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Upper and lower almost (τ_1, τ_2)-continuous multifunctions

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Abstract. This paper is concerned with the concepts of upper and lower almost (τ_1, τ_2)-continuous multifunctions. Furthermore, some characterizations of upper and lower almost (τ_1, τ_2)-continuous multifunctions are investigated.

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1. Introduction

It is well-known that the branch of mathematics called topology is related to all questions directly or indirectly concerned with continuity. Semi-open sets, preopen sets, α-open sets, β-open sets and δ-open sets play an important role in the researches of generalizations of continuity in topological spaces. By using these sets many authors introduced and studied various types of weak forms of continuity for functions and multifunctions. Singal and Singal [28] introduced the concept of almost continuous functions as a generalization of continuity. Munshi and Bassan [16] studied the notion of almost semi-continuous functions. Noiri [18] introduced and investigated the concept of almost α-continuous functions. Nasef and Noiri [17] introduced two classes of functions, namely almost precontinuous functions and almost β-continuous functions by utilizing the notions of preopen sets and β-open sets due to Mashhour et al [15] and Abd El-Monsef et al. [12], respectively. The class of almost precontinuity is a generalization of almost α-continuity. The class of almost β-continuity is a generalization of almost semi-continuity. Keskin and Noiri [13] introduced the concept of almost b-continuous functions by utilizing the notion of b-open

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sets due to Andrijević [1]. The class of almost b-continuity is a generalization of almost precontinuity and almost semi-continuity. The class of almost β-continuity is a generalization of almost b-continuity. Popa [22] introduced the concepts of upper and lower almost continuous multifunctions. Popa and Noiri [23] introduced the notions of upper and lower almost quasi-continuous multifunctions. Several characterizations of upper and lower almost quasi-continuous multifunctions were investigated in [19].

In 1996, Popa and Noiri [24] introduced and investigated the notions of upper and lower almost α-continuous multifunctions. In 1997, Popa et al. [26] introduced the concepts of upper and lower almost precontinuous multifunctions. In particular, several characterizations of upper and lower almost precontinuous multifunctions were presented in [27]. In 1999, Noiri and Popa [20] introduced the concepts of upper and lower almost β-continuous multifunctions. Some characterizations of upper and lower almost β-continuous multifunctions were investigated in [25]. In 2006, Ekici and Park [11] introduced and studied almost γ-continuous multifunctions. Noiri and Popa [21] introduced and investigated the notions of upper and lower almost m-continuous multifunctions as multifunctions from a set satisfying some minimal conditions into a topological space. In [3], the present author introduced and studied the concept of pairwise almost M-continuous functions in biminimal structure spaces. Laprom et al. [14] introduced and investigated the notion of almost $\beta(\tau_1, \tau_2)$-continuous multifunctions. Viriyapong and Boonpok [29] introduced and studied the concept of almost $(\tau_1, \tau_2)\alpha$-continuous multifunctions. Moreover, some characterizations of almost $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions, almost weakly (τ_1, τ_2)-continuous multifunctions, almost (Λ, sp)-continuous multifunctions, almost $\beta(\star)$-continuous multifunctions and almost $\star$-continuous multifunctions were established in [5], [8], [10], [6] and [4] respectively. In this paper, we introduce the concepts of upper and lower almost (τ_1, τ_2)-continuous multifunctions. Furthermore, several characterizations of upper and lower almost (τ_1, τ_2)-continuous multifunctions are investigated.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [9] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [9] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [9] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [9] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ and $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2$-Cl$(A) \subseteq \tau_1\tau_2$-Cl$(B)$.

(3) $\tau_1\tau_2$-Cl(A) is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-Cl(A).

(5) $\tau_1\tau_2$-Cl$(X - A) = X - \tau_1\tau_2$-Int$(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [29] (resp. $(\tau_1, \tau_2)s$-*open* [5], $(\tau_1, \tau_2)p$-*open* [5], $(\tau_1, \tau_2)\beta$-*open* [5]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))))$. The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is called $(\tau_1, \tau_2)r$-*closed,* $(\tau_1, \tau_2)s$-*closed,* $(\tau_1, \tau_2)p$-*closed,* $(\tau_1, \tau_2)\beta$-*closed.* Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $(\tau_1, \tau_2)s$-closed sets of X containing A is called the $(\tau_1, \tau_2)s$-*closure* [5] of A and is denoted by (τ_1, τ_2)-sCl(A). The union of all $(\tau_1, \tau_2)s$-open sets of X contained in A is called the $(\tau_1, \tau_2)s$-*interior* [5] of A and is denoted by (τ_1, τ_2)-sInt(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$-*open* [30] if $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is called $\alpha(\tau_1, \tau_2)$-*closed.* Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $(\tau_1, \tau_2)p$-closed (resp. $\alpha(\tau_1, \tau_2)$-closed) sets of X containing A is called the $(\tau_1, \tau_2)p$-*closure* (resp. $\alpha(\tau_1, \tau_2)$-*closure*) and is denoted by (τ_1, τ_2)-pCl(A) (resp. (τ_1, τ_2)-αCl(A)).

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [2] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and

$$F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3. Upper and lower almost (τ_1, τ_2)-continuous multifunctions

In this section, we introduce the notions of upper and lower almost (τ_1, τ_2)-continuous multifunctions. Moreover, several characterizations of upper and lower almost (τ_1, τ_2)-continuous multifunctions are discussed.

Definition 1. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost (τ_1, τ_2)-continuous if F has this property at each point of X.*

Lemma 2. *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) (τ_1, τ_2)*-sCl*$(A) = \tau_1\tau_2$*-Int*$(\tau_1\tau_2$*-Cl*$(A)) \cup A$ [5]*;*

(2) (τ_1, τ_2)*-sInt*$(A) = \tau_1\tau_2$*-Cl*$(\tau_1\tau_2$*-Int*$(A)) \cap A$*.*

Lemma 3. *Let A be a subset of a bitopological space (X, τ_1, τ_2). If A is $\tau_1\tau_2$-open in X, then (τ_1, τ_2)-sCl$(A) = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$.*

Theorem 1. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost (τ_1, τ_2)-continuous at $x \in X$;

(2) $x \in \tau_1\tau_2$-Int$(F^+(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))))$ for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$;

(3) $x \in \tau_1\tau_2$-Int$(F^+((\sigma_1, \sigma_2)$-sCl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$;

(4) $x \in \tau_1\tau_2$-Int$(F^+(V))$ for every $(\sigma_1, \sigma_2)r$-open set V of Y containing $F(x)$;

(5) for each $(\sigma_1, \sigma_2)r$-open set V of Y containing $F(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq V$.

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. There exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$. Thus, $x \in U \subseteq F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$ and hence $x \in \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Int}(V))))$.

(2) ⇒ (3): This follows from Lemma 3.

(3) ⇒ (4): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y containing $F(x)$. Then, it follows from Lemma 3 that $V = \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)) = (\sigma_1, \sigma_2)\text{-sCl}(V)$.

(4) ⇒ (5): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y containing $F(x)$. Then by (4), $x \in \tau_1\tau_2\text{-Int}(F^+(V))$ and there exists a $\tau_1\tau_2$-open set U of X containing x such that $x \in U \subseteq F^+(V)$; hence $F(U) \subseteq V$.

(5) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Since $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ is $(\sigma_1, \sigma_2)r$-open, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$. This shows that F is upper almost (τ_1, τ_2)-continuous at $x \in X$.

Definition 2. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower almost (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that*

$$F(x) \cap V \neq \emptyset,$$

there exists a $\tau_1\tau_2$-open set U of X containing x such that $\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)) \cap F(z) \neq \emptyset$ for each $z \in U$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower almost (τ_1, τ_2)-continuous if F has this property at each point of X.

Theorem 2. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost (τ_1, τ_2)-continuous at $x \in X$;

(2) $x \in \tau_1\tau_2\text{-}Int(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y *such that* $F(x) \cap V \neq \emptyset$;

(3) $x \in \tau_1\tau_2\text{-}Int(F^-((\sigma_1, \sigma_2)\text{-}sCl(V)))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y *such that*

$$F(x) \cap V \neq \emptyset;$$

(4) $x \in \tau_1\tau_2\text{-}Int(F^-(V))$ *for every* $(\sigma_1, \sigma_2)r$*-open set* V *of* Y *such that* $F(x) \cap V \neq \emptyset$;

(5) for each $(\sigma_1, \sigma_2)r$*-open set* V *of* Y *such that* $F(x) \cap V \neq \emptyset$*, there exists a* $\tau_1\tau_2$*-open set* U *of* X *containing* x *such that* $U \subseteq F^-(V)$.

Proof. The proof is similar to that of Theorem 1.

Definition 3. [7] *A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is said to be almost* (τ_1, τ_2)*-continuous at a point* $x \in X$ *if for each* $\sigma_1\sigma_2$*-open set* V *of* Y *containing* $f(x)$*, there exists a* $\tau_1\tau_2$*-open set* U *of* X *containing* x *such that* $f(U) \subseteq \sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))$*. A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is said to be almost* (τ_1, τ_2)*-continuous if* f *has this property at each point of* X.

Corollary 1. *For a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) f *is almost* (τ_1, τ_2)*-continuous at* $x \in X$;

(2) $x \in \tau_1\tau_2\text{-}Int(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y *containing* $f(x)$;

(3) $x \in \tau_1\tau_2\text{-}Int(f^{-1}((\sigma_1, \sigma_2)\text{-}sCl(V)))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y *containing* $f(x)$;

(4) $x \in \tau_1\tau_2\text{-}Int(f^{-1}(V))$ *for every* $(\sigma_1, \sigma_2)r$*-open set* V *of* Y *containing* $f(x)$;

(5) for each $(\sigma_1, \sigma_2)r$*-open set* V *of* Y *containing* $f(x)$*, there exists a* $\tau_1\tau_2$*-open set* U *of* X *containing* x *such that* $f(U) \subseteq V$.

Theorem 3. *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is upper almost* (τ_1, τ_2)*-continuous;*

(2) $F^+(V) \subseteq \tau_1\tau_2\text{-}Int(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y;

(3) $\tau_1\tau_2\text{-}Cl(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K)))) \subseteq F^-(K)$ *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y;

(4) $\tau_1\tau_2\text{-}Cl(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B))))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* B *of* Y;

(5) $F^+(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \tau_1\tau_2\text{-}Int(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B)))))$ *for every subset* B *of* Y;

(6) $F^+(V)$ *is* $\tau_1\tau_2$*-open in* X *for every* $(\sigma_1, \sigma_2)r$*-open set* V *of* Y*;*

(7) $F^-(K)$ *is* $\tau_1\tau_2$*-closed in* X *for every* $(\sigma_1, \sigma_2)r$*-closed set* K *of* Y*.*

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$. Thus, by Theorem 1, $x \in \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$ and hence $F^+(V) \subseteq \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$.

(2) ⇒ (3): Let K be any $\sigma_1\sigma_2$-closed set of Y. Then, $Y - K$ is $\sigma_1\sigma_2$-open in Y and by (2),

$$\begin{aligned} X - F^-(K) &= F^+(Y - K) \\ &\subseteq \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - K)))) \\ &= \tau_1\tau_2\text{-Int}(X - F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))) \\ &= X - \tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))). \end{aligned}$$

Thus, $\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))) \subseteq F^-(K)$.

(3) ⇒ (4): Let B be any subset of Y. Then, $\sigma_1\sigma_2\text{-Cl}(B)$ is a $\sigma_1\sigma_2$-closed set of Y and by (3), $\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B))))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B))$.

(4) ⇒ (5): Let B be any subset of Y. Then, we have

$$\begin{aligned} F^+(\sigma_1\sigma_2\text{-Int}(B)) &= X - F^-(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &\subseteq X - \tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B))))) \\ &= X - \tau_1\tau_2\text{-Cl}(F^-(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))) \\ &= \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))). \end{aligned}$$

(5) ⇒ (6): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y. By (5), we have $F^+(V) \subseteq \tau_1\tau_2\text{-Int}(F^+(V))$ and hence $F^+(V)$ is $\tau_1\tau_2$-open in X.

(6) ⇒ (7): The proof is obvious.

(7) ⇒ (1): Let $x \in X$ and V be any $(\sigma_1, \sigma_2)r$-open set of Y containing $F(x)$. Since $Y - V$ is $(\sigma_1, \sigma_2)r$-closed and by (7), $X - F^+(V) = F^-(Y - V)$ is $\tau_1\tau_2$-closed in X. Thus, $F^+(V)$ is $\tau_1\tau_2$-open and hence $x \in \tau_1\tau_2\text{-Int}(F^+(V))$. Then, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq V$. It follows from Theorem 1 that F is upper almost (τ_1, τ_2)-continuous.

Theorem 4. *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is lower almost* (τ_1, τ_2)*-continuous;*

(2) $F^-(V) \subseteq \tau_1\tau_2\text{-}Int(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(3) $\tau_1\tau_2\text{-}Cl(F^+(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K)))) \subseteq F^+(K)$ *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(4) $\tau_1\tau_2\text{-}Cl(F^+(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B))))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* B *of* Y*;*

(5) $F^-(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \tau_1\tau_2\text{-}Int(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B)))))$ *for every subset* B *of* Y*;*

(6) $F^-(V)$ *is* $\tau_1\tau_2$*-open in* X *for every* $(\sigma_1, \sigma_2)r$*-open set* V *of* Y*;*

(7) $F^+(K)$ *is* $\tau_1\tau_2$*-closed in* X *for every* $(\sigma_1, \sigma_2)r$*-closed set* K *of* Y*.*

Proof. The proof is similar to that of Theorem 3.

Corollary 2. *For a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) f *is almost* (τ_1, τ_2)*-continuous;*

(2) $f^{-1}(V) \subseteq \tau_1\tau_2\text{-}Int(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(3) $\tau_1\tau_2\text{-}Cl(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K)))) \subseteq f^{-1}(K)$ *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(4) $\tau_1\tau_2\text{-}Cl(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B))))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* B *of* Y*;*

(5) $f^{-1}(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \tau_1\tau_2\text{-}Int(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B)))))$ *for every subset* B *of* Y*;*

(6) $f^{-1}(V)$ *is* $\tau_1\tau_2$*-open in* X *for every* $(\sigma_1, \sigma_2)r$*-open set* V *of* Y*;*

(7) $f^{-1}(K)$ *is* $\tau_1\tau_2$*-closed in* X *for every* $(\sigma_1, \sigma_2)r$*-closed set* K *of* Y*.*

Theorem 5. *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is upper almost* (τ_1, τ_2)*-continuous;*

(2) $\tau_1\tau_2\text{-}Cl(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1, \sigma_2)\beta$*-open set* V *of* Y*;*

(3) $\tau_1\tau_2\text{-}Cl(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1, \sigma_2)s$*-open set* V *of* Y*.*

Proof. (1) ⇒ (2): Let V be any $(\sigma_1, \sigma_2)\beta$-open set of Y. Then, $\sigma_1\sigma_2$-Cl(V) is a $(\sigma_1, \sigma_2)r$-closed set of Y. Since F is upper almost (τ_1, τ_2)-continuous and by Theorem 3, $F^-(\sigma_1\sigma_2\text{-Cl}(V))$ is $\tau_1\tau_2$-closed in X. Thus, $\tau_1\tau_2\text{-Cl}(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$.

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (1): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then, K is $(\sigma_1, \sigma_2)s$-open in Y. Then by (3), $\tau_1\tau_2\text{-Cl}(F^-(K)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(K)) = F^-(K)$ and hence $F^-(K)$ is $\tau_1\tau_2$-closed in X. By Theorem 3, F is upper almost (τ_1, τ_2)-continuous.

Theorem 6. *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is lower almost* (τ_1, τ_2)*-continuous;*

(2) $\tau_1\tau_2\text{-}Cl(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1, \sigma_2)\beta$*-open set* V *of* Y*;*

(3) $\tau_1\tau_2\text{-}Cl(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1, \sigma_2)s$*-open set* V *of* Y*.*

Proof. The proof is similar to that of Theorem 5.

Corollary 3. *For a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) f *is almost* (τ_1, τ_2)*-continuous;*

(2) $\tau_1\tau_2\text{-}Cl(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1, \sigma_2)\beta$*-open set* V *of* Y*;*

(3) $\tau_1\tau_2\text{-}Cl(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1, \sigma_2)s$*-open set* V *of* Y*.*

Lemma 4. *For a bitopological space* (X, τ_1, τ_2)*, the following properties hold:*

(1) $(\tau_1, \tau_2)\text{-}\alpha Cl(V) = \tau_1\tau_2\text{-}Cl(V)$ *for every* $(\tau_1, \tau_2)\beta$*-open set* V *of* Y*;*

(2) $(\tau_1, \tau_2)\text{-}pCl(V) = \tau_1\tau_2\text{-}Cl(V)$ *for every* $(\tau_1, \tau_2)s$*-open set* V *of* Y*.*

Corollary 4. *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is upper almost* (τ_1, τ_2)*-continuous;*

(2) $\tau_1\tau_2\text{-}Cl(F^-(V)) \subseteq F^-((\sigma_1, \sigma_2)\text{-}\alpha Cl(V))$ *for every* $(\sigma_1, \sigma_2)\beta$*-open set* V *of* Y*;*

(3) $\tau_1\tau_2\text{-}Cl(F^-(V)) \subseteq F^-((\sigma_1, \sigma_2)\text{-}pCl(V))$ *for every* $(\sigma_1, \sigma_2)s$*-open set* V *of* Y*.*

Corollary 5. *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is lower almost* (τ_1, τ_2)*-continuous;*

(2) $\tau_1\tau_2\text{-}Cl(F^+(V)) \subseteq F^+((\sigma_1, \sigma_2)\text{-}\alpha Cl(V))$ *for every* $(\sigma_1, \sigma_2)\beta$*-open set* V *of* Y*;*

(3) $\tau_1\tau_2\text{-}Cl(F^+(V)) \subseteq F^+((\sigma_1, \sigma_2)\text{-}pCl(V))$ *for every* $(\sigma_1, \sigma_2)s$*-open set* V *of* Y*.*

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