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Sets Related to Openness and Continuity Decompositions in Primal Topological Spaces

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Abstract. This paper introduces and investigates several new classes of sets called $\mathcal{P}$-α-open sets, $\mathcal{P}$-semiopen sets, $\mathcal{P}$-preopen sets, and $\mathcal{P}$-β-open sets within the framework of primal topological spaces. Their properties and relationships with other open set generalizations are studied through examples. Additionally, the concepts of $\mathcal{P}_{\mathcal{R}}$-sets and $\mathcal{P}_{\mathcal{R}_\alpha}$-sets are defined and their characteristics examined. Also, the notions of $\mathcal{P}$-α-continuous, $\mathcal{P}$-semicontinuous, $\mathcal{P}$-precontinuous and $\mathcal{P}$-β-continuous mappings are initiated and their features and main characterizations determined. A new class of sets called $\widetilde{\Psi}_{\mathcal{P}}$-sets is also introduced in primal topological spaces using the $\Psi_{\mathcal{P}}$-operator. Their properties and relationships between $\widetilde{\Psi}_{\mathcal{P}}$-sets, α-open, semi-open, and pre-open are investigated. Theorems on arbitrary unions and finite intersections of $\widetilde{\Psi}_{\mathcal{P}}$ are discussed.

2020 Mathematics Subject Classifications: 54A05, 54B99, 94A60

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1. Introduction and Preliminaries

Topology is one of the important scientific fields in mathematics and physics. It can be used in many different areas of mathematics, including algebra, Riemann integration, Perron integration, operations research, probability theory, game theory, smoothness of functions, and measurement theory. Also, it is relevant to various fields in physics such as condensed matter physics, quantum field theory, physical cosmology, mechanical engineering, and materials science. Some applications of these mathematical concepts influence the mechanical properties of solids, as well as the electrical and mechanical characteristics that rely on the arrangement and network structures of molecules and primary units in materials.

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Over the past years, open-set generalizations have been discussed by many researchers. The initial concept of semi-open sets was introduced by Levine [1] in 1963. Njåstad [2] introduced several classes of almost open sets in 1965; specifically, they looked into the structure of α-open sets and provided several applications. Pre-open sets and pre-continuous functions were presented and investigated by Mashhour et al. in 1982, [3] presented and investigated the concepts of pre-open and pre-continuous functions. The new notions of β-open sets, β-continuous mappings, and β-open mappings were first presented by Abd El-Monsef et al. [4] in 1983.

Topology's application in social science and science has led to the development of many new ideas in addition to traditional structures. Kuratowski presented the concept of the ideal derived from a filter. The concept of an ideal can be thought of as the dual counterpart of a filter.

Likewise, among the new constructs in topology is the notion of a grill, which was first defined by Choquet in 1947 [5] and Thron [6] introduced proximity structures within the domain of grills. In 1977, Chattopadhyay and Thron [7] expanded the concepts of closure spaces in conjunction with grills. Furthermore, Chattopadhyay and colleagues [8] expanded the concept of grills to investigate merotopic spaces. Since that time, the grill structure has found extensive application within the field of topology. Roy and Mukherjee [9–11] conducted the initial attempt to identify the topological characteristics associated with grills. The authors in [12, 13] defined operators on grill topological space. Then, several variations on operators appeared from other researchers. Following that, topologists have defined new concepts related to grill topological space, their subsets, and continuity [14–18]. It is worth noting that the literature concerning grill structures is relatively limited compared to filter, ideal, and other topics; additionally, interdisciplinary applications of grill structures are scarce.

Njastad was the first to define the topology's compatibility with an ideal I [19]. In 1990, Jankovic and Hamlett [20, 21] obtained other characteristics of ideal topological spaces and Ψ-operator, for the ideal topological space $(\mathbb{X}, \delta, I)$, local function of $L \subseteq \mathbb{X}$ is defined as: $L^\star(I)$ (or simply $L^\star$) $= \{x \in \mathbb{X} : \mathcal{U} \cap L \notin I, \mathcal{U} \in \delta(x)\}$, where $\delta(x) = \{\mathcal{U} \in \delta : x \in \mathcal{U}\}$, whereas Ψ-operator is defined as $\Psi(L) = \mathbb{X} - (\mathbb{X} - L)^\star$. The Ψ-operator was used in 2007 by Modak and Bandyopadhyay [22] to define the notion of generalized open sets. In 2012, Al-Omari and Takashi [23] studied features of grill topological space and a different operator denoted by Ψ-operators where $\Psi(L) = \mathbb{X} - \Phi(\mathbb{X} - L)$. $\widetilde{\Psi}_{\mathcal{G}}$ was introduced and studied by Al-Omari and Takashi in [24], they also used the $\Psi_{\mathcal{G}}$-operator to define a new class of open sets.

Recently, the notion of primal topological space was introduced by Acharjee et al. [25, 26] as the dual structure of the grill and the authors obtained many fundamental properties of it. In 2023, Al-Omari, Acharjee, and Özkoç [26] defined and studied operator Ψ by using primal topological spaces as $\Psi(L) = \mathbb{X} - (\mathbb{X} - L)^\diamond$. The authors in [27] expand the class of primal lower pleasant functions to the setting of reflexive smooth Banach spaces. Furthermore, generalized primal topological spaces are a new category of generalized topology that Al-Saadi and Al-Malki recently introduced with the concept of the primal [28]. In [29], soft spaces were also investigated, and these concepts were applied in primal topological spaces.

In this paper, new classes of weaker sets are defined as some of the weak continuity on primal topological spaces, and some of their basic properties are investigated. Additionally, we investigated the relationship between them, and we provided examples of the opposite of relationships that were not satisfying. We define some new classes of functions and use these functions to introduce several interesting decomposition theorems of continuity. Finally, we use the $\Psi_{\mathcal{P}}$-operator to introduce and study $\widetilde{\Psi}_{\mathcal{P}}$-sets as a new class of sets in the structure of primal topological spaces, and we obtain their features. A counter-example to the theorems based on arbitrary unions and finite intersections is discussed. Moreover, we study the relationships between $\widetilde{\Psi}_{\mathcal{P}}$-sets and their analogous topological concepts.

Assume that $(\mathbb{X}, \delta)$ be a topological space (TS, for short), and $cl(L)$, $int(L)$, respectively, will be used to refer to the interior of L in $(\mathbb{X}, \delta)$ and the closure of L in $(\mathbb{X}, \delta)$. The family of all open neighborhoods of a point $x \in \mathbb{X}$ is denoted by $\mathcal{N}(x)$.

Definition 1.1. *([5]) A family $\mathcal{G}$ of $2^{\mathbb{X}}$ is called a grill on $\mathbb{X}$ if $\mathcal{G}$ satisfies the following conditions:*

(a) $\phi \notin \mathcal{G}$,

(b) If $L \in \mathcal{G}$ and $L \subseteq E$, then $E \in \mathcal{G}$,

(c) If $L \cup E \in \mathcal{G}$, then $L \in \mathcal{G}$ or $E \in \mathcal{G}$.

In [5], define an operator $\Phi : 2^{\mathbb{X}} \to 2^{\mathbb{X}}$ for a grill $\mathcal{G}$ on a TS $(\mathbb{X}, \delta)$, and for any $L \in 2^{\mathbb{X}}$, $\Phi(L) = \{x \in \mathbb{X} : \mathcal{U} \cap L \in \mathcal{G}, \forall \mathcal{U} \in \mathcal{N}(x)\}$.

Then, the author defined another operator, $\Psi : 2^{\mathbb{X}} \to 2^{\mathbb{X}}$, as $\Psi(L) = L \cup \Phi(L)$ for $L \subseteq \mathbb{X}$, is a Kuratowski closure operator, defining a distinct topology $\delta_{\mathcal{G}}$ on $\mathbb{X}$ that is, $\delta \subseteq \delta_{\mathcal{G}}$.

Definition 1.2. *Suppose that $(\mathbb{X}, \delta)$ is a TS. Hence, a subset L of $\mathbb{X}$ can be defined as:*

(a) α-open ([2]), if $L \subseteq int(cl(int(L)))$,

(b) semi-open([1]), if $L \subseteq cl(int(L))$,

(c) pre-open ([3]), if $L \subseteq int(cl(L))$,

(d) β-open ([4]) or semi-pre-open ([30]), if $L \subseteq cl(int(cl(L)))$,

(e) t-set ([31]), if $int(L) = int(cl(L))$,

(f) $\mathcal{R}$-set ([31]), if $L = L_1 \cap L_2$, where L_1 is an open set and L_2 is a t-set,

(g) t_α-set ([32]), if $int(L) = int(cl(int(L)))$,

(h) $\mathcal{R}_\alpha$-set ([32]), if $L = L_1 \cap L_2$, where L_1 is an open set and L_2 is a t_α-set.

For any collection of α-open (resp. semi-open, pre-open, and β-open) sets is denoted by δ^α (resp. $SO(\mathbb{X})$, $PO(\mathbb{X})$, and $\beta O(\mathbb{X})$).

Definition 1.3. *([25, 26]) For a collection $\mathcal{P} \subseteq 2^{\mathbb{X}}$ on $\mathbb{X} \neq \phi$. We define a primal on $\mathbb{X}$ as:*

(a) $\mathbb{X} \notin \mathcal{P}$,

(b) If $L \in \mathcal{P}$ and $E \subseteq L$, thus $E \in \mathcal{P}$,

(c) If $L \cap E \in \mathcal{P}$, then $L \in \mathcal{P}$ or $E \in \mathcal{P}$.

Definition 1.4. *([25, 26]) The TS $(\mathbb{X}, \delta)$ with a primal $\mathcal{P}$ defined on $\mathbb{X}$ as $(\mathbb{X}, \delta, \mathcal{P})$ is called a primal topological space (PTS, for short).*

Definition 1.5. *([25, 26]) Assume that the PTS is $(\mathbb{X}, \delta, \mathcal{P})$. Defined an operator $(.)^{\Diamond} : 2^{\mathbb{X}} \to 2^{\mathbb{X}}$ as $L^{\Diamond}(\mathbb{X}, \delta, \mathcal{P}) = \{x \in \mathbb{X} : (\forall \mathcal{U} \in \mathcal{N}(x))(L^c \cup \mathcal{U}^c \in \mathcal{P})\}$ for each subset L of $\mathbb{X}$ the primal by our needs, will be used $L^{\Diamond}_{\mathcal{P}}$ or $L^{\Diamond}(\mathbb{X}, \delta, \mathcal{P})$ to refer to this operator.*

Definition 1.6. *([25, 26]) Consider a map $cl^{\Diamond} : 2^{\mathbb{X}} \to 2^{\mathbb{X}}$ in a PTS $(\mathbb{X}, \delta, \mathcal{P})$, defined as $cl^{\Diamond}(L) = L \cup L^{\Diamond}$, where L is any subset of $\mathbb{X}$.*

Definition 1.7. *([25, 26]) In a PTS $(\mathbb{X}, \delta, \mathcal{P})$, the collection $\delta^{\Diamond} = \{L \subseteq \mathbb{X} : cl^{\Diamond}(L^c) = L^c\}$ is characterized as a topology on $\mathbb{X}$ that is generated by primal $\mathcal{P}$ and topology δ. The primal topology on $\mathbb{X}$ is the term for it and we can write $\delta^{\Diamond}_{\mathcal{P}}$ instead of $\delta^{\Diamond}$.*

Clearly, $\delta \subseteq \delta^{\Diamond}$ for any primal $\mathcal{P}$ on a topological (X, δ). We will use $\delta^{\Diamond}$-$int(L)$ to refer to the interior of L relative to $\delta^{\Diamond}$.

Theorem 1.8. *([25, 26]) If $(\mathbb{X}, \delta, \mathcal{P})$ is PTS. Consequently, the primal topology $\delta^{\Diamond}$ is finer than δ.*

Theorem 1.9. *([25, 26]) Considering a PTS $(\mathbb{X}, \delta, \mathcal{P})$, the following is true for any two subsets L and E of $\mathbb{X}$:*

(a) If $L^c \in \delta$, then $L^{\Diamond} \subseteq L$,

(b) $\phi^{\Diamond} = \phi$,

(c) $cl(L^{\Diamond}) = L^{\Diamond}$,

(d) $(L^{\Diamond})^{\Diamond} \subseteq L^{\Diamond}$,

(e) If $L \subseteq E$, then $L^{\Diamond} \subseteq E^{\Diamond}$,

(f) $L^{\Diamond} \cup E^{\Diamond} = (L \cup E)^{\Diamond}$,

(g) $(L \cap E)^{\Diamond} \subseteq L^{\Diamond} \cap E^{\Diamond}$.

Lemma 1.10. *([26]) In a PTS $(\mathbb{X}, \delta, \mathcal{P})$, if $L^c \notin \mathcal{P}$, then $L^{\Diamond} = \phi$.*

Theorem 1.11. *([25, 26]) Assume that $(\mathbb{X}, \delta, \mathcal{P})$ is a PTS. Then, the family $B_{\mathcal{P}} = \{T \cap P : T \in \delta$ and $P \notin \mathcal{P}\}$ is a base for the primal topology $\delta^{\Diamond}$ on $\mathbb{X}$.*

Definition 1.12. *([26]) Assume that $(\mathbb{X}, \delta, \mathcal{P})$ is a PTS. An operator $cl^{\Diamond}_{\mathcal{P}} : 2^{\mathbb{X}} \to 2^{\mathbb{X}}$ is defined as $cl^{\Diamond}_{\mathcal{P}}(L) = \{x \in \mathbb{X} : (\exists\, \mathcal{U} \in \delta(x))((\mathcal{U} - L)^c \notin \mathcal{P})\}$ for every $L \subseteq \mathbb{X}$.*

The theorem below demonstrates several characterizations of the operator $cl^{\Diamond}_{\mathcal{P}}$.

Theorem 1.13. *([26]) Assume that $(\mathbb{X}, \delta, \mathcal{P})$ is a PTS. Consequently, these characteristics are true:*

(a) If $L \subseteq \mathbb{X}$, then $\Psi_{\mathcal{P}}(L) = \mathbb{X} - (\mathbb{X} - L)^{\Diamond}$,

(b) If $L \subseteq \mathbb{X}$, then $\Psi_{\mathcal{P}}(L)$ is open,

(c) If $L \subseteq E$, then $\Psi_{\mathcal{P}}(L) \subseteq \Psi_{\mathcal{P}}(E)$,

(d) If $L, E \subseteq \mathbb{X}$, then $\Psi_{\mathcal{P}}(L \cap E) = \Psi_{\mathcal{P}}(L) \cap \Psi_{\mathcal{P}}(E)$,

(e) If $\mathcal{U} \in \delta^{\Diamond}$, then $\mathcal{U} \subseteq \Psi_{\mathcal{P}}(\mathcal{U})$,

(f) If $L \subseteq \mathbb{X}$, then $\Psi_{\mathcal{P}}(L) \subseteq \Psi_{\mathcal{P}}(\Psi_{\mathcal{P}}(L))$,

(g) If $L \subseteq \mathbb{X}$, then $L \cap \Psi_{\mathcal{P}}(L) = int^{\Diamond}(L)$.

Corollary 1.14. *([26]) Assume that $(\mathbb{X}, \delta, \mathcal{P})$ is a PTS. Then, $\mathcal{U} \subseteq \Psi_{\mathcal{P}}(\mathcal{U})$ for each open set $\mathcal{U} \in \delta$.*

Theorem 1.15. *([26]) Consider $(\mathbb{X}, \delta, \mathcal{P})$ as a PTS and $L \subseteq \mathbb{X}$. Then, the following properties hold:*

(a) $\Psi_{\mathcal{P}}(L) = \cup\{\mathcal{U} \in \delta : (\mathcal{U} - L)^c \notin \mathcal{P}\}$,

(b) $\Psi_{\mathcal{P}}(L) \supseteq \cup\{\mathcal{U} \in \delta : (\mathcal{U} - L)^c \cup (L - \mathcal{U})^c \notin \mathcal{P}\}$.

2. New Classes of Sets in Primal Topological Spaces

This section aims to describe, introduce, and examine several classes of open sets in primal topological spaces, as well as their fundamental characteristics and relationships.

Definition 2.1. *Suppose that a PTS $(\mathbb{X}, \delta, \mathcal{P})$. So, we may define a subset L of $\mathbb{X}$ as follows:*

(a) $\mathcal{P}$-open ([25, 26]), if $L \subseteq int(L^{\Diamond}_{\mathcal{P}})$,

(b) $\mathcal{P}$-α-open, if $L \subseteq int(cl^{\Diamond}(int(L)))$,

(c) $\mathcal{P}$-semi-open, if $L \subseteq cl^{\Diamond}(int(L))$,

(d) $\mathcal{P}$-pre-open, if $L \subseteq int(cl^{\Diamond}(L))$,

(e) $\mathcal{P}$-β-open, if $L \subseteq cl(int(cl^{\Diamond}(L)))$.

Theorem 2.2. *In a PTS $(\mathbb{X}, \delta, \mathcal{P})$, the next characteristics are true:*

(a) Each $\mathcal{P}$-α-open set is α-open,

(b) Each $\mathcal{P}$-semi-open set is semi-open,

(c) Each $\mathcal{P}$-pre-open set is pre-open,

(d) Each $\mathcal{P}$-β-open set is β-open.

Proof. (a) Suppose that L be a $\mathcal{P}$-α-open. Hence, $L \subset int(cl^{\Diamond}(int(L))) = int(int(L) \cup (int(L))^{\Diamond}) \subset int(cl(int(L)) \cup int(L)) \subset int(cl(int(L)))$. Thus, L is α-open.
(b) Suppose that L be a $\mathcal{P}$-semi-open. Hence, $L \subseteq cl^{\Diamond}(int(L)) = int(L) \cup (int(L))^{\Diamond} \subseteq int(L) \cup cl(int(L))$ (from Theorem 1.9) $= cl(int(L))$. Thus, L is semi-open.
(c) Suppose that L be a $\mathcal{P}$-pre-open. Hence, $L \subseteq int(cl^{\Diamond}(L)) = int(L \cup L^{\Diamond}) \subseteq int(L \cup cl(L)) = int(cl(L))$. Therefore, L is a pre-open set.
(d) Suppose that L be a $\mathcal{P}$-β-open set. Hence, $L \subseteq cl(int(cl^{\Diamond}(L))) = cl(int(L \cup L^{\Diamond})) \subset cl(int(cl(L) \cup L)) = cl(int(cl(L)))$. Therefore, L is a β-open set.

Remark 2.3. *In general, the following examples demonstrate that the opposite of Theorem 2.2 is not true.*

Example 2.4. *Assuming that* $\mathbb{X} = \{a_1, a_2, a_3\}$, $\delta = \{\phi, \{a_1\}, \mathbb{X}\}$, *with the primal* $\mathcal{P} = \{\phi, \{a_1\}, \{a_3\}, \{a_1, a_3\}\}$. *Thus,*
(a) $L = \{a_1, a_3\}$ *is a* α*-open set that is not* $\mathcal{P}$-α*-open, since* $L \subseteq int(cl(int(L))) = \mathbb{X}$, *but* $L \nsubseteq int(cl^{\Diamond}(int(L))) = \{a_1\}$.
(b) $L = \{a_1, a_3\}$ *is a semi-open set that is not* $\mathcal{P}$*-semi-open, since* $L \subseteq cl(int(L)) = \mathbb{X}$, *but* $L \nsubseteq cl^{\Diamond}(int(L)) = \{a_1\}$.
(c) $L = \{a_1, a_3\}$ *is a pre-open set that is not* $\mathcal{P}$*-pre-open, since* $L \subseteq int(cl(L)) = \mathbb{X}$, *but* $L \nsubseteq int(cl^{\Diamond}(L)) = \{a_1\}$.

Example 2.5. *Assuming that* $\mathbb{X} = \{a_1, a_2, a_3\}$, $\delta = \{\phi, \{a_1\}, \{a_2, a_3\}, \mathbb{X}\}$, *with the primal* $\mathcal{P} = \{\phi, \{a_1\}, \{a_3\}, \{a_1, a_3\}\}$. *Thus,* $L = \{a_1, a_3\}$ *is a* β*-open set, which is not* $\mathcal{P}$-β*-open since* $L \subseteq cl(int(cl(L))) = \mathbb{X}$, *but* $L \nsubseteq cl(int(cl^{\Diamond}(L))) = \{a_1\}$.

Remark 2.6. *Let* $(\mathbb{X}, \delta, \mathcal{P})$ *be a* PTS*, then the concept of openness and* $\mathcal{P}$*-openness are independence.*

Example 2.7. *(a) Assuming that* $\mathbb{X} = \{a_1, a_2, a_3\}$, $\delta = \{\phi, \{a_1\}, \{a_3\}, \{a_1, a_3\}, \mathbb{X}\}$, *and* $\mathcal{P} = \{\phi, \{a_1\}, \{a_2\}, \{a_1, a_2\}\}$. *Thus,* $(\mathbb{X}, \delta)$ *is a* TS*. Moreover,* $\mathcal{P}$ *is a primal on* $\mathbb{X}$*. Put* $\mathcal{U} = \{a_1, a_3\} \in \delta$*. However,* $\mathcal{U}^{\Diamond}_{\mathcal{P}} = \{a_2, a_3\}$ *in order that* $\mathcal{U}$ *is not* $\mathcal{P}$*-open.*
(b) Assuming that $\mathbb{X} = \{a_1, a_2, a_3\}$, $\delta = \{\phi, \mathbb{X}\}$ *and* $\mathcal{P} = \{\phi, \{a_1\}, \{a_2\}, \{a_1, a_2\}\}$*. Thus,* $(\mathbb{X}, \delta)$ *is a* TS*. Moreover,* $\mathcal{P}$ *is a primal on* $\mathbb{X}$*. Put* $L = \{a_2\}$*. Thus,* $L^{\Diamond}_{\mathcal{P}} = \phi$*, so that* L *is* $\mathcal{P}$*-open. However,* L *is not open in* $(\mathbb{X}, \delta)$.

Theorem 2.8. *If* $(\mathbb{X}, \delta, \mathcal{P})$ *is a* PTS*, the next characteristics apply for* $L \subseteq \mathbb{X}$.

(a) L *is* $\mathcal{P}$-α*-open if and only if it is* $\mathcal{P}$*-semi-open and* $\mathcal{P}$*-pre-open,*

(b) L *is* $\mathcal{P}$-β*-open, if* L *is* $\mathcal{P}$*-semi-open,*

(c) L *is* $\mathcal{P}$-β*-open, if* L *is* $\mathcal{P}$*-pre-open,*

(d) L *is* $\mathcal{P}$*-pre-open, if* L *is* $\mathcal{P}$*-open.*

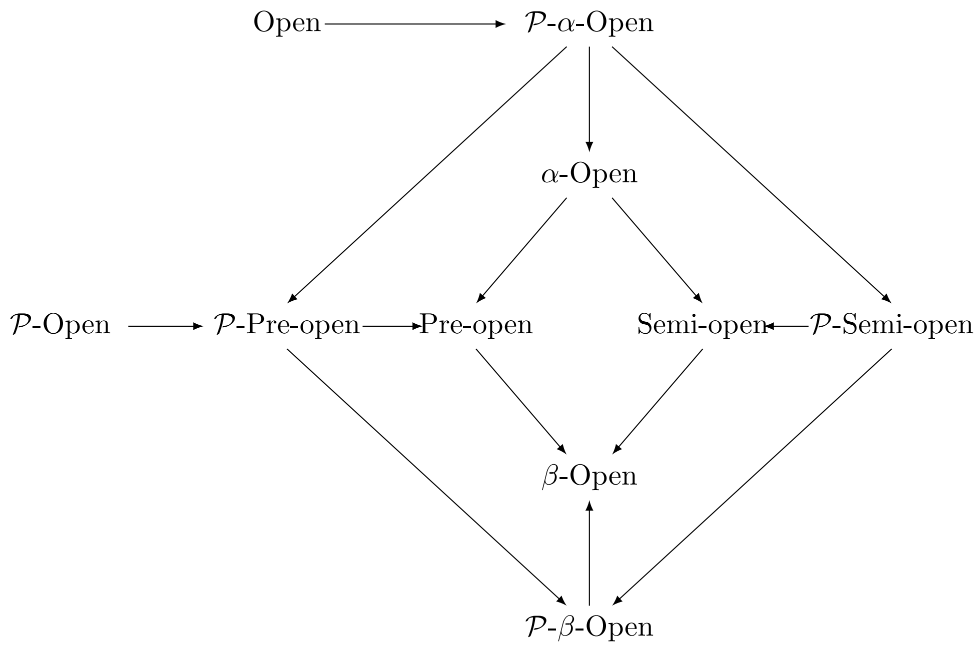
Proof. (a) Necessity. It is clear.
Sufficiency. Let L be a $\mathcal{P}$-semi-open and a $\mathcal{P}$-pre-open. Hence, $L \subseteq int(cl^{\Diamond}(L)) \subseteq$

$int(cl^\Diamond(cl^\Diamond(int(L)))) \subseteq int(cl^\Diamond(int(L)))$. Thus, L is $\mathcal{P}$-α-open.
(b) Since L is $\mathcal{P}$-semi-open and $\delta \subseteq \delta^\Diamond$, we already have $L \subseteq cl^\Diamond(int(L)) \subseteq cl(int(L)) \subseteq cl(int(cl^\Diamond(L)))$. Thus, L is $\mathcal{P}$-β-open.
(c) It is clear.
(d) Let L be a $\mathcal{P}$-open. Thus, $L \subset int(L^\Diamond_\mathcal{P}) \subset int(L \cup L^\Diamond_\mathcal{P}) = int(cl^\Diamond(L))$. Therefore, L is $\mathcal{P}$-pre-open.

Theorem 2.9. *Each open set in a PTS is $\mathcal{P}$-α-open.*

Proof. If L is any open set, then $L = int(L) \subset int((int(L))^\Diamond \cup int(L)) = int(cl^\Diamond(int(L)))$. Therefore, L is $\mathcal{P}$-α-open.

Remark 2.10. *The following figure represents several of the above-described sets, where the opposite of the figure may not be as correct as the next.*



Example 2.11. *Consider $\mathbb{X} = \{a_1, a_2, a_3\}$, $\delta = \{\phi, \{a_1\}, \mathbb{X}\}$, with the primal $\mathcal{P} = \{\phi, \{a_1\}, \{a_2\}, \{a_3\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}\}$. Thus, $L = \{a_1, a_3\}$ is a $\mathcal{P}$-α-open that is not open, since $L \subseteq int(cl^\Diamond(int(L))) = \mathbb{X}$, but $L \notin \delta$.*

Example 2.12. *Consider $\mathbb{X} = \{a_1, a_2, a_3, a_4\}$, $\delta = \{\phi, \{a_1\}, \{a_1, a_2\}, \{a_1, a_4\}, \{a_1, a_2, a_4\}, \mathbb{X}\}$, with the primal $\mathcal{P} = \{\phi, \{a_1\}, \{a_2\}, \{a_3\}, \{a_1, a_2\}, \{a_2, a_3\}, \{a_1, a_3\}, \{a_1, a_2, a_3\}\}$. Thus, $L = \{a_1, a_3, a_4\}$ is a $\mathcal{P}$-semi-open set that is not $\mathcal{P}$-α-open, since $L \subseteq cl^\Diamond(int(L)) = L$, but $L \nsubseteq int(cl^\Diamond(int(L))) = \{a_1, a_4\}$.*

Example 2.13. *Consider* $\mathbb{X} = \{a_1, a_2, a_3, a_4\}$, $\delta = \{\phi, \{a_1\}, \{a_1, a_2\}, \{a_1, a_4\}, \{a_1, a_2, a_4\}, \mathbb{X}\}$, *with the primal* $\mathcal{P} = \{\phi, \{a_1\}, \{a_2\}, \{a_3\}, \{a_1, a_2\}, \{a_2, a_3\}, \{a_1, a_3\}, \{a_1, a_2, a_3\}\}$. *Thus,* $L = \{a_1, a_3, a_4\}$ *is a* $\mathcal{P}$-β-*open set that is not* $\mathcal{P}$-*pre-open, since* $L \subseteq cl(int(cl^{\lozenge}(L))) = \mathbb{X}$, *but* $L \nsubseteq int(cl^{\lozenge}(L)) = \{a_1, a_4\}$.

Example 2.14. *Consider* $\mathbb{X} = \{a_1, a_2, a_3\}$, $\delta = \{\phi, \mathbb{X}\}$, *with the primal* $\mathcal{P} = \{\phi, \{a_1\}, \{a_2\}, \{a_3\}, \{a_1, a_2\}, \{a_2, a_3\}\}$. *Thus,*

(a) $L = \{a_3\}$ *is a set that is* $\mathcal{P}$-β-*open but not* $\mathcal{P}$-*semi-open, since* $L \subseteq cl(int(cl^{\lozenge}(L))) = \mathbb{X}$, *but* $L \nsubseteq cl^{\lozenge}(int(L)) = \phi$.

(b) $L = \{a_3\}$ *is a set that is* $\mathcal{P}$-*pre-open but not* $\mathcal{P}$-α-*open, since* $L \subseteq int(cl^{\lozenge}(L)) = \mathbb{X}$, *but* $L \nsubseteq int(cl^{\lozenge}(int(L))) = \phi$.

(c) $L = \{a_1, a_3\}$ *is a set that is* $\mathcal{P}$-*pre-open but not* $\mathcal{P}$-*open, since* $L \subseteq int(cl^{\lozenge}(L)) = \mathbb{X}$, *but* $L \notin \mathcal{P}$.

Example 2.15. *Consider* $\mathbb{X} = \{a_1, a_2, a_3\}$, $\delta = \{\phi, \mathbb{X}\}$, *with the primal set given by* $\mathcal{P} = \{\phi, \{a_1\}, \{a_2\},$
$\{a_3\}, \{a_1, a_2\}, \{a_1, a_3\}\}$. *Thus,* $L = \{a_3\}$ *is a set that is* $\mathcal{P}$-*open but not* $\mathcal{P}$-*semi-open, since* $L \subseteq int(L^{\lozenge}_{\mathcal{P}}) = \mathbb{X}$, *but* $L \nsubseteq cl^{\lozenge}(int(L)) = \phi$.

Definition 2.16. *A subset* L *of a* PTS $(\mathbb{X}, \delta, \mathcal{P})$ *is called* $\mathcal{P}$-*dense in* $\mathbb{X}$ *if* $cl^{\lozenge}(L) = \mathbb{X}$.

Theorem 2.17. *Assume that* $(\mathbb{X}, \delta, \mathcal{P})$ *be* PTS. *Thus, for a subset* L *of* $\mathbb{X}$, *the next is true:*

(a) If $\mathcal{P} = 2^{\mathbb{X}} \setminus \{\mathbb{X}\}$, L *is* $\mathcal{P}$-β-*open iff* L *is semi-open,*

(b) Let $\mathcal{P} = \{N \subseteq \mathbb{X} \setminus N$ *is the primal dense for all nowhere dense set*$\}$ *and* $L \subseteq \mathbb{X}$. *Then,* L *is* $\mathcal{P}$-β-*open iff* L *is* β-*open.*

Proof.
(a) When $\mathcal{P} = 2^{\mathbb{X}} \setminus \{\mathbb{X}\}$, thus $L^{\lozenge}_{\mathcal{P}} = \phi$ for any $L \subset \mathbb{X}$. Consequently, we have $cl(int(cl^{\lozenge}(L))) = cl(int(L^{\lozenge} \cup L)) = cl(int(L))$. Thus, $\mathcal{P}$-β-openness and semi-openness are equivalent.
(b) By Theorem 2.2, every $\mathcal{P}$-β-open set is β-open. If $\mathcal{P} = N$, then it is well-known that $L^{\lozenge} = cl(int(cl(L)))$. Therefore, if L is β-open, we obtain $L \subset cl(int(cl(L))) = L^{\lozenge} = cl^{\lozenge}(L)$, and hence $L \subset cl(int(cl(L))) = cl(int[cl(int(cl(L)))]) = cl(int(cl^{\lozenge}(L)))$.

3. $\mathcal{P}_{\mathcal{R}}$-sets and $\mathcal{P}_{\mathcal{R}_\alpha}$-sets

In this section, we focus on using the PTS of some defined sets, namely the $\mathcal{P}_{\mathcal{R}}$-sets and $\mathcal{P}_{\mathcal{R}_\alpha}$-sets. Furthermore, their characterizations and main features are determined, and their relationships with another set are investigated.

Definition 3.1. *Assume that* $(\mathbb{X}, \delta, \mathcal{P})$ *be* PTS. *Thus,* $L \subseteq \mathbb{X}$ *has the following definition:*

(a) Primal t-set (briefly, $\mathcal{P}_t$*-set), if* $int(L) = int(cl^{\lozenge}(L))$.

(b) Primal tα-set (briefly, $\mathcal{P}_{t\alpha}$-set), if $int(L) = int(cl^{\Diamond}(int(L)))$.

(c) Primal $\mathcal{R}$-set (briefly, $\mathcal{P}_{\mathcal{R}}$-set), if $L = L_1 \cap L_2$, *where* L_1 *is open and* L_2 *is* $\mathcal{P}_t$*-set.*

(d) Primal $\mathcal{R}_\alpha$-set (briefly, $\mathcal{P}_{\mathcal{R}_\alpha}$-set), if $L = L_1 \cap L_2$, *where* L_1 *is open and* L_2 *is* $\mathcal{P}_{t_\alpha}$*-set.*

Example 3.2. *Let* $\mathbb{X} = \{a_1, a_2\}$ *and* $\delta = \{\phi, \{a_1\}, \mathbb{X}\}$. *If* $\mathcal{P} = \{\phi, \{a_1\}\}$, *then* $\{a_1\}$ *is* $\mathcal{P}_t$*-set,* $\mathcal{P}_{t_\alpha}$*-set,* $\mathcal{P}_{\mathcal{R}}$*-set, and* $\mathcal{P}_{\mathcal{R}_\alpha}$*-set, since* $int(\{a_1\}) = int(cl^{\Diamond}(\{a_1\})) = int(cl^{\Diamond}(int(\{a_1\}))) = \{a_1\}$.

Theorem 3.3. *Assume that* $(\mathbb{X}, \delta, \mathcal{P})$ *be PTS. Therefore,*

(a) Each open set $\mathcal{U}$ *is* $\mathcal{P}_{\mathcal{R}}$*-set.*

(b) Each $\mathcal{P}_t$*-set is* $\mathcal{P}_{\mathcal{R}}$*-set.*

Proof. (a) Put $\mathcal{U} = \mathcal{U} \cap \mathbb{X}$. Thus, $int(cl^{\Diamond}(\mathcal{U})) = int(\mathcal{U})$.
(b) Let L be a $\mathcal{P}_t$-set. If we assume that $\mathcal{U} = \mathbb{X} \in \delta$, then $L = \mathcal{U} \cap L$, and hence L is $\mathcal{P}_{\mathcal{R}}$-set.

Remark 3.4. *The opposite of Theorem 3.3 is untrue in all cases, as proved in the following Examples.*

Example 3.5. *In Example 3.2, the set* $\{a_2\}$ *is* $\mathcal{P}_{\mathcal{R}}$*-set. However, it is not an open set.*

Example 3.6. *Consider* $\mathbb{X} = \{a_1, a_2, a_3\}$, $\delta = \{\phi, \{a_3\}, \mathbb{X}\}$ *and* $\mathcal{P} = \{\phi, \{a_1\}, \{a_2\}, \{a_1, a_2\}\}$. *Then,* $\{a_3\}$ *is* $\mathcal{P}_{\mathcal{R}}$*-set but not* $\mathcal{P}_t$*-set, since* $\{a_3\} = int(\{a_3\}) \neq int(cl^{\Diamond}(\{a_3\})) = \mathbb{X}$.

Proposition 3.7. *Suppose that* L *and* E *are subsets of the space* $(\mathbb{X}, \delta, \mathcal{P})$. *If* L *and* E *are* $\mathcal{P}_t$*-sets, then* $L \cap E$ *is a* $\mathcal{P}_t$*-set.*

Proof. Let L and E be $\mathcal{P}_t$-sets. We have $int(L \cap E) \subset int(cl^{\Diamond}(L \cap E)) \subset int(cl^{\Diamond}(L) \cap cl^{\Diamond}(E)) = int(cl^{\Diamond}(L)) \cap int(cl^{\Diamond}(E)) = int(L) \cap int(E) = int(L \cap E)$. Then $int(L \cap E) = int(cl^{\Diamond}(L \cap E))$, and hence $L \cap E$ is a $\mathcal{P}_t$-set.

The example below shows that the union of two $\mathcal{P}_t$-sets need not be a $\mathcal{P}_t$-sets.

Example 3.8. *Consider* $\mathbb{X} = \{a_1, a_2, a_3, a_4\}$, $\delta = \{\phi, \{a_1, a_3\}, \mathbb{X}\}$, *with the primal* $\mathcal{P} = \{\phi, \{a_1\}, \{a_2\}, \{a_3\},$
$\{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}, \{a_1, a_2, a_3\}\}$. *Then,* $L = \{a_1, a_3\}$ *and* $E = \{a_1, a_4\}$ *are* $\mathcal{P}_t$*-sets, since* $int(\{a_1, a_3\}) = int(cl^{\Diamond}(\{a_1, a_3\}) = \{a_1, a_3\}$ *and* $int(\{a_1, a_4\}) = int(cl^{\Diamond}(\{a_1, a_4\}) = \phi$, *but* $L \cup E = \{a_1, a_3\, a_4\}$ *is not* $\mathcal{P}_t$*-set.*

Proposition 3.9. *Assume that* $(\mathbb{X}, \delta, \mathcal{P})$ *is a PTS. The following statements are equivalent for a subset* L *of* $\mathbb{X}$*:*

(a) L *is open,*

(b) L *is* $\mathcal{P}$*-pre-open and* $\mathcal{P}_{\mathcal{R}}$*-set.*

Proof. (a)⇒(b) : Consider L as open. Thus, $L = int(L) \subset int(cl^{\Diamond}(L))$, and L is $\mathcal{P}$-pre-open. Also by Theorem 3.3, L is $\mathcal{P}_{\mathcal{R}}$-set.
(b) ⇒ (a): Consider L as a $\mathcal{P}_{\mathcal{R}}$-set. So $L = L_1 \cap L_2$, where L_1 is open, and $int(Q) = int(cl(Q))$. Thus, $L \subseteq L_1 = int(L_1)$. Also, L is $\mathcal{P}$-pre-open implies $L \subseteq int(cl(L)) \subset int(cl^{\Diamond}(L_2)) = int(L_2)$ by assumption. Consequently, $L \subseteq int(L_1) \cap int(L_2) = int(L_1 \cap L_2) = int(L)$, and so L is open.

Remark 3.10. *Suppose that $(\mathbb{X}, \delta, \mathcal{P})$ is a PTS. So, the concepts of $\mathcal{P}$-pre-open sets and $\mathcal{P}_{\mathcal{R}}$-sets are independent.*

Example 3.11. *(a) Consider $\mathbb{X} = \{a_1, a_2, a_3, a_4\}$, $\delta = \{\phi, \{a_1, a_3\}, \mathbb{X}\}$, with the primal $\mathcal{P} = \{\phi, \{a_1\}, \{a_2\}, \{a_3\},$
$\{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}, \{a_1, a_2, a_3\}\}$. Thus, $L = \{a_1, a_3, a_4\}$ is $\mathcal{P}$-pre-open but not $\mathcal{P}_{\mathcal{R}}$-sets, since $L \subseteq int(cl^{\Diamond}(L)) = \mathbb{X}$, but $L = \mathbb{X} \cap L$ and $\{a_1, a_3\} = int(L) \neq int(cl^{\Diamond}(L)) = \mathbb{X}$.*
(b) In Example 2.13, $\{a_1, a_3, a_4\}$ is $\mathcal{P}_{\mathcal{R}}$-sets but not $\mathcal{P}$-pre-open.

Theorem 3.12. *Suppose that $(\mathbb{X}, \delta, \mathcal{P})$ is a PTS. So,*

(a) Each open set is $\mathcal{P}_{\mathcal{R}_\alpha}$-set,

(b) Each $\mathcal{P}_{t_\alpha}$-set is $\mathcal{P}_{\mathcal{R}_\alpha}$-set.

Proof. Obvious.

Example 3.13. *(a) In Example 3.2, the set $\{a_2\}$ is $\mathcal{P}_{\mathcal{R}_\alpha}$-set. However, it is not open.*
(b) In Example 3.6, the set $\{a_3\}$ is $\mathcal{P}_{\mathcal{R}_\alpha}$-set. However, it is not $\mathcal{P}_{t_\alpha}$-set.

Proposition 3.14. *If L_1 and L_2 are $\mathcal{P}_{t_\alpha}$-sets, then $L_1 \cap L_2$ is a $\mathcal{P}_{t_\alpha}$-set.*

Proof. Let L_1 and L_2 be $\mathcal{P}_{t_\alpha}$-sets. Next, we have $int(L_1 \cap L_2) \subset int(cl^{\Diamond}(int(L_1 \cap L_2))) \subseteq int[cl^{\Diamond}(int(L_1)) \cap cl^{\Diamond}(int(L_2))] = int(cl^{\Diamond}(int(L_1))) \cap int(cl^{\Diamond}(int(L_2))) = int(L_1) \cap int(L_2) = int(L_1 \cap L_2)$. Then, $int(L_1 \cap L_2) = int(cl^{\Diamond}(int(L_1 \cap L_2)))$. Therefore, $L_1 \cap L_2$ is a $\mathcal{P}_{t_\alpha}$-set.

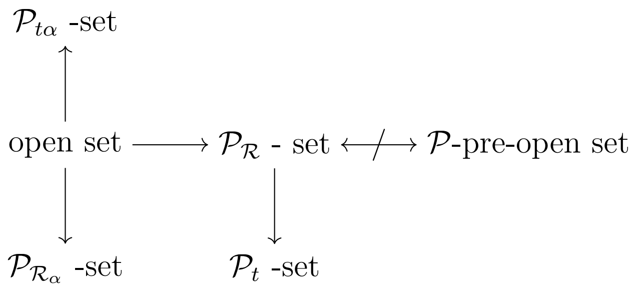
Proposition 3.15. *Assume that $(\mathbb{X}, \delta, \mathcal{P})$ is a PTS. The following statements are equivalent for a subset L of $\mathbb{X}$:*

(a) L is open,

(b) L is $\mathcal{P}$-α-open and $\mathcal{P}_{\mathcal{R}_\alpha}$-set.

Proof. (a) ⇒ (b): Suppose that L is an open set. Thus, $L = int(L) \subseteq cl^{\Diamond}(int(L))$ and $L = int(L) \subseteq int(cl^{\Diamond}(int(L)))$. Therefore, L is $\mathcal{P}$-α-open. Also by Theorem 3.12, L is $\mathcal{P}_{\mathcal{R}_\alpha}$-set.
(b)⇒ (a): Let L be the $\mathcal{P}_{\mathcal{R}_\alpha}$-set. So, $L = L_1 \cap L_2$, where L_1 is open and $int(L_2) = int(cl^{\Diamond}(int(L_2)))$. Thus, $L \subseteq L_1 = int(L_1)$. Also, L is $\mathcal{P}$-α-open implies $L \subseteq int(cl^{\Diamond}(int(L))) \subseteq int(cl^{\Diamond}(int(L_2))) = int(L_2)$ by assumption. Thus, $L \subseteq int(L_1) \cap int(L_2) = int(L_1 \cap L_2) = int(L)$, and L is open.

Remark 3.16. *The relationships between the above open sets are shown in the following figure:*



4. Decomposition of Generalized Continuity

In this section, we focus on defining some classes of primal continuous functions to obtain decompositions of continuity.

Definition 4.1. *A function $\mathcal{F} : (\mathbb{X}, \delta, \mathcal{P}) \to (\mathbb{Y}, \varsigma)$ is said to be $\mathcal{P}$-α-continuous (resp. $\mathcal{P}$-semicontinuous, $\mathcal{P}$-precontinuous, $\mathcal{P}$-β-continuous) if the inverse image of each open set in $\mathbb{Y}$ is $\mathcal{P}$-α-open (resp. $\mathcal{P}$-semi-open, $\mathcal{P}$-pre-open, $\mathcal{P}$-β-open) in $(\mathbb{X}, \delta, \mathcal{P})$.*

Theorem 4.2. *Let $\mathcal{F} : (\mathbb{X}, \delta, \mathcal{P}) \to (\mathbb{Y}, \varsigma)$ be a function. Hence, $\mathcal{F}$ is a $\mathcal{P}$-α-continuous iff it is $\mathcal{P}$-semicontinuous and $\mathcal{P}$-precontinuous.*

Proof. Clearly from Theorem 2.8.

Definition 4.3. *Let $\mathcal{F} : (\mathbb{X}, \delta, \mathcal{P}) \to (\mathbb{Y}, \varsigma)$ be a function. Hence, $\mathcal{F}$ is said to be α-continuous ([33]) (resp. semicontinuous ([1]), precontinuous ([3]), β-continuous ([4])) if the inverse image of any open set of $(\mathbb{Y}, \varsigma)$ is an α-open (resp. semi-open, pre-open, β-open) in $(\mathbb{X}, \delta, \mathcal{P})$.*

Proposition 4.4. *If a function $\mathcal{F} : (\mathbb{X}, \delta, \mathcal{P}) \to (\mathbb{Y}, \varsigma)$ is $\mathcal{P}$-α-continuous (resp. $\mathcal{P}$-semicontinuous, $\mathcal{P}$-precontinuous, $\mathcal{P}$-β-continuous), thus $\mathcal{F}$ is α-continuous (resp. semicontinuous, precontinuous, β-continuous).*

Proof. Clearly from Theorem 2.2.

Example 4.5. *Consider $\mathbb{X} = \{a_1, a_2, a_3, a_4\}$, $\delta = \{\phi, \{a_1\}, \{a_1, a_2\}, \{a_1, a_4\}, \{a_1, a_2, a_4\}, \mathbb{X}\}$, with the primal $\mathcal{P} = \{\phi, \{a_1\}, \{a_2\}, \{a_3\}, \{a_1, a_2\}, \{a_2, a_3\}, \{a_1, a_3\}, \{a_1, a_2, a_3\}\}$. We define a function $\mathcal{F} : (\mathbb{X}, \delta, \mathcal{P}) \to (\mathbb{X}, \delta)$ as follows: $\mathcal{F}(a_1) = a_1$, $\mathcal{F}(a_2) = a_2$, and $\mathcal{F}(a_3) = \mathcal{F}(a_4) = a_3$. Thus, $\mathcal{F}$ is not continuous since $\mathcal{F}^{-1}(\{a_1, a_3\}) = \{a_1, a_4\}$ is not $\mathcal{P}$-open. However, $\mathcal{F}$ is $\mathcal{P}$-semicontinuous.*

Definition 4.6. *A function $\mathcal{F} : (\mathbb{X}, \delta, \mathcal{P}) \to (\mathbb{Y}, \varsigma)$ is said to be $\mathcal{P}_{\mathcal{R}}$-continuous (resp. $\mathcal{P}_{\mathcal{R}_\alpha}$-continuous), if the inverse image of each open set in $\mathbb{Y}$ is $\mathcal{P}_{\mathcal{R}}$-set (resp. $\mathcal{P}_{\mathcal{R}_\alpha}$-set) in $(\mathbb{X}, \delta, \mathcal{P})$.*

Definition 4.7. *A function $\mathcal{F} : (\mathbb{X}, \delta) \to (\mathbb{Y}, \varsigma)$ is said to be $\mathcal{R}$-continuous [31] (resp. $\mathcal{R}_\alpha$-continuous [32]), if the inverse image of each open set in $\mathbb{Y}$ is $\mathcal{R}$-set (resp. $\mathcal{R}_\alpha$-set) in $(\mathbb{X}, \delta)$.*

Proposition 4.8. *If a function $\mathcal{F} : (\mathbb{X}, \delta, \mathcal{P}) \to (\mathbb{Y}, \varsigma)$ is $\mathcal{R}$-continuous (resp. $\mathcal{R}_\alpha$-continuous), it is also $\mathcal{P}_\mathcal{R}$-continuous (resp. $\mathcal{P}_{\mathcal{R}_\alpha}$-continuous).*

Proof. Straightforward.

Theorem 4.9. *Let $\mathcal{F} : (\mathbb{X}, \delta, \mathcal{P}) \to (\mathbb{Y}, \varsigma)$ be a function. Hence, the following are equivalent:*

(a) $\mathcal{F}$ is continuous;

(b) $\mathcal{F}$ is a $\mathcal{P}$-precontinuous and a $\mathcal{P}_\mathcal{R}$-continuous;

(c) $\mathcal{F}$ is a $\mathcal{P}$-α-continuous and a $\mathcal{P}_{\mathcal{R}_\alpha}$-continuous.

Proof. It is a direct result of Propositions 3.9 and Propositions 3.15.

5. $\widetilde{\Psi}_\mathcal{P}$-Sets

In this section, we describe a new class of sets in PTS that contain the class of all open sets, using the $\Psi_\mathcal{P}$-operator.

Definition 5.1. *A subset L of a PTS $(\mathbb{X}, \delta, \mathcal{P})$ is called $\widetilde{\Psi}_\mathcal{P}$-set if $L \subseteq cl(\Psi_\mathcal{P}(L))$.*

The family of all $\widetilde{\Psi}_\mathcal{P}$-sets in $(\mathbb{X}, \delta, \mathcal{P})$ is denoted by $\widetilde{\Psi}_\mathcal{P}(\mathbb{X}, \delta)$.

Theorem 5.2. *Suppose that $(\mathbb{X}, \delta, \mathcal{P})$ is a PTS. If $L \in \delta$, then $L \in \widetilde{\Psi}_\mathcal{P}(\mathbb{X}, \delta)$.*

Proof. By Corollary 1.14, $\delta \subset \widetilde{\Psi}_\mathcal{P}(\mathbb{X}, \delta)$ is obtained in the topological space $(\mathbb{X}, \delta, \mathcal{P})$.

In general, the following example demonstrates that the opposite of Theorem 5.2 is not true.

Example 5.3. *Let $\mathbb{X} = \{a_1, a_2, a_3\}$, and $\delta = \{\phi, \{a_1\}, \{a_2\}, \{a_1, a_2\}, \mathbb{X}\}$, with the primal $\mathcal{P} = \{\phi, \{a_1\}, \{a_2\}, \{a_1, a_2\}\}$. Now, $\Psi_\mathcal{P}(\{a_3\}) = \mathbb{X} - \{a_1, a_2\}^\diamond = \mathbb{X} - \phi = \mathbb{X}$. Thus, $cl(\Psi_\mathcal{P}(\{a_3\})) = \mathbb{X}$. Therefore, $\{a_3\} \subseteq cl(cl^\diamond_\mathcal{P}(\{a_3\}))$, but $\{a_3\}$ is not open in δ.*

Now, we show that any union of $\widetilde{\Psi}_\mathcal{P}$-sets is a $\widetilde{\Psi}_\mathcal{P}(\mathbb{X}, \delta)$.

Proposition 5.4. *Suppose that $\{L_\alpha : \alpha \in \Delta\}$ is a set of non-empty $\widetilde{\Psi}_\mathcal{P}$-sets in a PTS $(\mathbb{X}, \delta, \mathcal{P})$, then $\cup_{\alpha\in\Delta} L_\alpha \in \widetilde{\Psi}_\mathcal{P}(\mathbb{X}, \delta)$.*

Proof. For every $\alpha \in \Delta$, $L_\alpha \subseteq cl(\Psi_\mathcal{P}(L_\alpha)) \subsetneq cl(\Psi_\mathcal{P}(\cup_{\alpha\in\Delta} L_\alpha))$. This implies that $\cup_{\alpha\in\Delta} L_\alpha \subseteq cl(\Psi_\mathcal{P}(\cup_{\alpha\in\Delta} L_\alpha))$. Hence, $\cup_{\alpha\in\Delta} L_\alpha \in \widetilde{\Psi}_\mathcal{P}(\mathbb{X}, \delta)$.

The example below demonstrates an intersection of two $\widetilde{\Psi}_\mathcal{P}$-sets not necessarily a $\widetilde{\Psi}_\mathcal{P}$-set.

Example 5.5. *Assuming that* $\mathbb{X} = \{a_1, a_2, a_3\}$, $\delta = \{\phi, \{a_1, a_3\}, \mathbb{X}\}$, *with the primal* $\mathcal{P} = \{\phi, \{a_2\}, \{a_3\}, \{a_2, a_3\}\}$. *Then* $L = \{a_1, a_2\}$ *and* $E = \{a_1, a_3\}$ *are* $\widetilde{\Psi}_{\mathcal{P}}$*-sets, since* $\Psi_{\mathcal{P}}(\{a_1, a_2\}) = \mathbb{X} - \{a_3\}^{\Diamond} = \mathbb{X}$, $\{a_1, a_2\} \subseteq cl(\Psi_{\mathcal{P}}(\{a_1, a_2\})) = \mathbb{X}$ *and* $\Psi_{\mathcal{P}}(\{a_1, a_3\}) = \mathbb{X} - \{a_2\}^{\Diamond} = \mathbb{X}$, $\{a_1, a_3\} \subseteq cl(\Psi_{\mathcal{P}}(\{a_1, a_3\})) = \mathbb{X}$. *Therefore,* $L \cap E = \{a_1\}$ *is not* $\widetilde{\Psi}_{\mathcal{P}}$*-set, since* $\Psi_{\mathcal{P}}(\{a_1\}) = \mathbb{X} - \{a_2, a_3\}^{\Diamond} = \phi$, $\{a_1\} \nsubseteq cl(\Psi_{\mathcal{P}}(\{a_1\})) = \phi$.

We will demonstrate that the intersection of two $\widetilde{\Psi}_{\mathcal{P}}$-sets are not often be a $\widetilde{\Psi}_{\mathcal{P}}$-set, we will show that the intersection of a δ^{α} with a $\widetilde{\Psi}_{\mathcal{P}}$-set is a $\widetilde{\Psi}_{\mathcal{P}}$-set.

Theorem 5.6. *Suppose that we have a PTS* $(\mathbb{X}, \delta, \mathcal{P})$*, and let* L *belong to the set* $\widetilde{\Psi}_{\mathcal{P}}(\mathbb{X}, \delta)$. *Thus, if* $\mathcal{U}$ *is an element of* δ^{α}*, then it follows that the intersection of* $\mathcal{U}$ *with* L *also belongs to the set* $\widetilde{\Psi}_{\mathcal{P}}(\mathbb{X}, \delta)$.

Proof. We note that if G is open, for any $L \subseteq \mathbb{X}$, $G \cap cl(L) \subseteq cl(G \cap L)$. Let $\mathcal{U} \in \delta^{\alpha}$ and $L \in \widetilde{cl^{\Diamond}_{\mathcal{P}}}(\mathbb{X}, \delta)$. Then by Theorem 1.15 and Corollary 1.14 we have $\mathcal{U} \cap L \subseteq int(cl(int(\mathcal{U}))) \cap cl(\Psi_{\mathcal{P}}(L)) \subseteq int(cl(\Psi_{\mathcal{P}}(\mathcal{U}))) \cap cl(\Psi_{\mathcal{P}}(L)) \subseteq cl[int(cl(\Psi_{\mathcal{P}}(\mathcal{U}))) \cap \Psi_{\mathcal{P}}(L)] = cl[int(cl[\Psi_{\mathcal{P}}(\mathcal{U}) \cap \Psi_{\mathcal{P}}(L)])] = cl[\Psi_{\mathcal{P}}(\mathcal{U}) \cap \Psi_{\mathcal{P}}(L)] = cl[\Psi_{\mathcal{P}}(\mathcal{U} \cap L)]$. Hence, $\mathcal{U} \cap L \in \widetilde{\Psi}_{\mathcal{P}}(\mathbb{X}, \delta)$.

Corollary 5.7. *Let* $(\mathbb{X}, \delta, \mathcal{P})$ *be a PTS, and let* L *belong to* $\widetilde{\Psi}_{\mathcal{P}}(\mathbb{X}, \delta)$. *If* $\mathcal{U}$ *is an element of* δ*, then their intersection* $\mathcal{U} \cap L$ *also belongs to* $\widetilde{\Psi}_{\mathcal{P}}(\mathbb{X}, \delta)$.

For any non-empty relative to an open set $\mathcal{U} \cap L$, it holds that $(\mathcal{U} \cap L) \cap \mathcal{D} \in \mathcal{P}$ when $\mathcal{U} \in \delta$, then we refer to a set $\mathcal{D}$ as being relative $\mathcal{P}$-dense in the set L.

Theorem 5.8. *Let* $(\mathbb{X}, \delta, \mathcal{P})$ *be a PTS. A set* $L \notin \widetilde{\Psi}_{\mathcal{P}}(\mathbb{X}, \delta)$ *if and only if there exists an element* x *in* L *such that there is a neighborhood* $\mathcal{V}_x \in \delta$ *of* x *for which* $\mathbb{X} - L$ *is relative to* $\mathcal{P}$*-dense in* $\mathcal{V}_x$.

Proof. Suppose that a set $L \notin \widetilde{\Psi}_{\mathcal{P}}(\mathbb{X}, \delta)$. We need to show the existence of $x \in L$ and a neighborhood $\mathcal{V}_x \in \delta(x)$, then $(\mathbb{X} - L)$ is relative $\mathcal{P}$-dense in $\mathcal{V}_x$. Now, since $L \nsubseteq cl(\Psi_{\mathcal{P}}(L))$, and so there exists an element $x \in \mathbb{X}$ such that x is in L but not in the closure of $\Psi_{\mathcal{P}}(L)$. Consequently, there is a neighborhood $\mathcal{V}_x \in \delta(x)$ so that $\mathcal{V}_x \cap \Psi_{\mathcal{P}}(L) = \phi$. Hence, $\mathcal{V}_x \cap (\mathbb{X} - (\mathbb{X} - L)^{\Diamond}) = \phi$, and therefore, $\mathcal{V}_x \subseteq (\mathbb{X} - L)^{\Diamond}$. Now, consider any non-empty open set $\mathcal{U}$ in $\mathcal{V}_x$. Since $\mathcal{V}_x \subseteq (\mathbb{X} - L)^{\Diamond}$, it follows that $\mathcal{U} \cap (\mathbb{X} - L) \in \mathcal{P}$. This demonstrates that $(\mathbb{X} - L)$ is relatively $\mathcal{P}$-dense in $\mathcal{V}_x$.

Definition 5.9. *Let* $(\mathbb{X}, \delta, \mathcal{P})$ *is a PTS.* $\mathcal{P}$ *is said to be primal anti-codense if* $\delta - \{\phi\} \subseteq \mathcal{P}$.

Theorem 5.10. *In the PTS* $(\mathbb{X}, \delta, \mathcal{P})$*, if* $\mathcal{P}$ *is characterized as a primal anti-codense, then* $SO(\mathbb{X}, \delta^{\Diamond}) = \widetilde{\Psi}_{\mathcal{P}}(\mathbb{X}, \delta)$.

Proof. First assume that L is an element of $SO(\mathbb{X}, \delta^{\Diamond})$. According to Theorem 1.15, we have $L \subseteq \delta^{\Diamond}\text{-}cl(\delta^{\Diamond}\text{-}int(L)) = \delta^{\Diamond}\text{-}cl(\Psi_{\mathcal{P}}(L) \cap L)$. This implies that $L \subseteq cl(\Psi_{\mathcal{P}}(L) \cap L) \subseteq$

$cl(\Psi_{\mathcal{P}}(L))$. Consequently, we can conclude that L belongs to $\widetilde{\Psi}_{\mathcal{P}}(\mathbb{X},\delta)$. Therefore, $SO(\mathbb{X},\delta^{\Diamond}) \subseteq \widetilde{\Psi}_{\mathcal{P}}(\mathbb{X},\delta)$.
Conversely, assume that L is an element of $\widetilde{\Psi}_{\mathcal{P}}(\mathbb{X},\delta)$, and let x be an element of L. Consider a basic neighborhood $\mathcal{U}_1$ of x in $(\mathbb{X},\delta^{\Diamond})$. The neighborhood $\mathcal{U}_1$ can be represented as $\mathcal{U} - G$, where $\mathcal{U} \in \delta$ and $G \notin \mathcal{P}$. This implies that x is in $\mathcal{U}$, and consequently, $L \subseteq cl(\Psi_{\mathcal{P}}(L))$. Also, $\mathcal{U} \in \delta(x)$, which means $\mathcal{U} \cap \Psi_{\mathcal{P}}(L) \neq \phi$. Now, let $y \in \mathcal{U} \cap \Psi_{\mathcal{P}}(L)$. Then, there is exists a neighborhood $\mathcal{W}y$ of y such that $\mathcal{W}y - L \notin \mathcal{P}$ (by definition of $\Psi_{\mathcal{P}}(L)$). Now, assume that $\mathcal{U} \cap \mathcal{W}y = \mathcal{V}$. Consider $G_1 = \mathcal{V} - L \notin \mathcal{P}$. Since $\mathcal{V} \neq \phi$, $\mathcal{V} \in \delta$, and $\mathcal{V} - G_1 \subseteq L$, it follows that $\mathcal{V} \subseteq \mathcal{U}$. Consequently, $\mathcal{M} = \mathcal{V} - (G_1 \cup G) \subseteq L$ and $\mathcal{M} = \mathcal{V} - (G_1 \cup G) \neq \phi$, since $\mathcal{P}$ is a primal anti-codense, then $\mathcal{M} \subseteq L \cap (\mathcal{U} - G)$. Hence, we have shown that L includes a nonempty $\delta^{\Diamond}$-open set $\mathcal{M}$ included in $\mathcal{U} - G$. Choose $x \in L$, we have that $L \subseteq \delta^{\Diamond}\text{-}cl(\delta^{\Diamond}\text{-}int(L))$. Therefore, L is an element of $SO(\mathbb{X},\delta^{\Diamond})$. Thus, we have shown that $\widetilde{\Psi}_{\mathcal{P}}(\mathbb{X},\delta) \subseteq SO(\mathbb{X},\delta^{\Diamond})$. Thus, $SO(\mathbb{X},\delta^{\Diamond}) = \widetilde{\Psi}_{\mathcal{P}}(\mathbb{X},\delta)$.

Definition 5.11. *A subset L of a PTS $(\mathbb{X},\delta,\mathcal{P})$ is called a Ψ_L-set if $L \subseteq int(cl(\Psi_{\mathcal{P}}(L)))$.*

The set of all Ψ_L-sets in $(\mathbb{X},\delta,\mathcal{P})$ is represented as δ^L. By Definitions 5.1 and Definitions 5.9, it can be deduced that δ^L is a subset of $\delta^L \subseteq \widetilde{\Psi}_{\mathcal{P}}(\mathbb{X},\delta)$. We demonstrate that the collection δ^L forms a topology.

Theorem 5.12. *Suppose that $(\mathbb{X},\delta,\mathcal{P})$ is a PTS. If $\mathcal{P}$ is primal anti-codense, then the collection $\delta^L = \{L \subseteq \mathbb{X} : L \subseteq int(cl(\Psi_{\mathcal{P}}(L)))\}$ forms a topology on $\mathbb{X}$.*

Proof. We have show that both ϕ and $\mathbb{X}$ satisfy the conditions $\phi \subseteq int(cl(\Psi_{\mathcal{P}}(\phi)))$ and $\mathbb{X} \subseteq int(cl(\Psi_{\mathcal{P}}(\mathbb{X})))$, which means that ϕ and $\mathbb{X}$ belong to the collection δ^L.
Suppose that a family of sets $\{L_\alpha : \alpha \in \Delta\} \subseteq \delta^L$. For any $\alpha \in \Delta$, it holds that $\Psi_{\mathcal{P}}(L\alpha) \subseteq \Psi_{\mathcal{P}}(\cup L\alpha)$. Consequently, $L_\alpha \subseteq int(cl(\Psi_{\mathcal{P}}(L\alpha))) \subseteq int(cl(\Psi_{\mathcal{P}}(\cup L\alpha)))$ for any $\alpha \in \Delta$. This implies that $\cup L_\alpha \subseteq int(cl(\Psi_{\mathcal{P}}(\cup L\alpha)))$. Thus, $\cup L_\alpha$ is an element of δ^L.
Let L and E be two sets in δ^L. As $\Psi_{\mathcal{P}}(L)$ is open in $(\mathbb{X},\delta)$, we can apply Theorem 1.15, which leads to the conclusion that $L \cap E \subseteq int(cl(\Psi_{\mathcal{P}}(L))) \cap int(cl(\Psi_{\mathcal{P}}(E))) = int(cl(\Psi_{\mathcal{P}}(L) \cap \Psi_{\mathcal{P}}(E))) = int(cl(\Psi_{\mathcal{P}}(L \cap E)))$. Therefore, $L \cap E \subseteq int(cl(\Psi_{\mathcal{P}}(L \cap E)))$ and $L \cap E \in \delta^L$. This is the end of the proof.

Proposition 5.13. *Suppose that $(\mathbb{X},\delta,\mathcal{P})$ is a PTS. Then, $\Psi_{\mathcal{P}}(L) \neq \phi$ if and only if L has a non-empty $\delta^{\Diamond}$-interior.*

Proof. First, suppose that $\Psi_{\mathcal{P}}(L) \neq \phi$. However, according to Theorem 1.15, $\Psi_{\mathcal{P}}(L)$ we can write $\Psi_{\mathcal{P}}(L) = \cup\{\mathcal{U} \in \delta : (\mathcal{U} - L)^{\Diamond} \notin \mathcal{P}\}$. This implies that there exists a non-empty set $\mathcal{U} \in \delta$ for which $(\mathcal{U} - L)^c \notin \mathcal{P}$. Let $(\mathcal{U} - L)^c = T$, where $T \notin \mathcal{P}$. Now, $\mathcal{U} - T \subseteq L$, and since $\mathcal{U} - T$ is a $\delta^{\Diamond}$-open set according to Theorem 1.11, we can conclude that L includes a non-empty $\delta^{\Diamond}$-interior.
Conversely, assume that L includes a non-empty $\delta^{\Diamond}$-interior. This implies that $\mathcal{U} \in \delta$ and $T \notin \mathcal{P}$ such that $\mathcal{U} - T \subseteq L$. Consequently, $\mathcal{U} - L \subseteq T$. Let $H = \mathcal{U} - L \subseteq T$, and thus $H \notin \mathcal{P}$. Hence, $\cup\{\mathcal{U} \in \delta : (\mathcal{U} - L)^c \notin \mathcal{P}\} = \Psi_{\mathcal{P}}(L) \neq \phi$.

Corollary 5.14. *Suppose that $(\mathbb{X}, \delta, \mathcal{P})$ is a PTS. Then, $\{a_x\} \in \widetilde{\Psi}_{\mathcal{P}}(\mathbb{X}, \delta)$ if and only if $\{a_x\} \in \delta^L$.*

Proof. First, suppose that $\{a_x\} \in \widetilde{\Psi}_{\mathcal{P}}(\mathbb{X}, \delta)$. This means that $\{a_x\}$ is open in $(\mathbb{X}, \delta^{\Diamond})$ through Proposition 5.13. Since $\{a_x\} \subseteq \Psi_{\mathcal{P}}(\{a_x\})$ and $\Psi_{\mathcal{P}}(\{a_x\})$ is open in $(\mathbb{X}, \delta)$, we can conclude that $\{a_x\} \subseteq int(cl(\Psi_{\mathcal{P}}(\{a_x\})))$. Thus, we have shown that $\{a_x\} \in \delta^L$. Conversely, suppose that $\{a_x\} \subseteq int(cl(\Psi_{\mathcal{P}}(\{a_x\})))$ and $\{a_x\} \subseteq (cl(\Psi_{\mathcal{P}}(\{a_x\})))$. Therefore, it follows that $\{a_x\} \in \widetilde{\Psi}_{\mathcal{P}}(\mathbb{X}, \delta)$.

6. Conclusions

New classes of sets, namely $\mathcal{P}$-α-open, $\mathcal{P}$-semi-open, $\mathcal{P}$-pre-open, $\mathcal{P}$-β-open, $\mathcal{P}_{\mathcal{R}}$-sets, and $\mathcal{P}_{\mathcal{R}_\alpha}$-sets in *PTSs*, have been studied along with results about the relationships between class sets in *TSs* and a new class of sets in *PTSs*. Moreover, the relationships between these classes of subsets and some generalizations of open have been introduced. Many theorems are discussed together with the counterexamples. In addition, examples are provided in such a way as to illustrate the independence between openness and $\mathcal{P}$-openness, as well as the independence between $\mathcal{P}$-pre-open and $\mathcal{P}_{\mathcal{R}}$-sets. Furthermore, the decomposition of continuity by using a new class of sets in *PTSs* has been obtained. Finally, $\widetilde{\Psi}_{\mathcal{P}}$-sets were introduced and investigated by defining intriguing generalized open sets in *PTS* using the Ψ-operator, besides investigating some of their important properties.
In future work, the same concepts presented in this article can be introduced in the context of primal topological spaces [29, 34, 35] by the techniques in soft sets or rough sets. Additionally, we hope to relate these classes of sets to some concepts in different topological structures.

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