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# Modified Inertial Krasnosel'skii-Mann type Method for Solving Fixed Point Problems in Real Uniformly Convex Banach Spaces

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**Abstract.** We present an altered version of the inertial Krasnosel'skii-Mann algorithm and demonstrate convergence outcomes for mappings that are asymptotically nonexpansive within real, uniformly convex Banach spaces. To achieve our results, we skillfully construct the inequality in equation (6) and apply it accordingly. Our findings support and broadly generalize a number of significant findings from the literature. We demonstrate, as an application, the generation of maximal monotone operators' zeros via fixed point methods in Hilbert spaces. Additionally, we solve convex minimization issues using our fixed-point techniques.

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**Key Words and Phrases**: Asymptotically Nonexpansive Mappings, Modified Inertial Krasnosel'skii-Mann, Continuous Mappings, Convergence, Fixed Points, Banach Spaces

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## 1. Introduction

We consider a Banach space $\mathcal{X}$ and any self map of $\mathcal{X}$ to be $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ in this paper. The fixed points set of $\mathcal{T}$ is denoted by $F(\mathcal{T})$ and may be found with the formula $F(\mathcal{T}) := \{x\in\mathcal{X} : \mathcal{T}a = a\})$. We refer to the mapping $\mathcal{T}$ as follows:

(i) Nonexpansive, if $||\mathcal{T}a-\mathcal{T}b|| \le ||a-b||$ for all $a,b\in\mathcal{X}$,

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(ii) Asymptotically Nonexpansive, if $\forall a, b \in \mathcal{X}$, there exists a sequence $\{\kappa_n\} \subset [1, +\infty)$, with $\lim_{n\to\infty} \kappa_n = 1$, such that

$$||\mathcal{T}^n a - \mathcal{T}^n b|| \leq \kappa_n ||a - b|| \quad \forall n \geq 1. \tag{1}$$

(iii) Uniformly $L-$ Lipschitzian (see for example [19]), if there exists a real constant $L > 0$, such that for all $a, b \in \mathcal{X},\ n \geq 1$, the following holds

$$||\mathcal{T}^n a - \mathcal{T}^n b|| \leq L||a - b||.$$

We can easily observe that any nonexpansive maps with sequence $\kappa_n = 1 \forall\ n \geq 1$ are asymptotically nonexpansive maps. It is also bounded because $\kappa_n$ is convergent. All asymptotically nonexpansive mappings are therefore uniformly L-Lipschitzian, and so continuous.

Goebel and Kirk (see [9]) introduced the class of asymptotically nonexpansive mappings as a natural extension of the nonexpansive mappings class, and the construction of fixed points of nonexpansive mappings and their generalizations have historically attracted a great deal of research interest. This is due to the fact that nonexpansive mappings are strongly associated with numerous other mapping classes, such as the accretive operators, and the practical applications of their fixed point construction in image recovery, computer tomography, signal processing, and other fields are numerous.

In 1967, Browder [5] and Kato [12] independently introduced the class of accretive operators. Browder [5] proved a basic result in the theory of accretive operators: if $\mathcal{A}$ is Lipschitzian and accretive, then $\frac{dz}{dt} + \mathcal{T}\mathcal{A}z = 0,\ z(0) = z_0$ is a solved initial value problem.

Assume we have a Hilbert space $\mathcal{H}$. It is well known (see, for example, [1]) that if $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ is an accretive operator, then the resolvent of $A$, given by $J^\lambda_{\mathcal{A}} := (I + \lambda\mathcal{A})^{-1}$, is a nonexpansive operator for any real constant $\lambda > 0$. The zeros of $\mathcal{A}$ are obviously the fixed points of $J^\lambda_{\mathcal{A}}$. Thus, several application domains united by the theory of accretive operators are brought together by studying fixed points of nonexpansive mappings and their generalizations. This makes the study current.

The approximation of fixed points of nonexpansive mappings has been studied by a number of authors in various ways. An iteration approach for the production of fixed points of nonexpansive mappings was introduced by W. R. Mann in [16], to be precise: Given a real Hilbert space $\mathcal{H}$, let $\mathcal{C}$ be a nonempty convex subset of it. Starting from any random $x_0 \in \mathcal{C}$, the Mann's sequence is produced by

$$a_{n+1} = (1 - \xi_n)a_n + \xi_n \mathcal{T} a_n, \tag{2}$$

where the real sequence $\{\xi_n\} \subset (0, 1)$. The author demonstrated that the sequence $\{a_n\}$ converges weakly to a fixed point of $\mathcal{T}$ with the constraint $\sum \xi_n(1 - \xi_n) = +\infty$.

Studying convergence results to its fixed points becomes relevant when one realizes that asymptotically nonxepansive mappings are generalizations of nonexpansive mappings. Numerous scholars have examined convergence outcomes for fixed points of asymptotically nonexpansive mappings (see, for instance, to [9, 10]). Some authors used the modified Mann iteration sequence, which is defined as follows, to achieve this: The modified Mann iteration sequence (see, for example, [19]) is formed from an arbitrary $a_0 \in \mathcal{C}$ if $\mathcal{C}$ is a nonempty convex subset of a Banach space, $\mathcal{E}$, and $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is any map.

$$a_{n+1} = (1 - \xi_n)a_n + \xi_n \mathcal{T}^n a_n, \tag{3}$$

where a real sequence $\{\xi_n\} \subset (0, 1)$. The authors (see, for example, [9]) demonstrated convergence results to fixed points of $\mathcal{T}$ under specific restrictions on the iteration parameter.

## 2. Inertial Iteration Schemes

Many authors (see, for example, [1, 4, 6–8, 13, 14, 17, 21, 23, 24]) have recently investigated iteration schemes known as 'inertial iteration schemes' since the rate of convergence of iteration sequences is equally highly significant. The characteristic of these schemes is that they are known to be faster than well-known convergent iteration schemes because of the addition of a term called the inertial term. A few examples of inertial schemes are given in [1], where the authors presented the inertial Krasnosel'skii-Mann iteration as follows for a self nonexpansive mapping of a real Hilbert space $\mathcal{H}$:

$$\begin{cases} a_0, a_1 \in \mathcal{H} \\ b_k = a_k + t_k(a_k - a_{k-1}) \\ a_{k+1} = (1 - \xi_k)a_k + \xi_k T b_k, \ k = 0, 1, ... \end{cases} \tag{4}$$

The authors demonstrated the weak convergence of the scheme to fixed points of nonexpansive mappings under the criteria that $0 \le t_k \le t < 1$, for some $t \in (0, 1)$, and $\sum t_k \|a_k - a_{k-1}\|^2 < +\infty$. The inertial term is denoted by $t_k(a_k - a_{k-1})$. The interested reader might refer to [1, 4, 7, 15], etc. for additional arguments on the insertion of inertial terms to iteration schemes.

The concept of inertial technique was merged with the Halpern viscosity algorithms in [25], whereby modified inertial Mann algorithms were introduced. The following theorems were used by the authors to examine convergence results to fixed points of nonexpansive mappings using these algorithms:

**Theorem 1.** *[25] Let $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ be a nonexpansive mapping with $F(\mathcal{T}) \neq \emptyset$ and $\mathcal{C}$ a nonempty closed convex subset of a real Hilbert space $H$. The following criteria are met given a point $z \in \mathcal{C}$ and two sequences $\{\psi_n\}$ and $\{v_n\}$ in $(0, 1)$:*

*(D1)* $\sum v_n = \infty$ *and* $\lim v_n = 0$;

*(D2)* $\lim \frac{\delta_n}{v_n} \|a_n - a_{n-1}\| = 0$.

*Assume that* $a_{-1}, a_0 \in \mathcal{C}$ *are arbitrary. Use the following algorithm to define a sequence* $\{a_n\}$*:*

$$\begin{cases} w_n = a_n + \delta_n(a_n - a_{n-1}), \\ b_n = \psi_n w_n + (1-\psi_n)\mathcal{T}w_n \\ a_{n+1} = \upsilon_n z + (1-\upsilon_n)b_n, \ n \geq 0. \end{cases}$$

*The generated iterative sequence* $\{a_n\}$ *then strongly converges to* $a^* = P_{F(\mathcal{T})}z$.

**Theorem 2.** *[25] Let* $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ *be a nonexpansive mapping with* $F(\mathcal{T}) \neq \emptyset$*. Assume that* $\mathcal{C}$ *is a nonempty closed convex subset of a real Hilbert space* $\mathcal{H}$*. Define a* $\rho$*-contraction* $h : \mathcal{C} \to \mathcal{C}$ *such that* $\|h(a) - h(b)\| \leq \rho\|a - b\|$, $\forall a, b \in \mathcal{C}$*. Given two sequences in* $(0,1)$, $\{\psi_n\}$ *and* $\{\upsilon_n\}$*, the following conditions hold:*
$(D1)$ $\sum \upsilon_n = \infty$ *and* $\lim \upsilon_n = 0$;
$(D2)$ $\lim \frac{\delta_n}{\upsilon_n}\|a_n - a_{n-1}\| = 0$.
*Assume that* $a_{-1}, a_0 \in C$ *are arbitrary. Generate a sequence* $\{a_n\}$ *using the procedure below:*

$$\begin{cases} w_n = a_n + \delta_n(a_n - a_{n-1}), \\ b_n = \psi_n w_n + (1-\psi_n)\mathcal{T}w_n \\ a_{n+1} = \upsilon_n h(a_n) + (1-\upsilon_n)b_n, \ n \geq 0. \end{cases}$$

*The resultant iterative sequence* $\{a_n\}$ *then converges strongly to* $a^* = P_{F(\mathcal{T})}h(a^*)$.

**Observation 1:** Given condition (D1), condition (D2) sug gests that $\delta_n\|a_n - a_{n-1}\|$ approaches zero at a quicker rate than $\upsilon_n$. This suggests that $\lim \delta_n\|a_n - a_{n-1}\| = 0$. This further suggests that $\lim \delta_n\|a_n - a_{n-1}\|^2 = (\lim \delta_n\|a_n - a_{n-1}\|)(\lim \|a_n - a_{n-1}\|) = 0(\lim \|a_n - a_{n-1}\|) = 0$. Therefore, $\delta_n\|a_n - a_{n-1}\|^2$ is bounded. Consequently, it is weaker to impose a boundedness constraint on $\delta_n\|a_n - a_{n-1}\|^2$. This will be helpful for our outcomes in the follow-up.

**Observation 2:** It is possible that $\{w_n\}$ will not belong in $\mathcal{C}$ since $\mathcal{C}$ is a **convex subset**. This suggests that the schemes in [25] need precise definitions. A well-defined scheme can only exist if $\mathcal{C}$ is either the entire space or an affine subset of it.
An inertial accelerated algorithm for obtaining a fixed point in the fixed points set of an asymptotically nonexpansive mapping in a real uniformly convex Banach space that satisfies Opial criteria was recently explored by Murtala et al. (see [11]). More specifically, the authors suggested the following outcomes:

**Assumption 1.** *[11] Let* $\mathcal{X}$ *be a real uniformly convex Banach space.*

*(i) Choose sequences* $\{\xi_n\} \subset (0,1)$, $\{\beta_n\}$, $\{\delta_n\} \subset [0, +\infty)$ *and* $\sum_{n=1}^{\infty} \delta_n < +\infty$ *with* $\delta_n = o(\beta_n)$ *which means* $\lim_{n\to\infty} \frac{\delta_n}{\beta_n} = 0$.

*(ii) Let* $a_0, a_1 \in \mathcal{X}$ *be arbitrary points, for the iterates* $a_{n-1}$ *and* $a_n$ *for each* $n \geq 1$*, choose* $\theta_n$ *such that* $0 \leq \theta_n \leq \bar{\theta_n}$ *where, for* $\eta \geq 3$

$$\bar{\theta_n} := \begin{cases} min\{\frac{n-1}{n+\eta-1}, \frac{\delta_n}{\|a_n - a_{n-1}\|}\}, \ if \ a_n \neq a_{n-1} \\ \\ \frac{n-1}{n+\eta-1}, \ otherwise \end{cases}$$

According to the authors, the extrapolation phase described in [3] provides the concept of Assumption 1. Along with it, the authors added this: It is easy to see from Assumption 1 that for each $n \geq 1$, we have

$$\theta_n \|a_n - a_{n-1}\| \leq \delta_n,$$

which together with $\sum \delta_n < +\infty$ and $\lim_{n\to\infty} \frac{\delta_n}{\beta_n} = 0$, we obtain

$$\sum \theta_n \|a_n - a_{n-1}\| < +\infty$$

and

$$\lim_{n\to\infty} \frac{\theta_n}{\beta_n} \|a_n - a_{n-1}\| \leq \lim_{n\to\infty} \frac{\delta_n}{\beta_n} = 0.$$

Using Assumption 1, the authors stated and proved the following theorem:

**Theorem 3.** *[11] Let $\mathcal{X}$ be a Banach space that is real and uniformly convex, possessing Opial's property. With sequence $\{\kappa_n\} \subset [0, \infty)$, let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be an asymptotically nonexpansive mapping such that $\sum_{n=0}^{\infty} \kappa_n < \infty$ and $F(\mathcal{T}) \neq \emptyset$. Let $\{a_n\}$ be the sequence that is produced in this way:*

$$\begin{cases} a_0, a_1 \in \mathcal{X} \\ w_n = a_n + \theta_n(a_n - a_{n-1}), \\ d_{n+1} = \frac{1}{\lambda}(\mathcal{T}^n(w_n) - w_n) + \beta_n d_n, \\ b_n = w_n + \lambda d_{n+1}, \\ a_{n+1} = \mu\xi_n w_n + (1 - \mu\xi_n) y_n, \ n \geq 1, \end{cases}$$

*where $\mu \in (0, 1]$, $\lambda > 0$, assuming that Assumption 1 holds and set $d_1 = \frac{1}{\lambda}(\mathcal{T}^n w_0 - w_0)$. Then the sequence $\{a_n\}$ converges weakly to a point $a^* \in F(\mathcal{T})$, provided that the following conditions hold:*

*(C1)* $\sum_{n=0}^{\infty} \beta_n < +\infty$

*(C2)* $\liminf_{n\to\infty} \mu\xi_n(1 - \mu\xi_n) > 0$ *Moreover, $\{w_n\}$ satisfies*

*(C3)* $\{\mathcal{T}^n w_n - w_n\}$ *is bounded.*

**Observation 3:** The discussion in **Observation 1** also holds for $\theta_n\|a_n - a_{n-1}\|^2$, based on Assumption 1 and the fact that $\sum \theta_n\|a_n - a_{n-1}\| < \infty$. Moreover, $\lim \theta_n\|a_n - a_{n-1}\|^{2p}$ $= (\lim \theta_n\|a_n - a_{n-1}\|)$ holds for every positive integer $p > 1$,. When $(lim\|a_n - a_{n-1}\|^{2p-1}) =$ 0, $Limit\|a_n - a_{n-1}\|^{2p-1}) = 0$. It is therefore weaker to impose a boundedness constraint on $\theta_n\|a_n - a_{n-1}\|^{2p}$. This will help with the outcomes we get in the follow-up.

**Observation 4:** The computations and analysis performed in [11] are negatively impacted by the inequality that characterizes uniformly convex Banach spaces, which is unfortunately economically quoted in [11] (only for $p = 2$). For real uniformly convex Banach spaces, therefore, the conclusions in [11] are not applicable in general.

In this article, we modify the inertial iteration scheme introduced in [1] and prove convergence results for fixed points of asymptotically nonexpansive mappings in some real uniformly convex Banach spaces. We do this by imposing different sets of conditions, some of which are weaker than those imposed in [11]. Our motivation comes from the aforementioned works and observations. Compared to the class examined in [11], our class of spaces is more general (just for $p = 2$). Our improved inertial strategy for a real uniformly convex Banach $\mathcal{X}$ is as follows:

$$\begin{cases} a_0, a_1 \in X \\ b_n = a_n + \nu_n(a_n - a_{n-1}) \\ a_{n+1} = (1-\xi_n)b_n + \xi_n \mathcal{T}^n b_n, \ n = 1, 2, ... \end{cases} \tag{5}$$

## 3. Preliminaries

Assume that the Banach space $\mathcal{X}$ is real. It's common knowledge that if $\mathcal{D}$ is a nonempty convex subset of $\mathcal{X}$ and $h : \mathcal{X} \to \bar{\Re} := \Re \cup \{+\infty\}$ is a suitable functional, then $h$ is convex on $\mathcal{D}$ if

$$h(\lambda a + (1-\lambda)b) \leq \lambda h(a) + (1-\lambda)h(b)$$

for all $0 \leq \lambda \leq 1$ and $a, b \in \mathcal{D}$. In $\mathcal{D}$, $h$ is considered uniformly convex (refer to [27]) if and only if there is a function $\mu : \Re^+ := [0, +\infty) \to \Re^+$ with $\mu(t) = 0$, such that

$$h(\lambda a + (1-\lambda)b) \leq \lambda h(a) + (1-\lambda)h(b) - \lambda(1-\lambda)\mu(\|a-b\|)$$

for all $0 \leq \lambda \leq 1$ and $a, b \in \mathcal{D}$.

A definition and a few lemmas that will be helpful in the sequel are provided before we express and demonstrate our primary findings:

**Definition 1.** *(see e.g [20]) Consider the Banach space $\mathcal{E}$. When $\{a_n\}$ is a sequence in $\mathcal{D}(\mathcal{T})$ such that $\{a_n\}$ converges weakly to $z \in \mathcal{D}(\mathcal{T})$ and $\{\mathcal{T}a_n\}$ converges strongly to $z$, then $\mathcal{T}z = z$. This mapping $\mathcal{T} : \mathcal{D}(\mathcal{T}) \subseteq \mathcal{E} \to \mathcal{E}$ is said to be demiclosed at a point $z \in \mathcal{D}(\mathcal{T})$.*

**Lemma 1.** *(see e.g [10]) Assume that $\mathcal{E}$ is a uniformly convex Banach space, $\mathcal{D}$ is a nonempty closed convex subset of $\mathcal{E}$, and $\mathcal{T} : \mathcal{D} \to \mathcal{D}$ is an asymptotically nonexpansive mapping with a sequence $\{\kappa_n\} \subset [1, \infty)$, where $\lim_{n\to\infty} \kappa_n = 1$.. At zero, $I - \mathcal{T}$ is demiclosed.*

**Lemma 2.** *(see e.g [19]) For all $n \geq 1$, let $\{\xi_n\}, \{\beta_n\}$, and $\{\delta_n\}$ be sequences of nonnegative real numbers that fulfill the inequality*

$$\xi_{n+1} \leq (1+\delta_n)\xi_n + \beta_n,$$

*$\lim \xi_n$ exists if and only if $\sum \delta_n = +\infty$ and $\sum \beta_n = +\infty$. Additionally, if $\{\xi_n\}$ has a subsequence that strongly converges to zero, then $\lim \xi_n = 0$.*

**Lemma 3.** *(see [26]) Assume that $p > 1$ is a fixed real value. If $\mathcal{X}$ is p-uniformly convex, then the functional $\|.\|^p$ is uniformly convex on the entire Banach space $\mathcal{X}$. That is, $\mathcal{X}$ is $p-$uniformly convex if and only if $h : \Re^+ := [0, +\infty) \to \Re^+$, where $h(0) = 0$, exists and such a function*
$\|\lambda a+(1-\lambda)a\|^p \leq \lambda\|a\|^p+(1-\lambda)\|b\|^p-\lambda(1-\lambda)g(\|a-b\|)$ *for all* $0 \leq \lambda \leq 1$ *and* $a, b \in \mathcal{X}$.

According to [26], this lemma's result yields the inequality below: A real constant $c > 0$ exists such that $\|\lambda a + (1 - \lambda)b\|^p \leq \lambda\|a\|^p + (1 - \lambda)\|b\|^p - w_p(\lambda)c\|a - b\|^p$, where $w_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda)$For every $0 \leq \lambda \leq 1$ and $a, b \in X$.

## 4. Main Results

We now state and prove our main results.

**Theorem 4.** *Let $\mathcal{X}$ be a $p-$uniformly convex Banach space that satisfies Opial's condition, and let $p > 1$ be any positive integer. Given a non-empty fixed points set $F(\mathcal{T})$ and a sequence $\{\kappa_n\} \subset [1, \infty)$, let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be an asymptotically nonexpansive mapping such that $\sum_{n\to\infty} \kappa_n - 1 < \infty$. Let $\{a_n\}$ be the modified inertial Krasnosel'skii-Mann sequence for $a_0,\ a_1 \in \mathcal{X}$, where $\{\xi_n\}$ and $\{\nu_n\}$ are real sequences in $(0, 1)$, satisfying:*

*(i)* $\nu_n\|a_n - a_{n-1}\|^{2p} \leq D$, *for some real positive constant* $\mathcal{D}$,

*(ii)* $\liminf \xi_n(1 - \xi_n) > 0$,

*(iii)* $\sum \nu_n^{\frac{1}{2}} < +\infty$.

*Then $\{a_n\}$ converges weakly to a fixed point of $\mathcal{T}$.*

*Proof.* Condition (i) in our theorem is motivated by **Observation 3** above.
For any positive integer $p > 1$ and since $\nu_n \in (0, 1)$ we have
$(1 + \nu_n)\nu_n^p < \nu_n(1 + \nu_n)^p$, implying that $(1 + \nu_n)\nu_n^p - \nu_n(1 + \nu_n)^p < 0$.
Letting $w_p(.)$ to be the functional in Lemma 3 (consequence inequality), we have

$$w_p(1+\nu_n) = (1+\nu_n)(-\nu_n)^p+(1+\nu_n)^p(-\nu_n) = \begin{cases} -[(1 + \nu_n)\nu_n^p + (1 + \nu_n)^p\nu_n],\ if\ p\ is\ odd \\ \\ (1 + \nu_n)\nu_n^p - (1 + \nu_n)^p\nu_n,\ if\ p\ is\ even \end{cases}$$

So, for all positive integers $p > 1$, we have $w_p(1+\nu_n) < 0$ (since $(1+\nu_n)\nu_n^p-\nu_n(1+\nu_n)^p < 0$,) so that $-w_p(\nu + \nu_n) > 0$. Also, observe that

$$\begin{aligned} -w_p(1 + \nu_n) &= -[(1 + \nu_n)(-\nu_n)^p + (1 + \nu_n)^p(-\nu_n)] \\ &= (1 + \nu_n)^p(\nu_n) - (1 + \nu_n)(-\nu_n)^p \\ &\leq (1 + \nu_n)^p\nu_n + (1 + \nu_n)(\nu_n)^p \end{aligned} \tag{6}$$

Since $-w_p(1 + \nu_n) > 0$, the inequality (consequent inequality) in Lemma 3 holds for $\lambda = 1 + \nu_n$, for some nonnegative real constant $c = c_1$ and for all $a, b$ in any real $p$-uniformly convex Banach space with $p > 1$ being any positive integer.

Since $\{\kappa_n\} \subset [1,+\infty)$ converges to 1, there exists a positive real constant $D_1$, such that $\kappa_n^p \leq D_1$. By applying the Lagrange mean value theorem, it is easily verifiable that for $r > 1$, we have $r^p - 1 \leq pr^{p-1}(r-1)$. Let $c$ be a positive real constant and $a^* \in F(\mathcal{T})$. Using these, (5), (6) and Lemma 3 (consequence inequality), we have

$$\begin{aligned}
||a_{n+1} - a^*||^p &= ||(1-\xi_n)b_n + \xi_n \mathcal{T}^n b_n - a^*||^p \\
&= ||(1-\xi_n)(b_n - a^*) + \xi_n(\mathcal{T}^n b_n - a^*)||^p \\
&\leq (1-\xi_n)||b_n - a^*||^p + \xi_n||\mathcal{T}^n b_n - a^*||^p - w_p(\xi_n)c||\mathcal{T}^n b_n - b_n||^p \\
&\leq (1-\xi_n)||b_n - a^*||^p + \xi_n\kappa_n^p||b_n - a^*||^p - w_p(\xi_n)c||\mathcal{T}^n b_n - b_n||^p \\
&= [1+\xi_n(\kappa_n^p - 1)]||b_n - a^*||^p - w_p(\xi_n)c||\mathcal{T}^n b_n - b_n||^p \\
&= [1+\xi_n(\kappa_n^p - 1)]||(1+\nu_n)a_n - \ni_n a_{n-1} - a^*||^p - w_p(\xi_n)c||\mathcal{T}^n b_n - b_n||^p \\
&= [1+\xi_n(\kappa_n^p - 1)]||(1+\nu_n)(a_n - a^*) - \nu_n(a_{n-1} - a^*)||^p - w_p(\xi_n)c||\mathcal{T}^n b_n - b_n||^p \\
&\leq [1+\xi_n(\kappa_n^p - 1)][(1+\nu_n)\|a_n - a^*\|^p - \nu_n\|a_{n-1} - a^*\|^p - w_p(1+\nu_n)c_1\|a_n - a_{n-1}\|^p] \\
&\quad - w_p(\xi_n)c||\mathcal{T}^n b_n - b_n||^p \\
&\leq [1+\xi_n(\kappa_n^p - 1)](1+\nu_n)\|a_n - a^*\|^p - [1+\xi_n(\kappa_n^p - 1)]w_p(1+\nu_n)c_1\|a_n - a_{n-1}\|^p \\
&\quad - w_p(\xi_n)c||\mathcal{T}^n b_n - b_n||^p \\
&\leq \|a_n - a^*\|^p + [\nu_n + \xi_n(\kappa_n^p - 1) + \xi_n\nu_n(\kappa_n^p - 1)]\|a_n - a^*\|^p - \kappa_n^p w_p(1+\nu_n)c_1\|a_n - a_{n-1}\|^p \\
&\quad - w_p(\xi_n)c||\mathcal{T}^n b_n - b_n||^p \; since \; w_p(1+\nu_n) < 0 \\
&\leq \|a_n - a^*\|^p + [\nu_n + 2\xi_n(\kappa_n^p - 1)]\|a_n - a^*\|^p - \kappa_n^p w_p(1+\nu_n)c_1\|a_n - a_{n-1}\|^p \\
&\quad - w_p(\xi_n)c||\mathcal{T}^n b_n - b_n||^p \\
&\leq \|a_n - a^*\|^p + [\nu_n + 2\xi_n p\kappa_n^{p-1}(\kappa_n - 1)]\|a_n - a^*\|^p - \kappa_n^p w_p(1+\nu_n)c_1\|a_n - a_{n-1}\|^p \\
&\quad - w_p(\xi_n)c||\mathcal{T}^n b_n - b_n||^p \\
&\leq \|a_n - a^*\|^p + [\nu_n + 2pD_1(\kappa_n - 1)]\|a_n - a^*\|^p - \kappa_n^p w_p(1+\nu_n)c_1\|a_n - a_{n-1}\|^p \\
&\quad - w_p(\xi_n)c||\mathcal{T}^n b_n - b_n||^p \\
&\leq [1+\delta_n]\|a_n - a^*\|^p - \kappa_n^p w_p(1+\nu_n)c_1\|a_n - a_{n-1}\|^p - w_p(\xi_n)c||\mathcal{T}^n b_n - b_n||^p \\
&\leq [1+\delta_n]\|a_n - a^*\|^p - D_1 w_p(1+\nu_n)c_1\|a_n - a_{n-1}\|^p \\
&\quad - w_p(\xi_n)c||\mathcal{T}^n b_n - b_n||^p \qquad \text{since} \;\; w_p(1+\nu_n) < 0 \\
&\leq [1+\delta_n]\|a_n - a^*\|^p + D_1[\nu_n(1+\nu_n)^p + \nu_n^p(1+\nu_n)]c_1\|a_n - a_{n-1}\|^p \\
&\quad - w_p(\xi_n)c||\mathcal{T}^n b_n - b_n||^p \\
&\leq [1+\delta_n]\|a_n - a^*\|^p + D_1\nu_n[(1+\nu_n)^p + \nu_n^{p-1}(1+\nu_n)]c_1\|a_n - a_{n-1}\|^p \\
&\quad - w_p(\xi_n)c||\mathcal{T}^n b_n - b_n||^p \\
&\leq [1+\delta_n]\|a_n - a^*\|^p + D_1\nu_n[2^p + 2]c_1\|a_n - a_{n-1}\|^p - w_p(\xi_n)c||\mathcal{T}^n b_n - b_n||^p
\end{aligned}$$

$$\begin{aligned}
&= [1+\delta_n]\|a_n - a^*\|^p + 2D_1\nu_n[2^{p-1} + 1]c_1\|a_n - a_{n-1}\|^p - w_p(\xi_n)c||\mathcal{T}^n b_n - b_n||^p \\
&\leq [1+\delta_n]\|a_n - a^*\|^p + 2D_1[2^{p-1} + 1]c_1\nu_n^{\frac{1}{2}}[\nu_n\|a_n - a_{n-1}\|^{2p}]^{\frac{1}{2}} - w_p(\xi_n)c||\mathcal{T}^n b_n - b_n||^p \\
&\leq [1+\delta_n]\|a_n - a^*\|^p + 2D_1[2^{p-1} + 1]c_1\nu_n^{\frac{1}{2}}D^{\frac{1}{2}} - w_p(\xi_n)c||\mathcal{T}^n b_n - b_n||^p
\end{aligned}$$

$$= [1+\delta_n]\|a_n - a^*\|^p + M\nu_n^{\frac{1}{2}} - w_p(\xi_n)c||\mathcal{T}^n b_n - b_n||^p \tag{7}$$

where $M = 2c_1D_1[2^{p-1}+1]D^{\frac{1}{2}}$ and $\delta_n = \nu_n + 2pD_1(\kappa_n - 1)$ is such that $\sum \delta_n < +\infty$, since $\sum(\kappa_n - 1) < +\infty$ and condition (iii) holds. Using this, (7), Lemma 2 and condition (iii), we have that $\lim \|a_n - a^*\|^p$ exists. This implies $\{a_n - a^*\}$ and $\{a_n\}$ are norm bounded. Hence, there exists a real constant $D_2 > 0$, such that $\|a_n - a^*\|^p \leq D_2$. Using this in 7 and $\forall n \geq 0$, we have

$$\|a_{n+1} - a^*\|^p \leq \|a_n - a^*\|^p + \delta_n D_2 + M\nu_n^{\frac{1}{2}} - w_p(\xi_n)c||\mathcal{T}^n b_n - b_n||^p$$

From this, the fact that $\sum \delta_n < +\infty$ and condition (iii), we have

$$\begin{aligned}\sum_{n\geq 0} 2[\xi_n(1-\xi_n)]^p c||\mathcal{T}^n b_n - b_n||^p &\leq \sum_{n\geq 0} w_p(\xi_n)c||\mathcal{T}^n b_n - b_n||^p \\ &\leq \sum_{n\geq 0}[||a_n - a^*||^p - ||a_{n+1} - a^*||^p] \\ &\quad + D_2\sum_{n\geq 0}\delta_n + M\sum_{n\geq 0}\nu_n^{\frac{1}{2}} < \infty.\end{aligned}$$

This implies from conditions (ii) that $\lim ||\mathcal{T}^n b_n - b_n||^p = 0$. Hence

$$\lim ||\mathcal{T}^n b_n - b_n|| = 0. \tag{8}$$

From (5), (iii) and the fact that $\{a_n\}$ is norm bounded, we have

$$\|b_n - a_n\| = \nu_n\|a_n - a_{n-1}\| \leq \nu_n[\|a_n\| + \|a_{n-1}\|] \to 0. \tag{9}$$

Furthermore, we have from (5), (8) and (9) that

$$\begin{aligned}||a_{n+1} - a_n|| &= ||(1-\xi_n)(b_n - a_n) + \xi_n(\mathcal{T}^n b_n - a_n)|| \\ &\leq (1-\xi_n)\|b_n - a_n\| + \xi_n\|\mathcal{T}^n b_n - a_n\| \\ &= (1-\xi_n)\|b_n - a_n\| + \xi_n\|\mathcal{T}^n b_n - b_n + b_n - a_n\| \\ &\leq (1-\xi_n)\|b_n - a_n\| + \xi_n[\|\mathcal{T}^n b_n - b_n\| + \|b_n - a_n\|] \\ &= \|b_n - a_n\| + \xi_n\|\mathcal{T}^n b_n - b_n\| \\ &\leq \|b_n - a_n\| + \|\mathcal{T}^n b_n - b_n\| \to 0. \end{aligned} \tag{10}$$

We also have from (5) that

$$\begin{aligned}\xi_n\mathcal{T}^n(a_n + \nu_n(a_n - a_{n-1})) - \xi_n a_n &= a_{n+1} - a_n - (1-\xi_n)\nu_n(a_n - a_{n-1}) \\ \Leftrightarrow \xi_n[\mathcal{T}^n(a_n + \nu_n(a_n - a_{n-1})) - a_n] &= a_{n+1} - a_n - (1-\xi_n)\nu_n(a_n - a_{n-1}) \\ \Leftrightarrow [\mathcal{T}^n(a_n + \nu_n(a_n - a_{n-1})) - a_n] &= \frac{1}{\xi_n}[a_{n+1} - a_n - (1-\xi_n)\nu_n(a_n - a_{n-1})].\end{aligned}$$

This implies

$$\begin{aligned}\|\mathcal{T}^n(a_n+\nu_n(a_n-a_{n-1}))-a_n\| &= \frac{1}{\xi_n}\|[a_{n+1}-a_n-(1-\xi_n)\nu_n(a_n-a_{n-1})]\|\\ &\le \frac{1}{\xi_n}[\|a_{n+1}-a_n\|+\|(1-\xi_n)\nu_n(a_n-a_{n-1})\|]\\ &= \frac{1}{\xi_n}[\|a_{n+1}-a_n\|+(1-\xi_n)\nu_n\|a_n-a_{n-1}\|]\\ &\le \frac{1}{\xi_n}[\|a_{n+1}-a_n\|+\nu_n\|a_n-a_{n-1}\|].\end{aligned}$$

Since $\mathcal{T}$ is uniformly L-Lipschitzian(and hence $\mathcal{T}^n$ is continuous), $\{a_n\}$ is norm bounded, using (10), conditions (ii) and (iii), this yields

$$\lim\|\mathcal{T}^n a_n-a_n\|=0. \tag{11}$$

Using (10) and (11), we now have

$$\begin{aligned}||a_n-\mathcal{T}a_n|| &= \|a_n-a_{n+1}+a_{n+1}-\mathcal{T}^{n+1}a_{n+1}+\mathcal{T}^{n+1}a_{n+1}-\mathcal{T}a_n\|\\ &\le \|a_n-a_{n+1}\|+\|\mathcal{T}^{n+1}a_{n+1}-a_{n+1}\|+L\|\mathcal{T}^n a_{n+1}-a_n\|\\ &= \|a_n-a_{n+1}\|+\|\mathcal{T}^{n+1}a_{n+1}-a_{n+1}\|+L\|\mathcal{T}^n a_{n+1}-\mathcal{T}^n a_n+\mathcal{T}^n a_n-a_n\|\\ &\le \|a_n-a_{n+1}\|+\|\mathcal{T}^{n+1}a_{n+1}-a_{n+1}\|+L[\|\mathcal{T}^n a_{n+1}-\mathcal{T}^n a_n\|+\|\mathcal{T}^n a_n-a_n\|]\\ &\le \|a_n-a_{n+1}\|+\|\mathcal{T}^{n+1}a_{n+1}-a_{n+1}\|+L^2\|a_{n+1}-a_n\|+L\|\mathcal{T}^n a_n-a_n\|\\ &= (1+L^2)\|a_{n+1}-a_n\|+\|\mathcal{T}^{n+1}a_{n+1}-a_{n+1}\|+L\|\mathcal{T}^n a_n-a_n\|\to 0.\end{aligned} \tag{12}$$

Since $\{a_n\}$ is norm bounded, it possesses a subsequence $\{a_{n_k}\}$ which converges weakly to a point $u\in\mathcal{X}$. Since $X$ satisfies the Opial condition, a standard argument (see e.g [18]) yields that $\{a_n\}$ converges weakly to $\in\mathcal{X}$. The demiclosedness property of $\mathcal{T}$ (see Lemma 1 now yields that $u\in F(\mathcal{T})$. Setting $u=a^*$ above, our proof is complete.

**Theorem 5.** *Let $\mathcal{H}$ be a real Hilbert space and let $\mathcal{T}:\mathcal{H}\to\mathcal{H}$ be an asymptotically non-expansive mapping with a non-empty fixed points set $F(\mathcal{T})$ and sequence $\{\kappa_n\}\subset[1,+\infty)$, such that $\sum_{n\to\infty}\kappa_n-1<+\infty$. Then the modified inertial Krasnosel'skii-Mann sequence $\{a_n\}$ generated from $a_0,a_1\in\mathcal{H}$ by*

$$\begin{cases} b_n=a_n+\nu_n(a_n-a_{n-1})\\ a_{n+1}=(1-\xi_n)b_n+\xi_n\mathcal{T}^n b_n,\ n=1,2,...\end{cases}$$

*where $\{\xi_n\}$ and $\{nu_n\}$ are real sequences in $(0,1)$, satisfying:*

*(i)* $0<\alpha_1\le\xi_n\le\alpha_2<1$ *for some real constants* $\alpha_1,\alpha_2\in(0,1)$,

*(ii)* $\sum\nu_n<+\infty$

*(iii)* $\sum\nu_n\|a_n-a_{n-1}\|^2<+\infty$

*converges weakly to a fixed point of $\mathcal{T}$.*

*Proof.* Since $\kappa_n \to 1$, there exists a real constant $M_1 > 0$ such that $\kappa_n + 1 \le M_1$, $\forall\, n \ge 1$. Let $z \in F(\mathcal{T})$. Using (5) and the well-known identity $||(1-\lambda)a + \lambda b||^2 = (1-\lambda)||a||^2 + \lambda||b||^2 - \lambda(1-\lambda)||a-b||^2$ which holds in Hilbert spaces $\mathcal{H}$, $\forall a, b \in \mathcal{H}$ and $\forall \lambda \in [0,1]$, we have

$$
\begin{aligned}
||a_{n+1} - z||^2 &= ||(1-\xi_n)b_n + \xi_n \mathcal{T}^n b_n - z||^2 \\
&= ||(1-\xi_n)(b_n - z) + \xi_n(\mathcal{T}^n b_n - z)||^2 \\
&= (1-\xi_n)||(b_n - z)||^2 + \xi_n||\mathcal{T}^n b_n - z||^2 - \xi_n(1-\xi_n)||\mathcal{T}^n b_n - b_n||^2 \\
&\le (1-\xi_n)||b_n - z||^2 + \xi_n \kappa_n^2 ||b_n - z||^2 - \xi_n(1-\xi_n)||\mathcal{T}^n b_n - b_n||^2 \\
&= [1 + \xi_n(\kappa_n^2 - 1)]||b_n - z||^2 - \xi_n(1-\xi_n)||\mathcal{T}^n b_n - b_n||^2 \\
&= [1 + \xi_n(\kappa_n^2 - 1)]||(1+\nu_n)a_n - \nu_n a_{n-1} - z||^2 - \xi_n(1-\xi_n)||\mathcal{T}^n b_n - b_n||^2 \\
&= [1 + \xi_n(\kappa_n^2 - 1)]||(1+\xi_n)(a_n - z) - \nu_n(a_{n-1} - z)||^2 - \xi_n(1-\xi_n)||xi^n b_n - b_n||^2 \\
&= [1 + \xi_n(\kappa_n^2 - 1)][(1+\nu_n)\|a_n - z\|^2 - \nu_n\|a_{n-1} - z\|^2 + \nu_n(1+\nu_n)\|a_n - a_{n-1}\|^2] \\
&\quad -\xi_n(1-\xi_n)||\mathcal{T}^n b_n - b_n||^2 \\
&\le [1 + \xi_n(\kappa_n^2 - 1)](1+\nu_n)\|a_n - z\|^2 + [1 + \xi_n(\kappa_n^2 - 1)]\nu_n(1+\nu_n)\|a_n - a_{n-1}\|^2] \\
&\quad -\xi_n(1-\xi_n)||\mathcal{T}^n b_n - b_n||^2 \\
&\le ||a_n - z||^2 + [\nu_n + \xi_n(\kappa_n^2 - 1) + \xi_n \nu_n(\kappa_n^2 - 1)]\|a_n - z\|^2 \\
&\quad +2(1 + M_1^2)\nu_n\|a_n - a_{n-1}\|^2 \\
&\quad -\xi_n(1-\xi_n)||\mathcal{T}^n b_n - b_n||^2 \\
&\le [1 + \delta_n]\|a_n - z\|^2 + 2(1 + M_1^2)\nu_n\|a_n - a_{n-1}\|^2 \\
&\quad -\xi_n(1-\xi_n)||\mathcal{T}^n b_n - b_n||^2 \qquad (13) \\
&\le [1 + \delta_n]\|a_n - z\|^2 + 2(1 + M_1^2)\nu_n\|a_n - a_{n-1}\|^2,
\end{aligned}
$$

where $\delta_n = \nu_n + 2M_1(\kappa_n - 1)$ is such that $\sum \delta_n < +\infty$, since $\sum(\kappa_n - 1) < +\infty$ and condition (ii) holds. Using this, Lemma 2 and condition (iii), we have that $\lim \|a_n - a^*\|^2$ exists. This implies $\{a_n - a^*\}$ and $\{a_n\}$ are norm bounded. Hence, there exists a real constant $D_3 > 0$, such that $\|a_n - a^*\|^2 \le D_3$. Using this in (13), we have that

$$\|a_{n+1} - a^*\|^2 \le \|a_n - a^*\|^2 + \delta_n D_3 + 2(1 + M_1^2)\nu_n\|a_n - a_{n-1}\|^2 - \xi_n(1-\xi_n)||\mathcal{T}^n b_n - b_n||^2$$

From this and condition and (iii), we have

$$
\begin{aligned}
\sum_{n\ge 0} \alpha_1(1-\alpha_2)||\mathcal{T}^n b_n - b_n||^2 &\le \sum_{n\ge 0} \xi_n(1-\xi_n)||\mathcal{T}^n b_n - b_n||^2 \\
&\le \sum_{n\ge 0}[||a_n - a^*||^2 - ||a_{n+1} - a^*||^2] + D_3 \sum_{n\ge 0} \delta_n \\
&\quad +2(1 + M_1^2)\sum_{n\ge 0} \nu_n\|a_n - a_{n-1}\|^2 < \infty.
\end{aligned}
$$

This implies from condition (i) that $\lim ||\mathcal{T}^n b_n - b_n||^2 = 0$. Hence

$$\lim ||\mathcal{T}^n b_n - b_n|| = 0.$$

The rest of the proof now follows easily as in that of Theorem 4 above.

**Theorem 6.** *Let $\mathcal{H}$ be a real Hilbert space and let $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ be an asymptotically nonexpansive mapping with a non-empty fixed points set $F(\mathcal{T})$ and sequence $\{\kappa_n\} \subset [1, \infty)$, such that $\sum_{n\to\infty} \kappa_n - 1 < +\infty$. Then the modified inertial Krasnosel'skii-Mann sequence $\{a_n\}$ generated from $a_0, a_1 \in \mathcal{H}$ by*

$$\begin{cases} b_n = a_n + \nu_n(a_n - a_{n-1}) \\ a_{n+1} = (1 - \xi_n)b_n + \xi_n \mathcal{T}^n b_n, \ n = 1, 2, ... \end{cases}$$

*where $\{\xi_n\}$ and $\{\nu_n\}$ are real sequences in $(0, 1)$, satisfying:*

*(i)* $\liminf \xi_n(1 - \xi_n) > 0$,

*(ii)* $\sum {\nu_n}^{\frac{1}{2}} < +\infty$

*(iii)* $\nu_n\|a_n - a_{n-1}\|^4 \le D_4$ *for some positive real constant* $D_4$,

*converges weakly to a fixed point of $\mathcal{T}$.*

*Proof.* Computing as in Theorem 5 above, we arrive at

$$||a_{n+1} - z||^2 \le [1 + \delta_n]\|a_n - z\|^2 + 2(1 + M^2)\nu_n\|a_n - a_{n-1}\|^2 - \xi_n(1 - \xi_n)||\mathcal{T}^n b_n - b_n||^2.$$

This implies

$$\begin{aligned} ||a_{n+1} - z||^2 &\le [1 + \delta_n]\|a_n - p\|^2 + 2(1 + M^2)\nu_n^{\frac{1}{2}}\sqrt{\nu_n\|a_n - a_{n-1}\|^4} - \xi_n(1 - \xi_n)||\mathcal{T}^n b_n - b_n||^2 \\ &\le [1 + \delta_n]\|a_n - z\|^2 + 2(1 + M^2)\nu_n^{\frac{1}{2}}\sqrt{D_4} - \xi_n(1 - \xi_n)||\mathcal{T}^n b_n - b_n||^2 \end{aligned}$$

where $\delta_n = \nu_n + 2M(\kappa_n - 1)$ is such that $\sum \delta_n < +\infty$, since $\sum(\kappa_n - 1) < +\infty$ and condition (ii) holds. Using this, Lemma 2 and condition (ii), we have that $\lim \|a_n - a^*\|^2$ exists. This implies $\{a_n - a^*\}$ and $\{a_n\}$ are norm bounded. Hence, there exists a real constant $D_5 > 0$, such that $\|a_n - a^*\|^2 \le D_5$. Using this in (14), we have that

$$\|a_{n+1} - a^*\|^2 \le \|a_n - a^*\|^2 + \delta_n D_5 + 2(1 + M^2)\nu_n^{\frac{1}{2}}\sqrt{D_4} - \xi_n(1 - \xi_n)||\mathcal{T}^n b_n - b_n||^2.$$

From this and conditions (ii), we have

$$\begin{aligned} \sum_{n\ge0} \xi_n(1 - \xi_n)||\mathcal{T}^n b_n - b_n||^2 &\le \sum_{n\ge0}[||a_n - a^*||^2 - ||a_{n+1} - a^*||^2] + D_5\sum_{n\ge0}\delta_n \\ &\quad +2(1 + M^2)\sqrt{D_4}\sum_{n\ge0}\nu_n^{\frac{1}{2}} < \infty. \end{aligned}$$

This implies from condition (i) that $\lim ||\mathcal{T}^n b_n - b_n||^2 = 0$. Hence

$$\lim ||\mathcal{T}^n b_n - b_n|| = 0.$$

From (5), (ii) and the fact that $\{a_n\}$ is norm bounded, we have

$$\|b_n - a_n\| = \nu_n \|a_n - a_{n-1}\| \leq \nu_n^{\frac{1}{2}}[\|a_n\| + \|a_{n-1}\|] \to 0$$

The rest of the proof follows easily like in Theorem 4 above.

**Theorem 7.** *Let $\mathcal{H}$ be a real Hilbert space and let $\mathcal{A} : \mathcal{H} \to \mathcal{C} \subseteq \mathcal{H}$ be a maximally monotone operator such that $Zer(\mathcal{A}) \neq \emptyset$. Let $J_{\mathcal{A}}^{\lambda} := (I + \lambda\mathcal{A})^{-1}$ be the resolvent of $\mathcal{A}$. Then the modified inertial Krasnosel'skii-Mann sequence $\{a_n\}$ generated from $a_0, a_1 \in \mathcal{H}$ by*

$$\begin{cases} b_n = a_n + \nu_n(a_n - a_{n-1}) \\ a_{n+1} = (1 - \xi_n)y_n + \xi_n J_{\mathcal{A}}^{\lambda} b_n \end{cases}$$

*where $\{\xi_n\}$ and $\{\nu_n\}$ are real sequences in $(0, 1)$, satisfying:*

*(i)* $0 < \alpha_1 \leq \xi_n \leq \alpha_2 < 1$ *for some real constants* $\alpha_1, \alpha_2 \in (0, 1)$,

*(ii)* $\sum \nu_n < +\infty$

*(iii)* $\sum \nu_n \|a_n - a_{n-1}\|^2 < +\infty$.

*converges weakly to an element of $F(J_{\mathcal{A}}^{\lambda})$, which is also an element of $zer(\mathcal{A})$.*

*Proof.* Since $J_{\mathcal{A}}^{\lambda}$ is nonexpansive, the proof follows like that of Theorem 5 above, since every nonexpansive mapping is an asymptotically nonexpansive mapping with sequence $\kappa_n = 1 \ \forall \ n \geq 1$.

Theorem 7 can be applied in solving convex optimization problems of the form

$$min_{a \in \mathcal{H}}\{h(a)\},$$

where $h : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function. To do this, we recall the following:
If $h : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function, then its (convex) subdifferential at $a \in \mathcal{H}$ is defined by

$$\partial h(a) = \{b \in \mathcal{H} : h(u) \geq h(a) + \langle b, u - a\rangle \forall u \in \mathcal{H}\},$$

for all $a \in \mathcal{H}$, with $h(a) = +\infty$ and $\partial h(a) = \emptyset$ otherwise. When the convex subdifferential is seen as a set-valued mapping, then, it is maximally monotone (see [22]) and its resolvent is given by $J_{\partial h} = prox_h$ (see [2]), where $Prox_h : \mathcal{H} \to \mathcal{H}$ is defined by

$$prox_h(a) = argmin_{b \in \mathcal{H}}\{h(b) + \frac{1}{2}\|b - a\|^2\}$$

and is called the proximal operator of $h$. We now have the following:

**Corollary 1.** *Let* $h : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ *be a set-valued, proper, convex and lower semi-continuous function which is such that* $argmin_{a\in\mathcal{H}}\{h(a)\} \neq \emptyset$. *Then the modified inertial Krasnosel'skii-Mann sequence* $\{a_n\}$ *generated from* $a_0, a_1 \in \mathcal{H}$ *by*

$$\begin{cases} b_n = a_n + \nu_n(a_n - a_{n-1}) \\ a_{n+1} = (1-\xi_n)b_n + \xi_n prox_h^\lambda b_n \end{cases}$$

*where* $\{\xi_n\}$ *and* $\{\ni_n\}$ *are real sequences in* $(0,1)$, *satisfying:*

*(i) (i)* $0 < \alpha_1 \leq \xi_n \leq \alpha_2 < 1$ *for some real constants* $\alpha_1, \alpha_2 \in (0,1)$,

*(ii) (ii)* $\sum \nu_n < +\infty$

*(iii) (iii)* $\sum \nu_n \|a_n - a_{n-1}\|^2 < +\infty$

*converges weakly to an element of* $argmin_{a\in\mathcal{H}}\{h(a)\}$.

**Proof:** Setting $\partial h = \mathcal{A}$, the proof follows as in the proof of Theorem 7 since the zero of $\partial h$ is an element of $argmin_{a\in\mathcal{H}}\{h(a)$.

**Remark 1.** *If we let* $\{\sigma_n\} \subset (0,1)$ *such that* $\sum \sigma_n < \infty$ *and choose* $\nu_n \in [0, \bar{\nu_n}]$ *with* $\bar{\nu_n} = min\{\sigma_n^2, \frac{1}{n^2\|a_n - a_{n-1}\|^{2p}}\}$, *then conditions (i) and (iii) of Theorem 4 hold.*

**Remark 2.** *If we let* $\{\sigma_n\} \subset (0,1)$ *such that* $\sum \sigma_n < +\infty$ *and choose* $\nu_n \in [0, \bar{\nu_n}]$ *with* $\bar{\nu_n} = min\{\sigma_n, \frac{1}{n^2\|a_n - a_{n-1}\|^2}\}$, *then conditions (ii) and (iii) of Theorem 5 hold.*

**Remark 3.** *It is necessary to restrict* $p > 1$ *to be a positive integer in order to be able to evaluate* $w_p(1+\nu_n)$, *resulting from the application of the functional* $w_p(.)$ *defined in Lemma 3, which is crucial for the proofs of our theorems.*

**Remark 4.** *Amongst other things, our results take care of the comments made in* **Observation 2** *above.*

**Remark 5.** *We do not require any boundedess condition similar to that on* $\{\mathcal{T}^n w_n - w_n\}$ *in [11], for our results to hold.*

**Competing Interests:** The authors declare that there are no competing interests surrounding the research work carried out herein.

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