EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 3, 2024, 1539-1552
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Convex 2-Domination in Graphs

Sergio R. Canoy , Jr.[1,2], Ferdinand P. Jamil [1,2], Rona Jane G. Fortosa [1,2,*],
Jead M. Macalisang [1,2]

[1] *Department of Mathematics and Statistics, College of Science and Mathematics, MSU-Iligan Institute of Technology, 9200 Iligan City, Philippines*
[2] *Center of Mathematical and Theoretical Physical Sciences- PRISM, MSU-Iligan Institute of Technology, 9200 Iligan City, Philippines*

Abstract. Let G be a connected graph. A set $S \subseteq V(G)$ is *convex* 2*-dominating* if S is both convex and 2-dominating. The minimum cardinality among all convex 2-dominating sets in G, denoted by $\gamma_{2con}(G)$, is called the *convex* 2*-domination number* of G. In this paper, we initiate the study of convex 2- domination in graphs. We show that any two positive integers a and b with $6 \leq a \leq b$ are, respectively, realizable as the convex domination number and convex 2-domination number of some connected graph. Furthermore, we characterize the convex 2-dominating sets in the join, corona, lexicographic product, and Cartesian product of two graphs and determine the corresponding convex 2-domination number of each of these graphs.

2020 Mathematics Subject Classifications: 05C69
Key Words and Phrases: convex, 2-dominating, convex 2-domination, join, corona, lexicographic product, Cartesian product

1. Introduction

Domination is one of the well-studied concepts in Graph Theory. Variations of domination as well as concepts related to it have been introduced and explored in various aspects and in several graphs (including those graphs resulting from binary operations). Two of its variants utilized the concepts of geodetic and convex sets. For studies that involve these concepts one may consider [5], [4], [9], and [22]. For some variations of domination, one may refer to [3], [10], [13], [20], [23] and [25].

Recently, motivated by some historical and theoretical applications, Roman and Italian domination concepts have also been studied (see [1], [2], [11], [12], [14], [16], [18], [21], [26], [27], and [28]).

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v17i3.5189

Email addresses: sergio.cano@g.msuiit.edu.ph (S. Canoy, Jr.),
ferdinand.jamil@g.msuiit.edu.ph (F. Jamil),
ronajane.fortosa@g.msuiit.edu.ph (RJ. Fortosa), jead.macalisang@g.msuiit.edu.ph (J. Macalisang)

One can readily observe that a variant of the domination concept is usually obtained by combining a domination concept with other graph-theoretic concepts. For example, convex domination is simply the combination of the standard domination and the notion of convexity in graphs (see [8], [17], and [24]). In this paper, the concept of convex 2-domination will be introduced and investigated. This new concept is a combination of convexity and 2-domination. As used to model a protection strategy problem in a given network, every vertex (location) which is not in a 2-dominating set will be considered unsafe (as it contains no guard). Hence, the elements of a 2-dominating set are the ones considered safe locations in the network and each location contains a guard. To defend or secure a given network, every unsafe location must be adjacent to at least two safe locations. This ensures that when an unsafe location is attacked, a location is within the vicinity of the guards from these safe locations. The convexity property attached to the concept guarantees that every location in any shortest path connecting two safe locations is also safe. This newly defined concept is useful when studying a variation of Italian domination, in particular, convex Italian domination [19].

2. Terminologies and Notations

For a connected graph $G = (V(G), E(G)$ and vertices u and v of G, any shortest path joining u and v is called a u-v geodesic. The length of a u-v geodesic is the *distance* between u and v. This distance is denoted by $d_G(u, v)$. The set $I_G[u, v]$ consists of vertices u and v and those lying on any u-v geodesic. Here, $I_G(u, v) = I_G[u, v] \setminus \{u, v\}$.

The set $N_G(u)$ is called the *open neighborhood* of u, i.e., $N_G(u)$ consists of all $v \in V(G)$ such that $uv \in E(G)$. The *closed neighborhood* of u is the set $N_G[u] = N_G(u) \cup \{u\}$. If $S \subseteq V(G)$, then $N_G(S) = \cup_{v \in S} N_G(v)$ and $N_G[S] = N(S) \cup S$. The *degree* of a vertex v, denoted by $deg_G(v)$, is given by $deg_G(v) = |N(v)|$. A vertex of degree 1 is called an *end-vertex* or a *leaf*. If v is a leaf and $w \in N_G(v)$, then w is called a *support vertex*. A vertex v is an *extreme* or *simplicial* vertex in G if $N_G(v)$ induces a complete subgraph of G, that is, $Ext(G) = \{v \in V(G) : N_G(v)$ induces a complete subgraph of $G\}$. We denote by $\mathcal{L}(G)$, $\mathcal{S}(G)$, and $Ext(G)$ the sets containing the leaves, the support, and the extreme vertices, respectively, of graph G.

A set $S \subseteq V(G)$ is *non-connecting* in G if for every two vertices $x, y \in V(G) \setminus S$ with $d_G(x, y) = 2$, it holds that $N_G(x) \cap N_G(y) \cap S = \varnothing$.

A set $S \subseteq V(G)$ is *independent* if $d_G(x, y) \neq 1$ for every pair of distinct vertices $x, y \in S$. The set $\mathcal{I}nd(G)$ denotes the set of all vertices v of G such that $N_G(v)$ is an independent subset of $V(G)$. Clearly, $\mathcal{L}(G) \subseteq \mathcal{I}nd(G)$.

A set $S \subseteq V(G)$ is said to be *dominating* (resp. 2-*dominating*) in G if $N[S] = V(G)$ (resp. $|N_G(v) \cap S| \geq 2$ for every $v \in V(G) \setminus S$). The smallest cardinality of a dominating (resp. 2-dominating) set S is called the *domination number* (resp. 2-*domination number*) of G and is denoted by $\gamma(G)$ (resp. $\gamma_2(G)$). Any dominating (resp. 2-dominating) set with cardinality $\gamma(G)$ (resp. $\gamma_2(G)$) is called a γ-*set* (resp. γ_2-*set*) in G. If $\{v\}$ is a dominating set in G, then we call v a *dominating vertex* in G.

A set $S \subseteq V(G)$ is a *clique* if the graph $\langle S \rangle$ induced by S is a complete graph. A set $S \subseteq V(G)$ is *clique dominating* (resp. *clique* 2*-dominating*) if S is both a clique and dominating (resp. a clique and 2-dominating). The smallest cardinality of a clique dominating (resp. clique 2-dominating) set in G, denoted by $\gamma_{cl}(G)$ (resp. $\gamma_{2cl}(G)$), is called the *clique domination number* (resp. *clique* 2*-domination number*) of G. Any clique dominating (resp. clique 2-dominating) set in G with cardinality $\gamma_{cl}(G)$ (resp. $\gamma_{2cl}(G)$) is called a γ_{cl}-set (resp. γ_{2cl}-set) in G. We note that not every connected graph admits a clique dominating set (for example, P_n and C_n, where $n \geq 5$, do not admit a clique dominating set).

A set $S \subseteq V(G)$ is *convex* if for every two vertices $x, y \in S$, it holds that $I_G(x, y) \subseteq S$. A set $S \subseteq V(G)$ is *convex dominating* (resp. *convex* 2*-dominating*) if S is both convex and dominating (resp. 2-dominating). The minimum cardinality among all convex dominating (convex 2-dominating) sets in G, denoted by $\gamma_{con}(G)$ (resp. $\gamma_{2con}(G)$, is called the *convex domination number* (resp. *convex* 2*-domination number*) of G. Any convex dominating (resp. convex 2-dominating) set in G with cardinality $\gamma_{con}(G)$ (resp. $\gamma_{2con}(G)$) is called a γ_{con}-set (resp. γ_{2con}-set) in G.

3. Results

Proposition 1. *Let G be any connected graph on n vertices. Then*

$$\max\{\gamma_{con}(G), \gamma_2(G)\} \leq \gamma_{2con}(G) \leq n.$$

Moreover,

(*i*) $\gamma_{2con}(G) = 1$ *if and only if* $G = K_1$.

(*ii*) $\gamma_{2con}(G) = 2$ *if and only if* $G = K_2$ *or* $G = K_2 + H$ *for some graph* H *of order* $n-2$.

Proof. The definition of the convex 2-dominating set implies that $\max\{\gamma_{con}(G), \gamma_2(G)\} \leq \gamma_{2con}(G)$. Since $V(G)$ is convex 2-dominating, the given upper bound follows.

(*i*) Clearly, $\gamma_{2con}(K_1) = 1$. Suppose $\gamma_{2con}(G) = 1$, say $S = \{v\}$ is a γ_{2con}-set of G. If $G \neq K_1$, then there exists $w \in V(G)\backslash S$. Since $|S| = 1$, $|N_G(w) \cap S| \leq 1$, a contradiction. Thus, $G = K_1$.

(*ii*) Suppose $\gamma_{2con}(G) = 2$ and suppose $G \neq K_2$. Let $S = \{x, y\}$ be a γ_{2con}-set in G. Since S is convex, $xy \in E(G)$. Let $H = \langle V(G)\backslash S\rangle$. Since S is a 2-dominating set, $z \in N_G(x) \cap N_G(y)$ for all $z \in V(H)$. Hence, $G = \langle S\rangle + H \cong K_2 + H$.

For the converse, suppose that $G = K_2$. Then $\gamma_{2con}(G) = 2$. Next, suppose that $G = K_2 + H$ for some graph H. Then $D = V(K_2)$ is a convex 2-dominating set in G. Hence, $\gamma_{2con}(G) = |D| = 2$. □

Proposition 2. *Let G be a connected graph and let S be a convex 2-dominating set in G. Then each of the following holds:*

(*i*) $\mathcal{S}(G) \cup \mathcal{I}nd(G) \subseteq S$.

(*ii*) *If* $v \in V(G) \setminus S$, *then* $\langle N_G(v) \cap S \rangle$ *is a non-trivial complete graph.*

Proof.

(*i*) Let $p \in \mathcal{I}nd(G)$. If $p \in V(G) \setminus S$, then there exist $s, t \in S \cap N_G(p)$ because S is a 2-dominating set. This is not possible since $st \notin E(G)$ and S is convex. Therefore, $p \in S$, showing that $\mathcal{I}nd(G) \subseteq S$. Next, let $w \in \mathcal{S}(G)$ and let $z \in \mathcal{L}(G) \cap N_G(w)$. Since $z \in S$ and S is convex, there exists no $u \in S \setminus \{z\}$ such that $uw \in E(G)$. This forces $w \in S$. Since w was arbitrarily chosen, $\mathcal{S}(G) \subseteq S$.

(*ii*) Let $v \in V(G) \backslash S$. Since S is 2-dominating, $|N_G(v) \cap S| \geq 2$. Let $x, y \in N_G(v) \cap S$ with $x \neq y$. Since $v \in V(G) \backslash S$ and S is convex, $xy \in E(G)$. Thus, $\langle N_G(v) \cap S \rangle$ is a (non-trivial) complete subgraph of G. □

Corollary 1. *Let* G *be a connected non-trivial graph. If* $\gamma_{2con}(G) = 2$, *then* $\gamma(G) = 1$.

Observe that since $\gamma(P_3) = 1$ and $\gamma_{2con}(P_3) = 3$, the converse of Corollary 1 is not true.

Proposition 3. *Let* G *be a connected graph of order* $n \geq 2$. *If* $\gamma_{2con}(G) = n$, *then* $Ext(G) \setminus \mathcal{L}(G) = \varnothing$. *Moreover, if* $V(G) = \mathcal{S}(G) \cup \mathcal{I}nd(G)$, *then* $\gamma_{2con}(G) = n$.

Proof. Suppose that $\gamma_{2con}(G) = n$. Suppose there exists $v \in Ext(G) \setminus \mathcal{L}(G)$. Then $|N_G(v)| \geq 2$. Let $S = V(G) \backslash \{v\}$. Then S is a convex 2-dominating set in G. Hence, $\gamma_{2con}(G) \leq |S| = n - 1$, a contradiction. Thus, $Ext(G) \setminus \mathcal{L}(G) = \varnothing$.

The remaining part follows from Proposition 2. □

Corollary 2. *Let* G *be a connected graph of order* n. *Then each of the following holds.*

(*i*) $\gamma_{2con}(K_n) = \begin{cases} 1, & n = 1 \\ 2, & n \geq 2. \end{cases}$

(*ii*) $\gamma_{2con}(P_n) = n$, *for all* $n \geq 1$.

(*iii*) $\gamma_{2con}(C_n) = \begin{cases} 2, & n = 3 \\ n, & n \geq 4. \end{cases}$

(*iv*) $\gamma_{2con}(K_{1,n}) = n + 1$, *for all* $n \geq 1$.

Proposition 4. *Let* $G = K_{n_1, n_2, \ldots, n_k}$ *be the complete* k*-partite graph with* $2 \leq n_1 \leq n_2 \leq \cdots \leq n_k$, *where* $k \geq 2$. *Then*

$$\gamma_{2con}(G) = \begin{cases} n_1 + n_2 & \text{if } k = 2 \\ 3 & \text{if } k \geq 3. \end{cases}$$

Proof. Let $S_{n_1}, S_{n_2}, \ldots, S_{n_k}$ be the partite sets in G. Suppose first that $k = 2$. Suppose further that $D \neq V(G)$, say $v \in V(G) \setminus D$. We may assume that $v \in S_{n_1}$. Since D is 2-dominating, there exist $a, b \in S_{n_2} \cap N_G(v)$. This, however, implies that D is not convex, a contradiction. Thus, $D = V(G)$, showing that $\gamma_{2con}(G) = n_1 + n_2$.

Next, suppose that $k \geq 3$. Since G is non-trivial and $G \neq K_2 + H$ for any graph H, $\gamma_{2con}(G) \geq 3$ by Proposition 1. Pick any $v_i \in S_{n_i}$ for $i = 1, 2, 3$ and let $S = \{v_1, v_2, v_3\}$. Then $\langle S \rangle$ is complete. Hence, S is a convex set in G. Clearly, S is also a 2-dominating set. Therefore, $\gamma_{2con}(G) = |S| = 3$. □

Theorem 1. [15] For a cycle C_n on $n \geq 6$ vertices, $\gamma_{con}(C_n) = n$.

Theorem 2. *Let a and b be positive integers such that $6 \leq a \leq b$. Then there exists a connected graph G such that $\gamma_{con}(G) = a$ and $\gamma_{2con}(G) = b$.*

Proof. Consider the following cases:

Case 1: $a = b$.

Let $G = C_a$. Then $\gamma_{con}(G) = a = \gamma_{2con}(G)$, by Theorem 1 and Corollary 2(*iii*).

Case 2: $a < b$.

Let $m = b - a$. Consider the graph G in Figure 1. Let S be a γ_{con}-set in G. Since S is a convex dominating set, $v_a \in S$. By Observation 1, $\{v_1, v_2, \ldots, v_{a-1}\} \subseteq S$. Since S is a γ_{con}-set in G, $S = \{v_1, v_2, \ldots, v_a\}$. Hence, $\gamma_{con}(G) = |S| = a$. Next, let D be a γ_{2con}-set in G. Since $\mathcal{I}nd(G) = V(G)$, $\gamma_{2con}(G) = |D| = |V(G)| = a + m = b$ by Proposition 3.

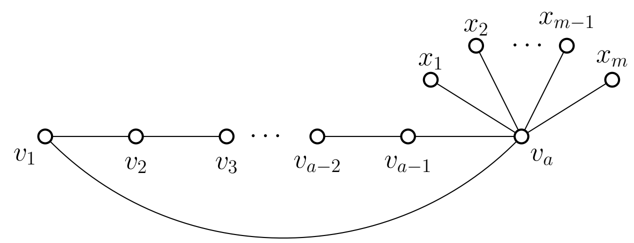

Figure 1: A graph G with $\gamma_{con}(G) = a$ and $\gamma_{2con}(G) = b$

This proves the assertion. □

The *join* of two graphs G and H, denoted by $G + H$, is the graph with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(G)\}$, where "∪" refers to a disjoint union of sets.

Theorem 3. *Let G and H be non-complete graphs. Then $S \subseteq V(G + H)$ is a convex 2-dominating set in $G + H$ if and only if one of the following holds:*

(*i*) $S = V(G + H)$.

(*ii*) S *is a clique* 2*-dominating set in* G.

(*iii*) S *is a clique* 2*-dominating set in* H.

(*iv*) $S = S_G \cup S_H$ *where* $\varnothing \neq S_G \subseteq V(G)$ *and* $\varnothing \neq S_H \subseteq V(H)$ *satisfy any of the following:*

(*a*) S_G *and* S_H *are cliques in* G *and* H*, respectively, where* $|S_G| \geq 2$ *and* $|S_H| \geq 2$.

(*b*) S_G *and* S_H *are dominating sets in* G *and* H*, respectively, with* $|S_G| = |S_H| = 1$.

(*c*) $|S_G| = 1$ *and* S_H *is a clique dominating set in* H *with* $|S_H| \geq 2$.

(*d*) $|S_H| = 1$ *and* S_G *is a clique dominating set in* G *with* $|S_G| \geq 2$.

Proof. Suppose S is a convex 2-dominating set in $G+H$ where $S \neq V(G+H)$. Suppose first that $S \subseteq V(G)$. Since S is a convex 2-dominating set in $G + H$, S must be a clique 2-dominating set in G. Similarly, if $S \subseteq V(H)$, S is a clique 2-dominating set in H. Hence, (*ii*) or (*iii*) holds. Next, suppose that $S_G = S \cap V(G) \neq \varnothing$ and $S_H = S \cap V(H) \neq \varnothing$. Since S is convex and G and H are non-complete, S_G and S_H are cliques in G and H, respectively. If $|S_G| \geq 2$ and $|S_H| \geq 2$, then (*a*) holds. Suppose $|S_G| = 1$ and $|S_H| = 1$. Since S is 2-dominating in $G + H$, it follows that S_G and S_H are dominating sets in G and H, respectively, showing that (*b*) holds. Suppose $|S_G| = 1$ and $|S_H| \geq 2$. Since $S_H \neq V(H)$ (otherwise $S = V(G + H)$) and S is convex 2-dominating in $G + H$, S_H is a clique dominating set in H. Hence, (*c*) holds. Similarly, (*d*) holds if $|S_G| \geq 2$ and $|S_H| = 1$.

The converse is clear. □

Corollary 3. *Let* G *and* H *be non-complete graphs such that* $\gamma(G) = \gamma(H) = 1$*. Then* $\gamma_{2con}(G + H) = 2$.

Lemma 1. *Let* G *be a non-trivial connected graph. If* G *admits a clique* 2*-dominating set, then* $1 + \gamma_{cl}(G) \leq \gamma_{2cl}(G)$.

Proof. Let D be a γ_{2cl}-set in G. Then, clearly, $|D| \geq 2$. Let $v \in D$ and set $D^* = D \setminus \{v\}$. If $|D| = 2$, then $|D^*| = 1$ and D^* is a dominating set in G. Suppose $|D| \geq 3$ Let $z \in V(G) \setminus D^*$. If $z \in D$, then $zw \in E(G)$ for every $w \in D^*$. Suppose $z \in V(G) \setminus D$. Since D is 2-dominating, $|N_G(z) \cap D| \geq 2$. It follows that $|N_G(z) \cap D^*| \geq 1$, showing that D^* is a clique dominating set in G. Thus, $\gamma_{cl}(G) \leq |D^*| = \gamma_{2cl}(G) - 1$. This proves the assertion. □

The next result follows from Theorem 3 and Lemma 1.

Corollary 4. *Let* G *and* H *be non-complete graphs such that* $\gamma(G) \neq 1$ *and* $\gamma(H) \neq 1$.

(*i*) *If* G *and* H *both admit a clique* 2*-dominating set* (*a clique dominating set*)*, then*

$$\gamma_{2con}(G + H) = \min\{1 + \gamma_{cl}(G), 1 + \gamma_{cl}(H), 4\}.$$

(*ii*) *If G admits a clique dominating set but H does not, then*

$$\gamma_{2con}(G+H) = \min\{1+\gamma_{cl}(G), 4\}.$$

(*iii*) *If G and H are disconnected graphs, then*

$$\gamma_{2con}(G+H) = \begin{cases} 4 & \text{if } E(G) \neq \varnothing \text{ and } E(H) \neq \varnothing \\ |V(G+H)| & \text{if } E(G) = \varnothing \text{ or } E(H) = \varnothing. \end{cases}$$

Theorem 4. *Let G be a non-complete graph and let n be a positive integer. A subset $S \subseteq V(K_n+G)$ is convex 2-dominating in K_n+G if and only if one of the following holds:*

(*i*) *$S \subseteq V(K_n)$ and $|S| \geq 2$.*

(*ii*) *S is a clique 2-dominating set in G.*

(*iii*) *$S = S_G \cup S_n$ where $\varnothing \neq S_G \subseteq V(G)$ and $\varnothing \neq S_n \subseteq V(K_n)$ satisfies any of the following:*

- (*a*) *$S_n = V(K_n)$ and $V(G) \setminus S_G$ is a non-connecting set in G, where, in addition, S_G is dominating if $n = 1$.*
- (*b*) *$S_n \neq V(K_n)$ with $|S_n| = 1$ and S_G is a clique dominating set in G.*
- (*c*) *$S_n \neq V(K_n)$ with $|S_n| \geq 2$ and S_G is a clique in G.*

Proof. Suppose S is a convex 2-dominating set in K_n+G. If $S \subseteq V(K_n)$, then $|S| \geq 2$, showing that (*i*) holds. Suppose that $S \subseteq V(G)$. Since G is non-complete and S is convex and 2-dominating in $G+H$, S is a clique dominating set in G. Hence, (*ii*) holds. Next, suppose $S_G = V(G) \cap S \neq \varnothing$ and $S_n = V(K_n) \cap S \neq \varnothing$. Suppose first that $S_n = V(K_n)$. Let $p, q \in S_G$ such that $d_G(p,q) = 2$. Since S is convex in K_n+G, $I_G(p,q) = I_{K_n+G}(p,q) \setminus V(K_n) \subseteq I_{K_n+G}(p,q) \subseteq S$. It follows that $I_G(p,q) \subseteq S_G$. Hence, $N_G(p) \cap N_G(q) \cap (V(G) \setminus S_G) = \varnothing$. This shows that $V(G) \setminus S_G$ is a non-connecting set in G. If $n = 1$, then S_G is a dominating set because S is a 2-dominating set in K_n+G. Thus, (*a*) holds. Now suppose that $S_n \neq V(K_n)$. Since S is convex, S_G is a clique in G. Moreover, since S is 2-dominating in K_n+G, S_G is a dominating set in G if $|S_n| = 1$. This shows that (*b*) or (*c*) holds.

The converse is easy. □

The next result is immediate from Theorem 4.

Corollary 5. *Let G be a non-complete graph and let n be a positive integer. Then*

$$\gamma_{2con}(K_n+G) = \begin{cases} 1+\psi_G & \text{if } n = 1 \\ 2 & \text{if } n \geq 2 \end{cases}$$

where $\psi_G = min\{|S| : S$ is a dominating set and $V(G) \setminus S$ is a non-connecting set in $G\}$.

Let G and H be connected graphs. The *corona* of G and H is the graph $G \circ H$ obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the ith vertex of G to every vertex of the ith copy of H. For convenience, we write H^v to denote the copy of H joined to v and write $H^v + v = H^v + \langle v \rangle$.

Remark 1. *Let G be a non-trivial connected graph. If S is a clique in G, then $V(G) \setminus S$ is a non-connecting set.*

Theorem 5. *Let G be a non-trivial connected graph and let H be any graph. Then S is a convex 2-dominating set in $G \circ H$ if and only if $S = V(G) \cup \left(\bigcup_{v \in V(G)} S_v \right)$ where S_v is dominating and $V(H^v) \setminus S_v$ is a non-connecting set in H^v for each $v \in V(G)$.*

Proof. Let S be a convex 2-dominating set in $G \circ H$. Let $A = S \cap V(G)$ and let $S_v = S \cap V(H^v)$ for each $v \in V(G)$. Then $S = A \cup \left(\bigcup_{v \in V(G)} S_v \right)$. Suppose $A \neq V(G)$. Then there exists $u \in V(G) \setminus A$. Since S is a dominating set, $S_u \neq \varnothing$. Since G is a non-trivial connected graph, $\langle S \rangle$ is disconnected, a contradiction to the assumption that S is convex. Hence, $A = V(G)$ and $S = V(G) \cup \left(\bigcup_{v \in V(G)} S_v \right)$. Next, let $v \in V(G)$. Since $v \in S$ and $S \cap [V(H^v) \cup \{v\}]$ is a convex 2-dominating set in $v + H^v$, S_v is a dominating set, and $V(H^v) \setminus S_v$ is a non-connecting set in H^v by Theorem 4$(iii)(a)$.

Conversely, suppose that S has the given form and satisfies the given property. Then S is a 2-dominating set in $v + H^v$. By Theorem 4$(iii)(a)$, $S_v \cup \{v\}$ is a convex 2-dominating set in $v + H^v$ for each $v \in V(G)$. Since G is connected, it follows that $\bigcup_{v \in V(G)} (S_v \cup \{v\}) = S$ is a convex set in $G \circ H$. Hence, S is a convex 2-dominating set in $G \circ H$. □

Corollary 6. *Let G be a nontrivial connected graph of order m and let H be any graph. Then*

$$\gamma_{2con}(G \circ H) = m(1 + \psi_H),$$

where $\psi_H = min\{|S| : S$ is a dominating set and $V(H) \setminus S$ is a non-connecting set in $H\}$.

Proof. Let $S = V(G) \cup \left(\bigcup_{v \in V(G)} S_v \right)$ be a γ_{2con}-set in $G \circ H$. By Theorem 5, S_v is a dominating set, and $V(H^v) \setminus S_v$ is a non-connecting set in H^v. Thus

$$\begin{aligned} \gamma_{2con}(G \circ H) &= |S| \\ &= |V(G)| + \sum_{v \in V(G)} |S_v| \\ &\geq m(1 + \psi_H). \end{aligned}$$

Next, let D_v be a dominating set in H^v such that $V(H^v) \setminus D_v$ is non-connecting and $|D_v| = \psi_H$ for each $v \in V(G)$. Then $S^* = V(G) \cup \left(\bigcup_{v \in V(G)} D_v \right)$ is a convex 2-dominating set in $G \circ H$ by Theorem 5. Hence,

$$\gamma_{2con}(G \circ H) \leq |S^*|$$

$$= |V(G)| + \sum_{v\in V(G)} |D_v|$$
$$= m(1+\psi_H).$$

This proves the desired equality. □

The *Cartesian product* $G \times H$ of two graphs G and H is the graph with $V(G \times H) = V(G) \times V(H)$ and $(u, u')(v, v') \in E(G \times H)$ if and only if either $uv \in E(G)$ and $u' = v'$ or $u = v$ and $u'v' \in E(H)$.

The next result obtained by Canoy and Garces characterizes convex sets in the Cartesian product of graphs.

Theorem 6. [7] Let G and H be connected graphs. A subset S of $V(G\Box H)$ is convex if and only if $S = S_1 \times S_2$, where S_1 and S_2 are convex sets in G and H, respectively.

Lemma 2. *Let G and H be connected graphs. If a subset $S = S_1 \times S_2$ of $V(G\Box H)$ is a 2-dominating set in $G\Box H$, then S_1 and S_2 are 2-dominating sets in G and H, respectively.*

Proof. Suppose that S is a 2-dominating set in $G\Box H$ and let $v \in V(G) \setminus S_1$. Pick any $p \in V(H)$. Since S is 2-dominating in $G\Box H$ and $(v, p) \notin S$, there exist $(u, q), (w, t) \in S\cap N_{G\Box H}((v, p))$. This implies that $p = q = t$, $u \neq w$, and $u, w \in S_1\cap N_G(v)$. Thus, S_1 is a 2-dominating set in G. A similar argument can be used to show that S_2 is a 2-dominating set in H. □

Theorem 7. *Let G and H be connected graphs. A subset S of $V(G\Box H)$ is a convex 2-dominating set in $G\Box H$ if and only if $S = S_1 \times S_2$ and*

(*i*) *S_1 is a convex 2-dominating set in G and $S_2 = V(H)$, or*

(*ii*) *S_2 is a convex 2-dominating set in H and $S_1 = V(G)$.*

Proof. Let S be a convex 2-dominating set in $G\Box H$. By Theorem 6, $S = S_1 \times S_2$, where S_1 and S_2 are convex sets in G and H, respectively. By Lemma 2, S_1 and S_2 are 2-dominating sets in G and H, respectively. Suppose $S_1 \neq V(G)$ and $S_2 \neq V(H)$. Pick any $x \in V(G) \setminus S_1$ and $a \in V(H) \setminus S_2$. Then $(x, d), (z, a) \notin S$ for all $d \in V(H)$ and $z \in V(G)$. It follows that $N_{G\Box H}((x, a)) \cap (S_1 \times S_2) = \varnothing$, implying that $S_1 \times S_2$ is a not a 2-dominating set in $G\Box H$, contrary to our assumption of the set. Hence, $S_1 = V(G)$ or $S_2 = V(H)$. This shows that (*i*) or (*ii*) holds.

For the converse, suppose that (*i*) holds. By Theorem 6, $S = S_1 \times S_2$ is a convex set in $G\Box H$. Let $(v, p) \in V(G\Box H) \setminus S$. Then $v \notin S_1$. Since S_1 is 2-dominating, there exist $u, w \in S_1$ such that $u, w \in N_G(v)$. Consequently, $(u, p), (w, p) \in N_{G\Box H}((v, p))\cap S$. Hence, S is a convex 2-dominating set in $G\Box H$. The same conclusion is obtained if (*ii*) holds. □

Corollary 7. *Let G and H be connected graphs of orders m and n, respectively. Then*

$$\gamma_{2con}(G\Box H) = \min\{m\gamma_{2con}(H), n\gamma_{2con}(G)\}.$$

Proof. Let S be a γ_{2con}-set in $G\square H$. By Theorem 7, $S = S_1 \times V(H)$ or $S = V(G) \times S_2$, where S_1 is a convex 2-dominating set in G and S_2 is a convex 2-dominating set in H. Thus,

$$\gamma_{2con}(G\square H) = |S| \geq \min\{n\gamma_{2con}(G), m\gamma_{2con}(H)\}.$$

Next, suppose that S_1' and S_2' are γ_{2con}-sets in G and H, respectively. Then by Theorem 7, $S' = S_1' \times V(H)$ and $S^* = V(G) \times S_2'$ are convex 2-dominating sets in $G\square H$. It follows that

$$\gamma_{2con}(G\square H) \leq \min\{|S'|, |S^*|\} = \min\{n\gamma_{2con}(G), m\gamma_{2con}(H)\}.$$

This proves the desired equality. □

The *lexicographic product* of two graphs G and H is the graph $G[H]$ with $V(G[H]) = V(G) \times V(H)$ and $(u_1, u_2)(v_1, v_2) \in E(G[H])$ if and only if either $u_1v_1 \in E(G)$ or $u_1 = v_1$ and $u_2v_2 \in E(H)$.

Theorem 8. [7] Let G and H be connected non-complete graphs and let C be a proper subset of $V(G[H])$. Then C is convex in $G[H]$ if and only if C is a clique.

Theorem 9. *Let G and H be connected non-complete graphs. A set $C = \cup_{x\in S}(\{x\}\times T_x) \subseteq V(G[H])$ is convex 2-dominating if and only if $C = V(G[H])$ or S and each T_x are cliques in G and H, respectively, and satisfy one of the following conditions:*

- (i) *S is a 2-dominating set in G and*
  - (a) $|S| \geq 3$ *or*
  - (b) *for each $x \in S$, T_x is dominating in H or $|T_y| \geq 2$ when $S = \{x, y\}$.*
- (ii) *S is a dominating set in G such that*
  - (c) *T_x is 2-dominating in H whenever $S = \{x\}$ and*
  - (d) *for each $v \in V(G) \setminus S$ with $|N_G(v) \cap S| = 1$, it holds that $|T_z| \geq 2$ for $z \in N_G(v) \cap S$.*

Proof. Suppose C is convex 2-dominating and $C \neq V(G[H])$. By Theorem 8, S and T_x are cliques in G and H, respectively, for each $x \in S$. Suppose S is 2-dominating in G and let $x \in S$. If $|S| \geq 3$, then $(i)(a)$ holds. Suppose $|S| = 2$, say $S = \{x, y\}$ and suppose that $|T_y| = 1$. Let $p \in V(H) \setminus T_x$ and $q \in T_y$. Then $(y, q) \in C \cap N_{G[H]}((x, p))$. Since C is 2-dominating in $G[H]$, there exists $(z, t) \in (C \setminus \{(y, q)\}) \cap N_{G[H]}((x, p))$. Since $T_y = \{q\}$, $z = x$ and $t \in T_x \cap N_H(p)$. Therefore, T_x is a (clique) dominating set in H, showing that $(i)(b)$ holds. Next, suppose that S is not 2-dominating. Since C is dominating, it follows that S is a (clique) dominating set in G. Suppose first that $|S| = 1$, say $S = \{x\}$. Let $d \in V(H) \setminus T_x$. Since C is 2-dominating and $(x, d) \notin C$, there exist $(v, l), (w, s) \in C \cap N_{G[H]}((x, d))$. This implies that $v = w = x$ and $l, s \in T_x \cap N_H(d)$. Thus, T_x is a (clique) 2-dominating set in H. Finally, let $v \in V(G) \setminus S$ with $|N_G(v) \cap S| = 1$, say $N_G(v) \cap S = \{z\}$. Since C is 2-dominating, $|T_z| \geq 2$, showing that (ii) holds.

The converse is easy. □

Corollary 8. *Let G and H be connected two non-complete graphs of orders m and n, respectively. If G and H do not admit a clique dominating set, then $\gamma_{2con}(G[H]) = mn$.*

Corollary 9. *Let G and H be connected non-complete graphs. Then $\gamma_{2con}(G[H]) = 2$ if and only if $\gamma_{cl}(G) = 1$ and $\gamma_{2cl}(H) = 2$ or $\gamma_{cl}(H) = 1$ and $\gamma_{2cl}(G) = 2$.*

Proof. Suppose $\gamma_{2con}(G[H]) = 2$ and let $C = \{(x, a), (y, b)\}$ be a γ_{2con}-set in $G[H]$. Then $(x, a), (y, b) \in E(G[H])$ because C is convex. If $x = y$, then $ab \in E(H)$. Hence, $\gamma_{cl}(G) = 1$ (x is a dominating vertex of G) and so $\gamma_{2cl}(H) = 2$ by Theorem 9$(ii)(c)$. Suppose $x \neq y$. Then $\{x, y\}$ is a 2-dominating set in G, i.e., $\gamma_{2cl}(G) = 2$. Since $|T_x| = |T_y| = 1$, T_x and T_y are clique dominating sets in H by Theorem 9$(i)(b)$. Thus, $\gamma_{cl}(H) = 1$.

The converse follows from Theorem 9. □

Corollary 10. *Let G and H be connected non-complete graphs. Then $\gamma_{2con}(G[H]) = 3$ if and only if one of the following holds:*

(i) $\gamma_{2cl}(G) = 3$.

(ii) $\gamma_{cl}(G) = 1$ and $\gamma_{2cl}(H) = 3$.

(iii) $\gamma_{2cl}(G) = 2$ and $\gamma_{cl}(H) = 2$.

(iv) $\gamma_{cl}(G) = \gamma_{cl}(H) = 1$, $\gamma_{2cl}(G) \neq 2$, and $\gamma_{2cl}(H) \neq 2$.

(v) $\gamma_{cl}(H) = 2$, $\gamma_{cl}(G) = 1$, and $\gamma_{2cl}(H) \neq 2$.

Proof. Suppose $\gamma_{2con}(G[H]) = 3$ and let $C = \{(w, p), (u, q), (v, r)\}$ be a γ_{2con}-set in $G[H]$. Then C is a clique by Theorem 8. If u, v, and w are distinct vertices of G, then $\gamma_{2cl}(G) = 3$, showing that (i) holds. If $w = u = v$, then $\{p, q, r\}$ is a clique in H. Hence, $\gamma_{cl}(G) = 1$ (w is a dominating vertex of G) and $\gamma_{2cl}(H) = 3$ by Theorem 9(ii). This shows that (ii) holds. Suppose now that $w = u$ and $v \neq w$. Then $Q = \{v, w\}$ is a clique dominating set in G. If Q is 2-dominating, then $\gamma_{2cl}(G) = 2$. Since $\gamma_{2con}(G[H]) \neq 2$, $\gamma_{cl}(H) \neq 1$. It follows that $\{p, q\}$ is a γ_{cl}-set in H, i.e., $\gamma_{cl}(H) = 2$. Hence, (iii) holds. Suppose Q is not 2-dominating. Let $z \in V(G) \setminus Q$ such that $z \notin N_G(w) \cap N_G(v)$. Since C is 2-dominating, this implies that $z \in N_G(w)$. It follows that w is a dominating vertex of G. Thus, $\gamma_{cl}(G) = 1$. By Corollary 9, $\gamma_{2cl}(H) \neq 2$. Suppose $\gamma_{cl}(H) = 1$ (p or q is a dominating vertex of H). Since $\gamma_{2con}(G[H]) \neq 2$, $\gamma_{2cl}(G) \neq 2$. Therefore, (iv) holds. If $\gamma_{cl}(H) \neq 1$, then $\gamma_{cl}(H) = 2$, showing that (v) holds.

For the converse, suppose first that (i) holds. Let $S = \{x, y, z\}$ be a γ_{2cl}-set in G and pick any $p \in V(H)$. Then $C_1 = \{(x, a), (y, a), (z, a)\}$ is a γ_{2con}-set of $G[H]$ by Theorem 9 and Corollary 9. Suppose (ii) holds, say $D = \{p, q, t\}$ is a γ_{2cl}-set in H. Let v be a dominating vertex in G. Then $C_2 = \{(v, p), (v, q), (v, t)\}$ is a γ_{2con}-set of $G[H]$ by Theorem 9 and Corollary 9. Next, suppose (iii) holds. Let $\{x, y\}$ be a γ_{2cl}-set in G and let $\{k, l\}$ be γ_{cl}-set in H. Then $C_3 = \{(x, k), (x, l), (y, k)\}$ is a γ_{2con}-set of $G[H]$ by Theorem 9 and Corollary 9. Suppose now that (iv) holds. Let w and p be dominating vertices of G and

H, respectively. Let $u \in N_G(w)$ and $s \in N_H(p)$. Let $C_4 = \{(w,p),(w,s),(u,p)\}$. By Theorem 9 and Corollary 9, C_3 is a γ_{2con}-set of $G[H]$. Lastly, suppose that (v) holds. Let w be a dominating vertex of G, $v \in N_G(w)$, and let $R = \{a,b\}$ be a γ_{cl}-set in H. Then $C_5 = \{(w,a),(w,b),(v,a)\}$ is a γ_{2con}-set of $G[H]$ by Theorem 9 and Corollary 9. Accordingly, $\gamma_{2con}(G[H]) = 3$. □

Corollary 11. *Let G and H be connected non-complete graphs such that $\gamma_{cl}(G) \geq 2$. Suppose H does not admit a clique dominating set.*

(i) If G admits a clique 2-dominating set, then $\gamma_{2con}(G[H]) \leq \min\{2\gamma_{cl}(G), \gamma_{2cl}(G)\}$.

(ii) If G does not admit a clique 2-dominating set, then $\gamma_{2con}(G[H]) \leq 2\gamma_{cl}(G)$.

The bound given in Corollary 11 is sharp. Indeed, $\gamma_{2con}(P_4[P_n]) = 4 = 2\gamma_{cl}(P_4)$ and $\gamma_{2con}(C_4[C_n]) = 4 = 2\gamma_{cl}(C_4)$ for all $n \geq 5$.

The next result is a rectification of the one obtained by Canoy and Garces in [7].

Theorem 10. [6] Let G be a connected graph and m a positive integer. A set $C = \cup_{v\in S}(\{x\}\times T_x) \subseteq V(G[K_m])$, where $S \subseteq V(G)$ and $T_x \subseteq V(K_m)$ for all $x \in S$, is convex in $G[K_m]$ if and only if S is convex in G and $T_x = V(K_m)$ for each $x \in S^0 = I(S) \cap S$.

Theorem 11. *Let G be a non-trivial connected graph and m a positive integer. A set $C = \cup_{v\in S}(\{x\}\times T_x) \subseteq V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(K_m)$ for all $x \in S$, is convex 2-dominating in $G[K_m]$ if and only if S is convex 2-dominating in G and $T_x = V(K_m)$ for each $x \in S^0 = I(S) \cap S$.*

Proof. Suppose C is convex 2-dominating in $G[K_m]$. By Theorem 10, S is convex in G and $T_x = V(K_m)$ for each $x \in S^0 = I(S) \cap S$. Let $v \in V(G) \setminus S$ and choose any $p \in V(K_m)$. Since $(v,p) \notin C$ and C is 2-dominating in $G[K_m]$, there exist two vertices $(w,t),(z,s) \in C \cap N_{G[K_m]}((v,p))$. This implies that $w, z \in S \cap N_G(v)$. Therefore, S is a 2-dominating set in G.

Conversely, suppose that S is convex in G and $T_x = V(K_m)$ for each $x \in S^0 = I(S)\cap S$. Then C is convex in $G[K_m]$ by Theorem 10. Let $(y,d) \in V(G[K_m])\setminus C$. Suppose $y \in S$. Since S is convex 2-dominating, it follows that $|S| \geq 2$ and $\langle S\rangle$ is connected. Let $u \in S \cap N_G(y)$. Pick any $a \in T_y$ and $b \in T_u$. Then $(y,a),(u,b) \in C \cap N_{G[K_m]}((y,d))$. Next, suppose that $y \notin S$. Since S is 2-dominating in G, there exist distinct vertices $v, w \in S \cap N_G(y)$. Choose any $p \in T_v$ and $q \in T_w$. Then $(v,p),(w,q) \in C \cap N_{G[K_m]}((y,d))$. Thus, C is a 2-dominating set in $G[K_m]$. □

Corollary 12. *Let G be a non-trivial connected graph and let $m \geq 2$. Then*

$$\gamma_{2con}(G[K_m]) = \min\{|S| + (m-1)|S^0| : S \text{ is a convex 2-dominating set in } G\}.$$

Conclusion

Convex 2-domination has been introduced and initially studied in this paper. It was shown that every support vertex and every vertex with an independent open neighborhood belong to every convex 2-dominating set in a connected graph. The convex 2-domination number of a connected graph is at least equal to its convex domination number. Moreover, the difference of these two parameters can be made arbitrarily large. Convex 2-domination has been investigated for the join and corona of two graphs as well as for the lexicographic and Cartesian products of graphs. For some graphs (especially for some join and lexicographic product of graphs), convex 2-domination is related to clique domination. The newly defined concept can be studied for other graphs. It is also interesting to determine the complexity of the convex 2-domination problem.

References

[1] S. Banerjee, J.M. Keil, and D. Pradhan. Perfect Roman domination in graphs. *Theoretical Computer Science*, 796:1–21, 2019.

[2] R.A. Beeler, T.W. Haynes, and S.T. Hedetnieme. Double Roman domination. *Discrete Applied Mathematics*, 211:23–29, 2016.

[3] B. Brešar and S. Brezovnik. Grundy domination and zero forcing in regular graphs. *Bulletin of the Malaysian Mathematical Sciences Society*, 44(6):3637–3661, 2021.

[4] G. Cagaanan and S. Canoy Jr. On the geodetic covers and geodetic bases of the composition g [km]. *Ars Combinatoria*, 79:33–45, 2006.

[5] G. Cagaanan and S. Canoy Jr. Bounds for the geodetic number of the cartesian product of graphs. *Utilitas Mathematica*, 79:91–98, 2009.

[6] S.R. Canoy. A short note on convexity and convex domination in $G[K_m]$. *Applied Mathematical Sciences*, 8(115):5737–5741, 2014.

[7] S.R. Canoy and I.J.L. Garces. Convex sets under some graph operations. *Graphs and Combinatorics*, 18(4):787–793, 2002.

[8] G. Chartrand, J.F. Fink, and P. Zhang. Convexity in oriented graphs. *Discrete applied mathematics*, 116(1-2):115–126, 2002.

[9] G. Chartrand and P. Zhang. Convexity in graphs. *Congr. Numerantium*, 136:19–32, 1999.

[10] M. Chellali. Bounds on the 2-domination number in cactus graphs. *Opuscula Mathematica*, 26(1):5–12, 2006.

[11] M. Chellali, T.W. Haynes, and S.T. Hedetnieme. Roman {2}- domination. *Discrete Applied Math*, 204:22–28, 2016.

[12] M. Chellali, T.W. Haynes, S.T. Hedetniemi, and A.A. McRae. Roman 2-domination. *Discrete Applied Mathematics*, 204:22–28, 2016.

[13] E.J. Cockayne, R.M. Dawes, and S.T. Hedetniemi. Total domination in graphs. *Networks*, 10(3):211–219, 1980.

[14] E.J. Cockayne, P.A. Deryer, S.M. Hedetnieme, and S.T. Hedetnieme. Roman Domination in Graphs. *Discrete Mathematics*, 278(13):11–22, 2004.

[15] J. Cyman, M. Lemańska, and J. Raczek. Graphs with convex domination number close to their order. *Discussiones Mathematicae Graph Theory*, 26(2):307–316, 2006.

[16] M. Dettlaff and M. Lemańskaand J.A. Rodríguez-Velázquez. Secure italian domination in graphs. *Journal of Combinatorial Optimization*, 41:56–72, 2021.

[17] M. Farber and R.E. Jamison. Convexity in graphs and hypergraphs. *SIAM Journal on Algebraic Discrete Methods*, 7(3):433–444, 1986.

[18] R.J. Fortosa and S.R. Canoy. Convex roman dominating function in graphs. *European Journal of Pure and Applied Mathematics*, 16(3):1705–1716, 2023.

[19] R.J. Fortosa, S.R. Canoy, and F.P. Jamil. Convex italian in graphs. *Kyungpook Mathematical Journal, submitted.*

[20] W. Goddard and M.A. Henning. Domination in planar graphs with small diameter. *Journal of Graph Theory*, 40(1):1–25, 2002.

[21] G. Hao, P. Jalilolghadr, and D.A. Mojdeh. Perfect double italian domination of a graph. *AKCE International Journal of Graphs and Combinatorics*, 20(3):247–257, 2023.

[22] F. Harary and J. Nieminen. Convexity in graphs. *Journal of Differential Geometry*, 16(2):185–190, 1981.

[23] T.W. Haynes, S. Hedetniemi, and P. Slater. *Fundamentals of domination in graphs.* CRC press, 2013.

[24] M. Lemanska. Weakly convex and convex domination numbers. *Opuscula Mathematica*, 24(2):181–188, 2004.

[25] B. Omamalin, S.R. Canoy Jr., and H. Rara. Locating total dominating sets in the join, corona and composition of graphs. *Applied Mathematical Sciences*, 8(48):2363–2374, 2014.

[26] L.M. Paleta and F.P. Jamil. On perfect italian domination in graphs. *Discrete Mathematics, Algorithms and Applications*, page 2350085, 2023.

[27] C.S. Revelle and K.E. Rosing. Defendens imperium romanum: a classical problem in military strategy. *American Mathematical Monthly*, 107(7):585–594, 2000.

[28] I. Stewart. Defend the Roman empire! *Scientific American*, 281(6):136–138, 1999.