$ where $n = p, p^2, p^3, pq, p^2q, p^2q^2, pqr$ for distinct prime numbers p, q, and r. The distance matrix is formed by partitioning the ring $\mathbb{Z}_n$ into a zero set, a nontrivial zero divisor set, and a unit set. Next, the adjacency between the vertices of each set is determined, so that the distance between the vertices of $\mathbb{Z}_n$ is obtained. Based on these results, for the next reseraches, we give some of the following open problems:

(i) $W(PG(\mathbb{Z}_{p^k}))$, with prime number p and natural number $k > 3$.

(ii) $W(PG(\mathbb{Z}_{p^kq}))$, with distinct prime numbers p, q and natural number $k > 2$.

(iii) $W(PG(\mathbb{Z}_{p^kq^h}))$, with distinct prime numbers p, q and natural numbers $k, h > 2$.

(iv) $W(PG(\mathbb{Z}_{p_1p_2\cdots p_n}))$, with distinct prime numbers $p_1, \ldots, p_n$.

Acknowledgements

This research was funded by a "Lector-Head Doctoral Grant" from Faculty of Mathematics and Natural Sciences University of Brawijaya with Contract Agreement Letter No: 4160.7/UN10.F09/PN/2023.

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