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Convex Roman Dominating Functions on Graphs under some Binary Operations

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Abstract. Let G be a connected graph. A function $f : V(G) \to \{0, 1, 2\}$ is a *convex Roman dominating function* (or CvRDF) if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$ and $V_1 \cup V_2$ is convex. The weight of a convex Roman dominating function f, denoted by $\omega_G^{CvR}(f)$, is given by $\omega_G^{CvR}(f) = \sum_{v\in V(G)} f(v)$. The minimum weight of a CvRDF on G, denoted by $\gamma_{CvR}(G)$, is called the *convex Roman domination number* of G. In this paper, we specifically study the concept of convex Roman domination in the corona and edge corona of graphs, complementary prism, lexicographic product, and Cartesian product of graphs.

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Key Words and Phrases: Convex set, Roman dominating function, Roman domination number, convex Roman dominating function, convex Roman domination number, corona, edge corona, complementary prism, lexicographic product, Cartesian product

1. Introduction

Roman domination was first introduced by Cockayne, Dreyer and Hedetnieme in [8] which was inspired by the defense strategy of the Roman emperor Constantine the Great during the 4th Century AD (see [23] and [24]). After several years, lots of variations on this concept have been introduced and studied (see [1], [2], [3], [4], [5], [7], [13], [16], [19], [20], and [22]).

The concept of convex domination in graphs was first introduced by Lemanska in 2004 [18]. Convex domination is a concept in graph theory that combines the notions of convexity and domination. Studies related on convexity and dominaton in graphs can be found in [14], [15], [6], [9], [11], [12], and [21]. A subset of vertices in a graph is said to be a convex dominating set if it is both a convex set and a dominating set, which means

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that every vertex on the shortest path between any two vertices in the set is also in the set and every vertex in the graph is either in the set or adjacent to a vertex in the set.

The convex Roman domination was introduced and initially investigated in [10] where properties of convex Roman dominating functions and convex Roman domination number of some graphs and the join of two graphs have been obtained.

In this present paper, authors continued the study of convex Roman domination, specifically on the corona, edge corona, complementary prism, lexicographic product, and Cartesian product of graphs.

Let G be a connected graph. For vertices u and v in G, a u-v geodesic is any shortest path in G joining u and v. The length of a u-v geodesic is called the *distance* $d_G(u,v)$ between u and v. For every two vertices u and v of G, the symbol $I_G[u,v]$ is used to denote the set of vertices lying on any of the u-v geodesics.

The set of neighbors of a vertex $u \in G$, denoted by $N_G(u)$, is called the *open neighborhood* of u. The *closed neighborhood* of u is the set $N_G[u] = N_G(u) \cup \{u\}$. The *degree* of a vertex v denoted $deg_G(v)$ in a graph G is the number of vertices in G that are adjacent to v. Hence, $deg_G(v) = |N(v)|$. The largest degree among the vertices of G is called the *maximum degree* of G and is denoted by $\triangle(G)$. The minimum degree of G is denoted by $\delta(G)$. A graph G is *connected* if every pair of its vertices can be joined by a path.

A set $S \subseteq V(G)$ is said to be a *dominating set* of a graph G if every vertex $v \in V(G)$ is either an element of S or is adjacent to an element of S. Thus, $N[S] = V(G)$. The smallest cardinality of a dominating set S is called the *domination number* of G and is denoted by $\gamma(G)$. That is $\gamma(G) = \min\{|S| : S \text{ is a dominating set of } G\}$. Any dominating set S of G with $|S| = \gamma(G)$ is called a γ*-set* of G.

If S is a clique (the induced graph $\langle S\rangle$ is complete) and a dominating set, then S is called a *clique dominating set* in G. A *clique domination number* $\gamma_{cl}(G)$ of G is the smallest cardinality of a clique dominating set in G.

A set $S \subseteq V(G)$ is *convex* if for every two vertices $x, y \in S$, $I_G[x,y] \subseteq S$. The largest cardinality of a proper convex set in G, denoted by $con(G)$, is called the *convexity number* of G. A set $S \subseteq V(G)$ is *convex dominating* if S is both convex and dominating. The minimum cardinality among all convex dominating sets in G, denoted by $\gamma_{con}(G)$ is called the *convex domination number* of G. Any convex dominating set S of G with $|S| = \gamma_{con}(G)$ is called a γ_{con}-set of G.

A function $f : V(G) \to \{0,1,2\}$ is a *Roman dominating function* (or just RDF) if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight* of an RDF f is given by $\omega_G(f) = \sum_{v\in V(G)} f(v)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of an RDF on G. Any RDF f on G with $\omega_G(f) = \gamma_R(G)$ is called a γ_R-function. If f is an RDF on G and $V_i = \{v \in V(G) : f(v) = i\}$ for $i \in \{0,1,2\}$, then we denote f by $f = (V_0, V_1, V_2)$. In this case, $\omega_G(f) = |V_1| + 2|V_2|$.

A Roman dominating function $f = (V_0, V_1, V_2)$ on G is a *convex Roman dominating function* (or CvRDF) if $V_1 \cup V_2$ is convex. The weight of a convex Roman dominating function $f = (V_0, V_1, V_2)$ on G is given by $\omega_G^{CvR}(f) = |V_1| + 2|V_2|$. The minimum weight of a CvRDF on G, denoted by $\gamma_{CvR}(G)$, is called the *convex Roman domination number* of G. Any CvRDF f on G with $\omega_G^{CvR}(f) = \gamma_{CvR}(G)$ is called a γ_{CvR}*-function.*

2. Known Results

The following results are useful in this study.

Proposition 1. [10] *Let n be a positive integer. Then*

$$\gamma_{CvR}(P_n) = \begin{cases} 1, & n = 1 \\ 2, & n = 2, 3 \\ n, & n \geq 4. \end{cases}$$

Proposition 2. [10] Let G be a non-trivial connected graph and let $f = (V_0, V_1, V_2)$ be a γ_{CvR}-function on G. Then the following hold:

(i) If $|V_0| = 0$, then $|V_2| = 0$.

(ii) If $|V_0| = 1$, then $|V_2| = 1$.

(iii) $|V_1| = 0$ if and only if V_2 is a γ_{con}-set in G

Theorem 1. [15] *Let G be a connected graph and K_m the complete graph of order m. Then a proper subset $C = S_1 \cup S_2$ of $V(G + K_m)$, where $S_1 \subseteq V(G)$ and $S_2 \subseteq V(K_m)$, is convex in $G + K_m$ if and only if either*

(i) *S_1 induces a complete subgraph of G, or*

(ii) *$S_1 = V(G) \setminus S$ and $S_2 = V(K_m)$ for some non-connecting set S in G.*

Theorem 2. [14] *Let G be a connected graph and K_m the complete graph of order $n \geq 2$. A subset $C = \bigcup_{x \in S}(\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$, is a convex set in $G[K_m]$ if and only if S is a convex set in G and $T_x = V(K_m)$ for each $x \in S^0 = I(S) \cap S$.*

Theorem 3. [15] *Let G and H be connected non-complete graphs. A subset $C = \bigcup_{x \in S}(\{x\} \times T_x)$ of $V(G[H])$ is a convex set in $G[H]$ if and only if S is a clique set in G and T_x is a clique in H for each $x \in S$.*

Theorem 4. [17] *Let G and H be connected non-complete graphs. A subset $C = \bigcup_{x \in S}(\{x\} \times T_x)$ of $V(G[H])$ is a convex dominating set in $G[H]$ if and only if S is a clique dominating set in G and T_x is a clique in H for each $x \in S$.*

Theorem 5. [15] *Let G and H be two connected graphs. A set $C \in V(G \Box H)$ is a convex set in $G \Box H$ if and only if $C = C_G \times C_H$, where C_G and C_H are convex sets in G and H respectively.*

Theorem 6. [17] *Let G and H be connected graphs. A subset C of $V(G \Box H)$ is a convex dominating set in $G \Box H$ if and only if $C = C_1 \times C_2$ and one of the following conditions holds:*

(*i*) *C_1 is a convex dominating set in G and $C_2 = V(H)$, or*

(*ii*) *C_2 is a convex dominating set in H and $C_1 = V(G)$.*

Theorem 7. [10] Let G be a connected graph on n vertices. Then each of the the following statements holds.
(*i*) $\gamma_{CvR}(G) = 1$ if and only if $G = K_1$
(*ii*) $\gamma_{CvR}(G) = 2$ if and only if $G = K_2$ or $G = K_1 + H$ for some graph H

Corollary 1. [10] For any connected graph G of order n, $\gamma_{CvR}(G) = 2$ if and only if $G \neq K_1$ and $\gamma(G) = 1$.

Proposition 3. [10] There exists no connected graph G with $\gamma_{CvR}(G) = 3$.

Proposition 4. [10] For any connected graph G of order n,

$$1 \leq \gamma_{con}(G) \leq \gamma_{CvR}(G) \leq \min\{n, 2\gamma_{con}(G)\}.$$

3. Results

Let G and H be connected graphs. The *corona* of G and H is the graph $G \circ H$ obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the *ith* vertex of G to every vertices of the *ith* copy of H. For convenience, we write H^v to denote the copy of H joined to v and write $H^v + v = H^v + \langle v \rangle$.

Let G be a graph. A non-empty subset S of $V(G)$ is a *non-connecting set* in G if it satisfies the following condition: For every pair of vertices $u, v \in V(G) \setminus S$ with $d_G(u, v) = 2$, we have $N_G(u) \cap N_G(v) \cap S = \varnothing$. A non-connecting set with minimum cardinality is called a *minimum non-connecting set.*

Let $f = (V_0, V_1, V_2)$ be a CvRDF on $G \circ H$. For each $v \in V(G)$, let $S_k^v = V_k \cap V(H^v)$ where $k = 0, 1, 2$.

Theorem 8. *Let G be a non-trivial connected graph and let H be any graphs. Then $f = (V_0, V_1, V_2)$ is a CvRDF on $G \circ H$ if and only if each of the following conditions hold:*

(*i*) $V_0 \cap V(G) = \varnothing$

(*ii*) *For each $v \in V_2 \cap V(G)$, $S_1^v \cup S_2^v$ induces a complete subgraph of H^v or $V(H^v) \setminus (S_1^v \cup S_2^v)$ is a non-connecting set in H^v.*

(*iii*) *For each $v \in V_1 \cap V(G)$ such that $S_1^v \neq V(H^v)$, $S_2^v \neq \varnothing$, $S_0^v \subseteq N_{H^v}(S_2^v)$, and $S_1^v \cup S_2^v$ induces a complete subgraph of H^v or $V(H^v) \setminus (S_1^v \cup S_2^v)$ is a non-connecting set in H^v and*

Proof. Let $f = (V_0, V_1, V_2)$ be a CvRDF of $G \circ H$. Since $V_1 \cup V_2$ is a dominating set of $G \circ H$,

$$V(v + H^v) \cap (V_1 \cup V_2) \neq \varnothing \tag{1}$$

for each $v \in V(G)$. Suppose that there exists $v \in V_0 \cap V(G)$. Then (1) implies that $S_1^v \cup S_2^v \neq \varnothing$. Pick $w \in V(G) \setminus \{v\}$. If $w \in V_1 \cup V_2$, then the convexity of $V_1 \cup V_2$ implies that $v \in I_{G \circ H}[z, w] \subseteq V_1 \cup V_2$ for all $z \in S_1^v \cup S_2^v$. Suppose that $w \in V_0$. Then by (1), $S_1^w \cup S_2^w \neq \varnothing$, and $w, v \in I_{G \circ H}[a, b] \subseteq V_1 \cup V_2$, for all $a \in S_1^v \cup S_2^v$ and $b \in S_1^w \cup S_2^w$. In any case, we get a contradiction. Thus, $V_0 \cap V(G) = \varnothing$, showing that (i) holds.

Next, let $v \in V_2 \cap V(G)$. By Theorem 1, convexity of $V_1 \cup V_2$ implies that $S_1^v \cup S_2^v$ induces a complete subgraph of H^v or $V(H^v) \setminus (S_1^v \cup S_2^v)$ is a non-connecting set in H^v. Hence, (ii) holds. Suppose $v \in V_1 \cap V(G)$ such that $S_1^v \neq V(H^v)$. Then $S_2^v \neq \varnothing$ and $S_0^v \subseteq N_{H^v}(S_2^v)$ because f is an RDF on $G \circ H$. Again, by Theorem 1, $S_1^v \cup S_2^v$ induces a complete subgraph of H^v or $V(H^v) \setminus (S_1^v \cup S_2^v)$ is a non-connecting set in H^v, showing that (iii) holds.

Conversely, suppose that (i), (ii) and (iii) hold. Let $x \in V_0$. By (i), $x \in S_0^v$ for some $v \in V(G)$. If $v \in V_2$, then $x \in N_{G \circ H}(v)$. Suppose $v \in V_1$. By (iii), $S_2^v \neq \varnothing$ and $x \in N_{G \circ H}(S_2^v)$. Thus $f = (V_0, V_1, V_2)$ is an RDF on $G \circ H$. Now, let $p, q \in V_1 \cup V_2$ and let $v, w \in V(G)$ such that $p \in V(v + H^v)$ and $q \in V(w + H^w)$. Consider the following cases.
Case 1. $v = w$.
If $p = v$ or $q = v$, then $I_{G \circ H}[p, q] = \{p, q\} \subseteq V_1 \cup V_2$. Suppose $p, q \in V(H^v)$. If $d_{H^v}(p, q) = 1$, then $I_{G \circ H}[p, q] = \{p, q\} \subseteq V_1 \cup V_2$. If $d_{H^v}(p, q) = 2$, then

$$I_{G \circ H}[p, q] = \{p, q, v\} \cup (N_{H^v}(p) \cap N_{H^v}(q)) \subseteq S_1^v \cup S_2^v \cup \{v\} \subseteq V_1 \cup V_2$$

since $V(H^v) \setminus (S_1^v \cup S_2^v)$ is a non-connecting set in H^v (by (ii) and (iii)). If $d_{H^v}(p, q) > 2$, then $I_{G \circ H}[p, q] = \{p, q, v\} \subseteq V_1 \cup V_2$.

Case 2. $v \neq w$.
Consider the following subcases.
Subcase 1. $p = v$ and $q = w$.
Then $V(G) \subseteq V_1 \cup V_2$ by (i). Since every p-q geodesic in $G \circ H$ is a p-q geodesic in G, it follows that $I_{G \circ H}[p, q] = I_G[p, q] \subseteq (V_1 \cup V_2)$.

Subcase 2. $p = v$ and $q \in V(H^w)$ (or $q = w$ and $p \in V(H^v)$).
Then $I_{G \circ H}[p, q] = I_G[v, w] \cup \{q\} \subseteq (V_1 \cup V_2)$.

Subcase 3. $p \in V(H^v)$ and $q \in V(H^w)$.
Then $I_{G \circ H}[p, q] = I_G[v, w] \cup \{p, q\} \subseteq V_1 \cup V_2$.

Therefore, $V_1 \cup V_2$ is a convex set in $G \circ H$. Accordingly, f is a CvRDF on $G \circ H$. □

Corollary 2. *Let G be a non-trivial connected graph of order n and let H be any graph. Then*

$$\gamma_{CvR}(G \circ H) = 2n.$$

Proof. Let $V_2' = V(G)$, $V_0' = \bigcup_{v \in V(G)} V(H^v)$, and $V_1' = \varnothing$. By Theorem 8, $g = (V_0', V_1', V_2')$ is a CvRDF on $G \circ H$. Thus, $\gamma_{CvR}(G \circ H) \leq \omega_{G \circ H}^{CvR}(g) = |V_1'| + 2|V_2'| = 2n$.

Next, let $f = (V_0, V_1, V_2)$ be a γ_{CvR}-function on $G \circ H$. Let $V_1^* = V_1 \cap V(G)$ and $V_2^* = V_2 \cap V(G)$ and let $|V_1^*| = k$. Then $|V_2^*| = n - k$. Let $D = \{v \in V_1^* : S_1^v \neq \varnothing\}$. Then $S_2^v \neq \varnothing$ for all $v \in V_1 \setminus D$ by (iii). Hence,

$$\gamma_{CvR}(G \circ H) = \omega_{G \circ H}^{CvR}(f)$$

$$
\begin{aligned}
&= |V_1| + 2|V_2| \\
&= k + \sum_{v\in V(G)} |S_1^v| + 2\left[(n-k) + \sum_{v\in V(G)} |S_2^v|\right] \\
&= 2n - k + \sum_{v\in V(G)} |S_1^v| + 2\sum_{v\in V(G)} |S_2^v| \\
&\geq 2n - k + \sum_{v\in D} |S_1^v| + 2\sum_{v\in V_1\setminus D} |S_2^v| \\
&\geq 2n - k + |D| + 2|V_1| - 2|D| \\
&= 2n + k - |D| \\
&\geq 2n.
\end{aligned}
$$

This proves the desired equality. □

Given graphs G and H we write H^{uv} to denote that copy of H that is being joined with the end vertices of the edge $uv \in E(G)$ in the *edge corona* $G \diamond H$. If $H = \{x\}$, then we write $V(H^{uv}) = \{x^{uv}\}$.

Recall that for subsets A and B of $V(G)$, we have $d_G(A,B) = \min\{d_G(a,b) : a \in A \text{ and } b \in B\}$.

Let $f = (V_0, V_1, V_2)$ be a CvRDF on $G \diamond H$. For each $v \in V(G)$, let $S_1^{uv} = V_1 \cap V(H^{uv})$ and $S_2^{uv} = V_2 \cap V(H^{uv})$. Denote $V_G^0 = V(G) \cap V_0$, $V_G^1 = V(G) \cap V_1$, and $V_G^2 = V(G) \cap V_2$.

Note that since $\gamma(K_2 \diamond H) = 1$ for any graph H, it follows from Corollary 1 that $\gamma_{CvR}(K_2 \diamond H) = 2$.

Theorem 9. *Let G be a non-trivial connected graph such that $G \neq K_2$ and let H be any graph. Then $f = (V_0, V_1, V_2)$ is a CvRDF on $G \diamond H$ if and only if each of the following conditions holds.*

(i) $|\{u,v\} \cap (V_G^1 \cup V_G^2)| \neq 0$ *for each* $uv \in E(G)$.

(ii) *For each pair of distinct edges uv and zw of G, $I_G(x,y) \subseteq V_G^1 \cup V_G^2$ whenever $x \in \{u,v\} \cap (V_G^1 \cup V_G^2)$ and $y \in \{z,w\} \cap (V_G^1 \cup V_G^2)$.*

(iii) *For every pair of distinct edges e and e' of G with $S_1^e \cup S_2^e \neq \varnothing$ and $S_1^{e'} \cup S_2^{e'} \neq \varnothing$, $v, z \in V_G^1 \cup V_G^2$ whenever v and z are incident with e and e', respectively, with $d_G(v,z) = d_G(\{u,v\},\{z,w\})$.*

(iv) *For each $uv \in E(G)$ such that $\{u,v\} \cap (V_1 \cup V_2) = \{v\}$, it holds that*

  (a) *$S_1^{uv} \cup S_2^{uv}$ a clique in H^{uv} whenever $S_1^{uv} \cup S_2^{uv} \neq \varnothing$ and*

  (b) *$v \in V_G^2$ or $S_2^{uv} \neq \varnothing$ and $S_0^{uv} \subseteq N_{H^{uv}}(S_2^{uv})$.*

(v) *For each $uv \in E(G)$ such that $\{u,v\} \subseteq (V_1 \cup V_2)$ and $S_1^{uv} \cup S_2^{uv} \neq V(H^{uv})$, it holds that*

(c) $V(H^{uv}) \setminus (S_1^{uv} \cup S_2^{uv})$ *is a non-connecting set in* H^{uv} *and*

(d) $\{u, v\} \cap V_G^2 \neq \varnothing$ *or* $S_2^{uv} \neq \varnothing$ *and* $S_0^{uv} \subseteq N_{H^{uv}}(S_2^{uv})$.

Proof. Let $u, v \in V(G)$ such that $uv \in E(G)$. Then $|\{u, v\} \cap (V_G^1 \cup V_G^2)| \neq 0$ because $V_1 \cup V_2$ is convex in $G \diamond H$ and $G \neq K_2$. Thus, (i) holds.

Let uv and zw be distinct edges of G and let $x \in \{u, v\} \cap (V_G^1 \cup V_G^2)$ and $y \in \{z, w\} \cap (V_G^1 \cup V_G^2)$. Since $V_1 \cup V_2$ is convex in $G \diamond H$, it follows that $I_G(x, y) = I_{G \diamond H}(x, y) \subseteq V_1 \cup V_2$. Therefore, $I_G(x, y) \subseteq V_G^1 \cup V_G^2$, showing that (ii) holds.

Let e and e' be distinct edges of G with $S_1^e \cup S_2^e \neq \varnothing$ and $S_1^{e'} \cup S_2^{e'} \neq \varnothing$ and suppose that v and z are incident with e and e', respectively, with $d_G(v, z) = d_G(\{u, v\}, \{z, w\})$. Let $x \in S_1^e \cup S_2^e$ and $y \in S_1^{e'} \cup S_2^{e'}$. Then $I_G[v, z] \subseteq I_{G \diamond H}[x, y]$. Since $V_1 \cup V_2$ is convex, $I_{G \diamond H}[x, y] \subseteq V_1 \cup V_2$. Hence, $v, z \in V_G^1 \cup V_G^2$. This shows that (iii) holds.

Now, let $uv \in E(G)$ such that $\{u, v\} \cap (V_1 \cup V_2) = \{v\}$. Since $V_1 \cup V_2$ is convex, (a) holds. Also, (b) holds since f is a CvRDF on $G \diamond H$. Thus, (iv) holds.

Next, let $uv \in E(G)$ such that $\{u, v\} \subseteq (V_1 \cup V_2)$ and $S_1^{uv} \cup S_2^{uv} \neq V(H^{uv})$. By Theorem 1, (c) holds. Since f is CvRDF on $G \diamond H$, (d) also holds. Hence, (v) holds.

Conversely, assume that (i), (ii), (iii), and (iv) hold. Let $x \in V_0$ and let $uv \in E(G)$ such that $x \in V(\{u, v\} + H^{uv})$. Consider the following cases:

Case 1. Suppose that $x \in V_G^0$, say $x = u$.
Suppose $v \notin V_G^2$. Then $S_2^{uv} \neq \varnothing$ by $(iv)(b)$. Let $p \in S_2^{uv}$. Then $p \in V_2 \cap N_{G \diamond H}(u)$.

Case 2. Suppose that $x \in S_0^{uv}$.
If $u \in V_2$ or $v \in V_2$, then $xu \in E(G \diamond H)$ or $xv \in E(G \diamond H)$. Suppose $u, v \notin V_2$. Then, by $(iv)(b)$, there exists $q \in S_2^{uv} \cap N_{H^{uv}}(x)$. Hence, $q \in V_2 \cap N_{G \diamond H}(x)$.

Thus, by Case 1 and Case 2, $f = (V_0, V_1, V_2)$ is an RDF on $G \diamond H$.

Next, let $x, y \in V_1 \cup V_2$ $(x \neq y)$ and $uv, zw \in E(G)$ such that $x \in V(\{u, v\} + H^{uv})$ and $y \in V(\{z, w\} + H^{zw})$. Consider the following cases:

Case 1. $uv = zw$.
If $x = u$ and $y = v$, then $I_{G \diamond H}[x, y] = \{x, y\} \subseteq V_1 \cup V_2$. Suppose that $x, y \in S_1^{uv} \cup S_2^{uv}$. If $|\{u, v\} \cap (V_1 \cup V_2)| = 1$, then $S_1^{uv} \cup S_2^{uv}$ induces a complete subgraph of H^{uv} by $(iv)(a)$. Therefore, $I_{G \diamond H}[x, y] = \{x, y\} \subseteq V_1 \cup V_2$. Suppose that $|\{u, v\} \cap (V_1 \cup V_2)| = 2$. Clearly, $I_{G \diamond H}[x, y] \subseteq V_1 \cup V_2$ if $S_1^{uv} \cup S_2^{uv} = V(H^{uv})$. Suppose $S_1^{uv} \cup S_2^{uv} \neq V(H^{uv})$. Then $V(H^{uv}) \setminus S_1^{uv} \cup S_2^{uv}$ is a non-connecting set in H^{uv} by $(v)(c)$. Hence, $N_{H^{uv}}(x) \cap N_{H^{uv}}(y) \subseteq S_1^{uv} \cup S_2^{uv}$. Therefore, $I_{G \diamond H}[x, y] \subseteq V_1 \cup V_2$.

Case 2. $uv \neq zw$.
Suppose that $x \in \{u, v\}$ and $y \in \{z, w\}$. By (ii), $I_{G \diamond H}[x, y] = I_G[x, y] \subseteq V_G^1 \cup V_G^2$. Suppose that $x \in S_1^{uv} \cup S_2^{uv}$ and $y \in S_1^{zw} \cup S_2^{zw}$. Suppose uv and zw are adjacent, say $v = z$. Then $v \in V_G^1 \cup V_G^2$ by (iii). Hence, $I_{G \diamond H}[x, y] = \{x, v, y\} \subseteq V_1 \cup V_2$. Next, suppose that uv and zw are non-adjacent. Let a and b be incident with $e = uv$ and $e' = zw$, respectively, with $d_G(a, b) = d_G(\{u, v\}, \{z, w\})$. Then $a, b \in V_G^1 \cup V_G^2$, by (iii). Moreover, $I_G[a, b] \subseteq V_G^1 \cup V_G^2$, by (ii). Therefore, $I_{G \diamond H}(x, y) = I_G[a, b] \subseteq V_1 \cup V_2$. Finally, suppose that $x \in S_1^{uv} \cup S_2^{uv}$ and $y \in \{z, w\}$. Suppose that a and b are the vertices described earlier. If $b = y$, then

$I_{G\diamond H}(x,y) = I_G(a,b) \cup \{a\} \subseteq V_1 \cup V_2$, by (ii). Again, by (ii), $I_{G\diamond H}(x,y) = I_G[a,b] \subseteq V_1 \cup V_2$ if $b \neq y$.

Therefore, $V_1 \cup V_2$ is convex in $G \diamond H$. Accordingly, f is a CvRDF on $G \diamond H$. □

Lemma 1. *Let G be a non-complete connected graph and H be any graph of order n. If $W_0 = Ext(G)$, $W_1 \cup W_2 = V(G) \setminus Ext(G)$ and $\{u,v\} \cap W_2 \neq \varnothing$ for each $uv \in E(G)$ such that $|\{u,v\} \cap Ext(G)| \neq 2$, then $f|_G = (W_0, W_1, W_2)$ is a CvRDF on G.*

Proof. Let $x \in W_0$. Since G is non-complete, there exists $y \in (V(G) \setminus Ext(G)) \cap N_G(x)$. By assumption, this implies that $y \in W_2$, showing that f is an RDF on G. Moreover, since $V(G) \setminus Ext(G)$ is convex in G, it follows that f is a CvRDF on G. □

Henceforth, we refer f in Lemma 1 as a CvRDF* on G.

Corollary 3. *Let G be a non-complete connected graph and H any graph of order n. Then*

$$\gamma_{CvR}(G \diamond H) \leq \min\{\omega_G^{CvR}(f) : f = (W_0, W_1, W_2) \text{ is a } CvRDF^* \text{ on } G\}.$$

Proof. Let $k = \min\{\omega_G^{CvR}(f) : f = (W_0, W_1, W_2) \text{ is a } CvRDF^* \text{ on } G\}$. Let $g = (W_0, W_1, W_2)$ be a CvRDF* on G such that $\omega_G^{CvR}(g) = k$. Let $V_0 = Ext(G) \cup (\bigcup_{e \in E(G)} V(H^e))$, $V_1 = W_1$, $V_2 = W_2$, and let $h = (V_0, V_1, V_2)$. Clearly, $h|_G = g$. Hence, h satisfies (i). Also, (ii), (iii) and (iv) of Theorem 9 hold. Thus, by Theorem 9, h is a CvRDF on $G \diamond H$. Moreover,

$$\gamma_{CvR}(G \diamond H) \leq \omega_{G\diamond H}(h) = |V_1| + 2|V_2| = |W_1| + 2|W_2| = \gamma^*_{CvR}(G). \quad \square$$

Remark 1. *The bound given in Corollary 3 is sharp. It can be verified that for any graph H and positive integer $n \geq 3$, the following holds:*

$$\begin{aligned} \gamma_{CvR}(P_n \diamond H) &= \min\{\omega_{P_n}^{CvR}(f) : f \text{ is a } CvRDF^* \text{ on } P_n\} \\ &= \begin{cases} \frac{3n-4}{2}, & \text{if } n \text{ is even} \\ \frac{3n-5}{2}, & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

For a graph G, the *complementary prism*, denoted $G\overline{G}$, is formed from the disjoint union of G and its complement $\overline{G}$ by adding a perfect matching between corresponding vertices of G and $\overline{G}$. For each $v \in V(G)$, let $\overline{v}$ denote the vertex corresponding to v in $\overline{G}$. Formally, the graph $G\overline{G}$ is formed from $G \cup G$ by adding the edge $v\overline{v}$ for every $v \in V(G)$.

Proposition 5. *Let G be a graph on n vertices. Then each of the following holds.*
(i) $\gamma_{CvR}(G\overline{G}) = 2$ if and only if $G = K_1$.
(i) $\gamma_{CvR}(G\overline{G}) = 4$ if and only if $G = K_2$ or $\overline{K}_2$.

Proof. (i) Suppose that $G = K_1$. Then $G\overline{G} = K_2$. By Theorem 7, we are done.

(ii) Suppose that $G = K_2$ or $G = \overline{K}_2$. Then $G = P_4$ and by Theorem 1, $\gamma_{CvR}(G\overline{G}) = \gamma_{CvR}(P_4) = 4$. Conversely, suppose that $\gamma_{CvR}(G\overline{G}) = 4$. Let $f = (V_0, V_1, V_2)$ be a γ_{CvR}-function. Then $\omega_{G\overline{G}}^{CvR}(f) = |V_1| + 2|V_2| = 4$. Then $|V_2| \leq 2$. If

$|V_2| = 0$, then $V_1 = V(G\overline{G})$. Hence, $G\overline{G} = P_4$. This implies that $G \in \{K_2, \overline{K}_2\}$. Suppose that $|V_2| = 1$, say $V_2 = \{u\}$. Then $|V_1| = 2$. WLOG, asssume that $u \in V(G)$. Suppose $|V(G)| \geq 3$ and let $v, w \in V(G) \setminus \{u\}$. Since $\overline{v}, \overline{w} \notin N_{G\overline{G}}(u)$, $V_1 = \{\overline{v}, \overline{w}\}$. Because $V_1 \cup V_2$ is convex, one of the following holds:

$$\begin{aligned} \overline{u} \in I_{G\overline{G}}[u, z] &\subseteq V_1 \cup V_2, \text{ where } z \in \{\overline{v}, \overline{w}\}, \\ v \in I_{G\overline{G}}[u, z] &\subseteq V_1 \cup V_2, \text{ where } z \in \{\overline{v}, \overline{w}\}, \\ w \in I_{G\overline{G}}[u, z] &\subseteq V_1 \cup V_2, \text{ where } z \in \{\overline{v}, \overline{w}\}. \end{aligned}$$

In any case, we have a contradiction. Thus, $|V(G)| \leq 2$. But by statement (i), $|V(G)| = 2$. This means that $G \in \{K_2, \overline{K_2}\}$. If $|V_2| = 2$, then $|V_1| = 0$. By Proposition 2, V_2 is a γ_{con}-set in $G\overline{G}$. Let $V_2 = \{x, y\}$. WLOG, assume that $x \in V(G)$. If $\overline{x} = y$, then $N_G[x] = V(G)$ and $N_{\overline{G}}[y] = V(\overline{G})$. This is possible only if $G = K_1$, a contradiction. Thus $y \in V(G)$ and $xy \in E(G)$. Now, for each $z \in V_0 = V(G) \setminus \{x, y\}$, $x\overline{z} \notin E(G\overline{G})$ and $y\overline{z} \notin E(G\overline{G})$. Hence, $V(G) \setminus \{x, y\} = \varnothing$. Therefore, $G \in \{K_2, \overline{K}_2\}$. □

Proposition 6. *For any connected graph G of order n,*

$$2 \leq \gamma_{CvR}(G\overline{G}) \leq 2\min\{\gamma_{con}(G\overline{G}), n\}.$$

In particular,

(i) *$\gamma_{CvR}(G\overline{G}) = 2\gamma_{con}(G\overline{G})$ if there exists a γ_{CvR}-function $f = (V_0, V_1, V_2)$ with $V_1 = \varnothing$.*

(ii) *If $G = K_n$, then $\gamma_{CvR}(G\overline{G}) = 2n$.*

Proof. By Proposition 4, $\gamma_{CvR}(G\overline{G}) \leq 2\gamma_{con}(G\overline{G})$. Define $f = (V_0, V_1, V_2)$, where $V_2 = V(G)$, $V_1 = \varnothing$, and $V_0 = V(\overline{G})$. Then f is a CvRDF of $G\overline{G}$. Thus $\gamma_{CvR}(G\overline{G}) \leq 2|V_2| = 2n$. □

It is worth noting that if $G = P_4$, $\gamma_{con}(G\overline{G}) = 8 > 4$. If $G = P_5$, $\gamma_{con}(G\overline{G}) = 4 < 5$.

The *lexicographic product* of two graphs G and H is the graph $G[H]$ with $V(G[H]) = V(G) \times V(H)$ and $(u_1, u_2)(v_1, v_2) \in E(G[H])$ if and only if either $u_1v_1 \in E(G)$ or $u_1 = v_1$ and $u_2v_2 \in E(H)$.

For $S \subseteq V(G[H])$, we write

$$S_G = \{x \in V(G) : (x, a) \in S \text{ for some } a \in V(H)\}.$$

S_G is referred to as the *G-projection* of S in $G[H]$.

Let $f = (V_0, V_1, V_2)$ be a CvRDF on $G[H]$. We write

$$\begin{aligned} S_G^0 &= \{x \in V(G) : (x, a) \in V_0 \text{ for some } a \in V(H)\}, \\ S_G^1 &= \{x \in V(G) : (x, a) \in V_1 \text{ for some } a \in V(H)\}, \\ S_G^2 &= \{x \in V(G) : (x, a) \in V_2 \text{ for some } a \in V(H)\}, \\ V_G^0 &= S_G^0 \setminus (S_G^1 \cup S_G^2), \end{aligned}$$

$$\begin{aligned} V_G^1 &= S_G^1 \setminus (S_G^0 \cup S_G^2), \\ V_G^2 &= V(G) \setminus (V_G^0 \cup V_G^1), \text{ and} \\ S_f^0 &= I(S_G^1 \cup S_G^2) \cap (S_G^1 \cup S_G^2). \end{aligned}$$

Note that if $V_G^1 \neq V(G)$, then $V_G^2 \neq \varnothing$ because f is a Roman dominating function on $G[H]$.

Theorem 10. *Let G be a non-trivial connected graph and K_m a complete graph. Then $f = (V_0, V_1, V_2)$ is a CvRDF on $G[K_m]$ if and only if each of the following conditions hold:*

(i) $g|_f = (V_G^0, V_G^1, V_G^2)$ *is a CvRDF on G.*

(ii) $S_G^1 \cup S_G^2$ *is convex in G.*

(iii) $\{x\} \times V(K_m) \subseteq V_1 \cup V_2$ *for $x \in S_f^0$.*

(iv) For each $v \in (S_G^0 \setminus S_G^2) \cap S_G^1$, there exists $w \in S_G^2 \cap N_G(v)$.

Proof. Let $f = (V_0, V_1, V_2)$ be a CvRDF on $G[K_m]$. Consider the function $g_f = (V_G^0, V_G^1, V_G^2)$ on G. Let $x \in V_G^0$ and let $q \in V(K_m)$. Then $(x, q) \in V_0$. Since f is a CvRDF on $G[K_m]$, there exists $(y, t) \in V_2$ such that $(x, q)(y, t) \in E(G[K_m])$. This implies that $xy \in E(G)$ and $y \in V_G^2$. Thus, $g|_f$ is an RDF on G. Let $u, v \in V_G^1 \cup V_G^2$ such that $u \neq v$. Let $x \in I_G(u, v)$. Let $P(u, v) = [u, u_1, u_2, \ldots, u_k, v]$ be a u-v geodesic in G with $x = u_r$ for some $1 \leq r \leq k$. Consider the following cases:

Case 1. $u, v \in V_G^1$ (or $u, v \in V_G^2$)
Pick any $t \in V(K_m)$. Then $(u, t) \in V_1$. Then $P((u, t), (v, t) = [(u, t), (u_1, t), \ldots, (u_{r-1}, t), (u_r, t), \ldots, (u_k, t), (v, t)]$ is a (u, t)-(v, t) geodesic in $G[K_m]$. Since $V_1 \cup V_2$ is a convex set in $G[K_m]$, it follows that $(u_r, t) \in V_1 \cup V_2$. This implies that $x = u_r \in V_G^1 \cup V_G^2$. A similar argument is used to show that $x \in V_G^1 \cup V_G^2$ whenever $u, v \in V_G^2$.

Case 2. $u \in V_G^1$ and $v \in V_G^2$
Pick any $s \in V(K_m)$ such that $(v, s) \in V_2$. Then $(u, s) \in V_1$ and $P((u, s), (v, s)) = [(u, s), (u_1, s), \ldots, (u_{r-1}, s), (u_r, s), \ldots, (u_k, s), (v, s)]$ is a (u, s)-(v, s) geodesic in $G[K_m]$. Since $V_1 \cup V_2$ is a convex set in $G[K_m]$, it follows that $(u_r, s) \in V_1 \cup V_2$. This implies that $x = u_r \in V_G^1 \cup V_G^2$.

Therefore, g is a CvRDF on G, showing that (i) holds.

Let $x, y \in S_G^1 \cup S_G^2$ with $x \neq y$ and let $z \in I_G(x, y)$. Let $P(x, y) = [x, x_1, x_2, \ldots, x_k, y]$ be an x-y geodesic in G where $z = x_j$ for some $1 \leq j \leq k$. Let $a, b \in V(K_m)$ such that $(x, a), (y, b) \in V_1 \cup V_2$. Then $P((x, a), (y, b)) = [(x, a), (x_1, a), (x_2, a), \ldots, (x_k, a), (y, b)]$ is an (x, a)-(y, b) geodesic in $G[K_m]$. Since $V_1 \cup V_2$ is a convex set in $G[K_m]$, $(x_j, a) \in V_1 \cup V_2$. Hence, $x_j \in S_G^1 \cup S_G^2$. This shows that $S_G^1 \cup S_G^2$ is convex in G, i.e., (ii) holds.

Next, let $x \in S_f^0$ and let $p \in V(K_m)$. Then $x \in S_G^1 \cup S_G^2$ and there exists $y, z \in S_G^1 \cup S_G^2$ such that $x \in I_G(y, z)$. Again, by convexity of $V_1 \cup V_2$, $(x, p) \in V_1 \cup V_2$. This shows that (iii) holds.

Finally, let $v \in (S_G^0 \setminus S_G^2) \cap S_G^1$ and let $a \in V(K_m)$ such that $(v,a) \in V_0$. Since f is a CvRDF on $G[K_m]$, there exists $(w,b) \in V_2$ such that $(v,a)(w,b) \in E(G[K_m])$. Hence, $w \in S_G^2$ and $v \in N_G(w)$. Hence, (iv) holds.

Conversely, assume that (i), (ii), (iii), and (iv) hold. Let $(v,p) \in V_0$. Then $v \in S_G^0$. If $v \in S_G^2$, then $(v,q) \in V_2$ for some $q \in V(K_m)$ and $(v,p)(v,q) \in E(G[K_m])$. Suppose $v \notin S_G^2$. Suppose further that $v \in S_G^1$. Then by (iv), there exists $w \in S_G^2$ such that $v \in N_G(w)$. Let $c \in V(K_m)$ such that $(w,c) \in V_2$. Then $(v,p)(w,c) \in E(G[H])$. Next, suppose that $v \notin S_G^1 \cup S_G^2$. Then $v \in V_G^0$. By (i), there exists $z \in V_G^2$ such that $vz \in E(G)$. Let $d \in V(K_m)$ such that $(z,d) \in V_2$. Then $(v,p)(z,d) \in E(G[K_m])$. Thus f is an RDF on $G[K_m]$.

Now, let $V_1 \cup V_2 = \bigcup_{x \in S} [\{x\} \times T_x]$. Then $S = S_G^1 \cup S_G^2$ and $T_x \subseteq V(K_m)$ for each $x \in S$. Morever, by (iii), $T_x = V(K_m)$ for each $x \in I_G(S) \cap S$. Thus, by Theorem 2, $V_1 \cup V_2$ is convex in $G[K_m]$. Therefore, f is a CvRDF on $G[K_m]$. □

Lemma 2. *Let G be a non-trivial connected graph with $G \neq K_2$ and let m be any positive integer. If $h = (W_0, W_1, W_2)$ is a CvRDF on G such that*

$$k = \omega_{CvR}(h) + (m-1)\left|S_h^0\right| = \min\left\{\omega_G^{CvR}(h') + (m-1)\left|S_{h'}^0\right|\right\},$$

then $W_1 \subseteq S_h^0$.

Proof. Suppose there exists $x \in W_1 \setminus S_h^0$. Suppose $x \in N_G(W_0)$, say $y \in W_0 \cap N_G(x)$. Since h is an RDF on G, there exists $v \in W_2$ such that $y \in N_G(v)$. By convexity of $W_1 \cup W_2$, $xv \in E(G)$. Let $W_0' = W_0 \cup \{x\}$, $W_1' = W_1 \setminus \{x\}$, and $W_2' = W_2$. Then $h' = (W_0', W_1', W_2')$ is an RDF on G. Let $p, q \in W_1' \cup W_2'$ such that $p \neq q$. Then $p, q \in W_1 \cup W_2$ and $I_G(p,q) \subseteq W_1 \cup W_2$ since $W_1 \cup W_2$ is convex. Let $z \in I_G(p,q)$. Since $x \in W_1 \setminus S_h^0$, $x \notin I_G(W_1 \cup W_2)$. Thus, $z \neq x$. Hence, $z \in W_1' \cup W_2'$, i.e., $I_G(p,q) \subseteq W_1' \cup W_2'$, showing that $W_1' \cup W_2'$ is convex in G. Therefore, h' is a CvRDF on G and $\omega_G^{CvR}(h') = |W_1'| + 2|W_2'| = |W_1| - 1 + 2|W_2| < \omega_G^{CvR}(h)$, a contradiction. Thus, $x \notin N_G(W_0)$. Next, suppose $x \in N_G(W_2)$, say $\{z\} \in W_2 \cap N_G(x)$. Following the argument above, this is also not possible. Thus, $x \notin N_G(W_2)$. Therefore, $N_G(x) \subseteq W_1$. Suppose that $N_G(x) \subseteq W_1 \setminus S_h^0$. Suppose there exists $y \in N_G(x) \cap (W_1 \setminus S_h^0)$. Since $|V(G)| \geq 3$, there exists $z \in N_G(x) \cup N_G(y)$. Moreover, since $x, y \notin S_h^0$, $z \in N_G(x) \cap N_G(y)$. It follows that $y, z \in W_1 \setminus S_h^0$. Let $W_0^* = W_0 \cup \{x, z\}$, $W_1^* = W_1 \setminus \{x, z, y\}$, and $W_2^* = W_2 \cup \{y\}$. Then $h^* = (W_0^*, W_1^*, W_2^*)$ is a CvRDF on G and $\omega_G^{CvR}(h^*) < \omega_G^{CvR}(h)$. Since $S_{h^*}^0 \subseteq S_h^0$, it follows that $\omega_G^{CvR}(h^*) + (m-1)\left|S_{h^*}^0\right| < k$, a contradiction. Therefore, $N_G(x) \cap (W_1 \cap S_h^0) \neq \varnothing$. Let $v_x \in N_G(x) \cap (W_1 \cap S_h^0)$. Let $V_0 = W_0 \cup \{x\}$, $V_1 = W_1 \setminus \{x, v_x\}$ and $V_2 = W_2 \cup \{v_x\}$. Then $f = (V_0, V_1, V_2)$ is a CvRDF on G and $\omega_G^{CvR}(f) = \omega_G^{CvR}(h)$. Since $x \notin S_f^0$, $|S_f^0| < |S_h^0|$. Thus, $\omega_G^{CvR}(f) + (m-1)\left|S_f^0\right| < k$, a contradiction. Therefore, $W_1 \setminus S_h^0 = \varnothing$, i.e, $W_1 \subseteq S_h^0$. □

Corollary 4. *Let G be a non-trivial connected graph and K_m be a complete graph of order $m \geq 1$. Then*

$$\gamma_{CvR}(G[K_m]) = \min\left\{\omega_G^{CvR}(g) + (m-1)\left|S_g^0\right| : g \text{ is a CvRDF on } G\right\}.$$

Proof. Let $k = \min\left\{\omega_G^{CvR}(g) + (m-1)\left|S_g^0\right| : g \text{ is a CvRDF on } G\right\}$. Let $f = (V_0, V_1, V_2)$ be a γ_{CvR}-function on $G[K_m]$. Then $g = (V_G^0, V_G^1, V_G^2)$ is a CvRDF on G, by Theorem 10 (i). For each $x \in S_G^1$, let $D_x = \{(x,p) \in V_1 : p \in V(K_m)\}$. For each $y \in S_G^2$, let $R_y = \{(y,q) \in V_2 : q \in V(K_m)\}$. Since f is a γ_{CvR}-function, $S_f^0 \subseteq V_G^1$. Hence, $S_f^0 \cap S_G^1 = S_f^0$ and $S_f^0 \cap S_G^2 = \varnothing$. Consequently,

$$\begin{aligned}
\gamma_{CvR}(G[K_m]) &= \omega_{G[K_m]}(f) \\
&= |V_1| + 2|V_2| \\
&= \sum_{x \in S_G^1 \setminus S_f^0} |D_x| + \sum_{x \in S_f^0} |D_x| + 2 \sum_{y \in S_G^2} |R_y| \\
&\geq |S_G^1 \setminus S_f^0| + m|S_f^0| + 2|S_G^2| \\
&= |S_G^1| + 2|S_G^2| + (m-1)|S_f^0| \\
&\geq |V_G^1| + 2|V_G^2| + (m-1)|S_g^0| \\
&= \omega^{CvR}(g) + (m-1)|S_g^0| \geq k.
\end{aligned}$$

Let $h = (W_0, W_1, W_2)$ be a CvRDF on G such that $k = \min\{\omega_G^{CvR}(g) + (m-1)\left|S_g^0\right| : g \text{ is a CvRDF on } G\}$. By Lemma 2, $W_1 \subseteq S_g^0$. Let $p \in V(K_m)$. Set $V_1 = (W_1 \times \{p\}) \cup \left[((W_1 \cup W_2) \cap S_g^0) \times (V(K_m) \setminus \{p\})\right]$, $V_2 = W_2 \times \{p\}$, and $V_0 = (W_0 \times V(K_m)) \cup ((W_1 \setminus S_g^0) \times V(K_m))$. Let $f = (V_0, V_1, V_2)$. Then $V_G^0 = W_0$, $V_G^1 = W_1$, and $V_G^2 = W_2$. Hence, $g = h$ is a CvRDF on G. Also, $S_G^1 \cup S_G^2 = W_1 \cup W_2$ is convex in G since h is a CvRDF on G. Clearly, (iii) and (iv) of Theorem 10 is satisfied. Hence, f is a CvRDF on $G[K_m]$ and

$$\begin{aligned}
\gamma_{CvR}(G[K_m]) \leq \omega_{G[K_m]}(f) &= |V_1| + 2|V_2| \\
&= |W_1| + (m-1)|(W_1 \cup W_2) \cap S_g^0| + 2|W_2| \\
&\leq |W_1| + 2|W_2| + (m-1)|S_g^0| \\
&= \omega_G^{CvR}(g) + (m-1)|S_g^0| = k.
\end{aligned}$$

This proves the desired equality. □

For each $x \in S_G^1 \cup S_G^2$, we write

$$\begin{aligned}
T_x^0 &= \{p \in V(H) : (x,p) \in V_0\}, \\
T_x^1 &= \{p \in V(H) : (x,p) \in V_1\}, \text{ and} \\
T_x^2 &= \{p \in V(H) : (x,p) \in V_2\}.
\end{aligned}$$

Theorem 11. *Let G and H be connected non-complete graphs with $\gamma_{cl}(G) \geq 2$. Then $f = (V_0, V_1, V_2)$ is CvRDF on $G[H]$ if and only if each of the following conditions hold:*

(i) *$S_G^1 \cup S_G^2$ is a clique dominating set in G.*

(ii) *$T_x^1 \cup T_x^2$ is a clique in H for each $x \in S_G^1 \cup S_G^2$.*

(iii) T_x^2 *is a (clique) dominating set in* H *for each* $x \in S_G^0 \setminus N_G(S_G^2)$.

Proof. Let $f = (V_0, V_1, V_2)$ be a CvRDF on $G[H]$ such that $\gamma_{cl}(G) \geq 2$. Then (i) and (ii) hold by Theorem 4. Now, let $x \in S_G^0 \setminus N_G(S_G^2)$ and let $p \in V(H) \setminus (T_x^1 \cup T_x^2)$. Since f is a CvRDF on $G[H]$, there exists $(x,q) \in V_2$ such that $(x,p)(x,q) \in E(G[H])$. Hence, $x \in S_G^2$ and there exists $q \in T_x^2$ such that $pq \in E(H)$. It follows that T_x^2 is a dominating set in H, showing that (iii) holds.

For the converse, suppose that (i), (ii), and (iii) hold. Let $(x,p) \in V_0$. Then $x \in S_G^0$. If $x \in (N_G(S_G^2))$, then there exists $y \in S_G^2 \cap N_G(x)$. Let $r \in T_y^2$. Then $(y,r) \in V_2 \cap N_{G[H]}((x,p))$. If $x \notin (N_G(S_G^2))$, then there exists $q \in T_x^2 \cap N_H(p)$ by (iii). Hence, $(x,q) \in V_2 \cap N_{G[H]}\big((x,p)\big)$, Therefore, f is an RDF on $G[H]$.

Now, let $V_1 \cup V_2 = \bigcup_{x \in S} [\{x\} \times T_x]$. Then $S = S_G^1 \cup S_G^2$ and $T_x = T_x^1 \cup T_x^2$. By (i), (ii) and Theorem 3, $V_1 \cup V_2$ is convex in $G[H]$. Therefore, f is a CvRDF on $G[H]$. □

Corollary 5. *Let* G *and* H *be connected non-complete graphs with* $\gamma_{cl}(G) \geq 2$. *Then*

$$\gamma_{CvR}(G[H]) = 2\gamma_{cl}(G).$$

Proof. Let D be a γ_{cl}-set in G and let $p \in V(H)$. Let $V_1 = \varnothing$, $V_2 = D \times \{p\}$, and $V_0 = [(V(G) \setminus D) \times V(H)] \cup [D \times (V(H) \setminus \{p\})]$. Then $S_G^0 = V(G) \setminus D$, $S_G^1 = \varnothing$, and $S_G^2 = D$. By assumption, $S_G^1 \cup S_G^2 = D$ is a clique dominating set in G. Also, $T_x^2 = \{p\}$ is a clique set in H for each $x \in S_G^2$. Moreover, $S_G^0 \setminus N_G(S_G^2) = \varnothing$. Hence, (i), (ii), and (iii) of Theorem 11 hold. Therefore, $f = (V_0, V_1, V_2)$ is a CvRDF on $G[H]$ and

$$\begin{aligned} \gamma_{CvR}(G[H]) &\leq \omega_{G[H]}^{CvR}(f) \\ &= 2|V_2| \\ &= 2\gamma_{cl}(G). \end{aligned}$$

Now, let $f = (V_0, V_1, V_2)$ be a γ_{CvR}-function on $G[H]$. By Theorem 11, $S_G^1 \cup S_G^2$ is a clique dominating set in G. Since G is non-complete and $\gamma_{cl}(G) \geq 2$, $|S_G^2| \geq 2$. Furthermore, S_G^2 is a clique dominating set in G. Therefore,

$$\begin{aligned} \gamma_{CvR}(G[H]) = \omega_{G[H]}^{CvR}(f) &= |V_1| + 2|V_2| \\ &= \sum_{x \in S_G^1} |T_x^1| + 2 \sum_{x \in S_G^2} |T_x^2| \\ &\geq |S_G^1| + 2|S_G^2| \\ &\geq 2\gamma_{cl}(G). \end{aligned}$$

This proves the desired equality. □

Theorem 12. *Let* G *and* H *be non-complete connected graphs with* $\gamma(G) = 1$. *Then*

$$\gamma_{CvR}(G[H]) = \begin{cases} 2, & \text{if } \gamma(H) = 1 \\ 4, & \text{if } \gamma(H) \neq 1 \end{cases}$$

Proof. If $\gamma(H) = 1$, then $\gamma(G[H]) = 1$. By Corollary 1, $\gamma_{CvR}(G[H]) = 2$. Next, let $\gamma(H) \neq 2$. Let v be a dominating vertex of G. Pick any $w \in N_G(v)$ and $p \in V(H)$. Let $V_0 = [V(G) \setminus \{v, w\} \times V(H)] \cup [\{v, w\} \times V(H) \setminus \{p\}]$, $V_1 = \varnothing$, and $V_2 = \{v, w\} \times \{p\}$. Let $(x, q) \in V_0$. If $x \in V(G) \setminus \{v, w\}$, then $xv \in E(G)$. Hence, $(v, p) \in V_2$ and $(x, q)(v, p) \in E(G[H])$. If $x = v$, then $(w, p) \in V_2 \cap N_{G[H]}((x, q))$ and if $x = w$, then $(v, p) \in V_2 \cap N_{G[H]}((x, q))$. Therefore, $g = (V_0, V_1, V_2)$ is an RDF on $G[H]$. Now, $V_1 \cup V_2 = V_2$ and $\langle V_2 \rangle \cong K_2$. Hence, $V_1 \cup V_2$ is convex in $G[H]$. This shows that g is a CvRDF on $G[H]$. Since $\omega_{G[H]}^{CvR}(g) = 2|V_2| = 4$, $\gamma_{CvR}(G[H]) = 4$ by Proposition 3. □

The *Cartesian product* $G \times H$ of two graphs G and H is the graph with $V(G \times H) = V(G) \times V(H)$ and $(u, u')(v, v') \in E(G \times H)$ if and only if either $uv \in E(G)$ and $u' = v'$ or $u = v$ and $u'v' \in E(H)$.

Lemma 3. *Let G and H be a connected graphs. If $f = (V_0, V_1, V_2)$ is a CvRDF on $G \square H$ then $V_1 \cup V_2 = (S_G^1 \cup S_G^2) \times (S_H^1 \cup S_H^2)$.*

Proof. If $(x, p) \in V_1$, then $x \in S_G^1$ and $p \in S_H^1$. Thus $(x, p) \in S_G^1 \times S_H^1$. Also, if $(x, p) \in V_2$, $x \in S_G^2$ and $p \in S_H^2$. Thus $(x, p) \in S_G^2 \times S_H^2$. Hence, $V_1 \cup V_2 \subseteq (S_G^1 \cup S_G^2) \times (S_H^1 \cup S_H^2)$.

Now, let $(z, q) \in (S_G^1 \cup S_G^2) \times (S_H^1 \cup S_H^2)$. Suppose that $(z, q) \in V_0$. Suppose $z \in S_G^1$ and $q \in S_H^1$. Since f is an RDF, there exists $(w, t) \in V_2 \cap N_{G[H]}((z, q))$. Suppose $w = z$. Then $tq \in E(H)$. Let $y \in V(G)$ such that $(y, q) \in V_1$. Let $P(y, z) = [y_1, y_2, \ldots, y_k]$ where $y_1 = y$ and $y_k = z$ be a y-z geodesic in G for some $k \geq 1$. Then, $P((y, q), (z, t)) = [(y_1, q), (y_2, q), \ldots, (y_k, q), (z, t)]$ is also (y, q)-(z, t) geodesic in $G \square H$, a contradiction to our assumption that $V_1 \cup V_2$ is convex. Suppose $w \neq z$. Then $wz \in E(G)$ and $t = q$. Since $z \in S_G^1$, there exists $r \in V(H)$ such that $(z, r) \in V_1$. Let $P(q, r) = [q_1, q_2, \ldots, q_m, z]$ where $q_1 = q$ and $q_m = r$ be a $q - r$ geodesic in H. Then $P((w, q)(z, r)) = [(w, q), (z, q_1), (z, q_2), \ldots, (z, q_m)]$ is a (w, q)-(z, r) geodesic in $G \square H$, a contradiction to our assumption that $V_1 \cup V_2$ is convex. Similar arguments can be used to show that a contradiction is obtained when $z \in S_G^1$, $q \in S_H^2$ or $z \in S_G^2$, $q \in S_H^1$ or $z \in S_G^1$, $q \in S_H^2$.

Therefore $(z, q) \in V_1 \cup V_2$ showing that $(S_G^1 \cup S_G^2) \times (S_H^1 \cup S_H^2) \subseteq V_1 \cup V_2$. This proves the desired equality. □

Theorem 13. *Let G and H be a connected graphs. Then $f = (V_0, V_1, V_2)$ is a CvRDF on $G \square H$ if and only if the following conditions hold:*

(i) *for each $x \in S_G^0$ and $p \in T_x^0$, there exists $q \in T_x^2 \cap N_H(p)$ or $y \in S_G^2 \cap N_G(x)$ with $q = p \in T_y^2$*

(ii) *$V_1 \cup V_2 = (S_G^1 \cup S_G^2) \times (S_H^1 \cup S_H^2)$ and*

  (a) *$S_G^1 \cup S_G^2$ is a convex dominating set in G and $S_H^1 \cup S_H^2 = V(H)$ or*

  (b) *$S_H^1 \cup S_H^2$ is a convex dominating set in G and $S_G^1 \cup S_G^2 = V(G)$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a CvRDF on $G\Box H$ and let $x \in S^0_G$ and $p \in T^0_x$. Then $(x, p) \in V_0$. This shows that (i) holds. By Lemma 3, $V_1 \cup V_2 = (S^1_G \cup S^2_G) \times (S^1_H \cup S^2_H)$. Hence, by Theorem 6, (ii) holds.

Conversely, suppose that (i) and (ii) hold. By (i). Let $(x, p) \in V_0$. Then $x \in S^0_G$ and $p \in T^0_x$. By (i), there exists $(y, q) \in N_{G\Box H}\big((x, p)\big)$. This implies that f is an RDF on $G\Box H$. By (a) and (b), $S^1_G \cup S^2_G$ and $S^1_H \cup S^2_H$ are convex sets in G and H, respectively. Hence, by Theorem 5, $V_1 \cup V_2$ is convex in $G\Box H$. Therefore, f is a CvRDF on $G\Box H$. $\Box$

Corollary 6. *Let G and H be connected graphs of orders m and n, respectively. Then*

$$\gamma_{CvR}(G\Box H) \le \min\{n \cdot \gamma_{CvR}(G), m \cdot \gamma_{CvR}(H)\}.$$

Proof. Let $g = (W_0, W_1, W_2)$ be a γ_{CvR}-function on G. Set $V_0 = W_0 \times V(H)$, $V_1 = W_1 \times V(H)$, and $V_2 = W_2 \times V(H)$. Let $f = (V_0, V_1, V_2)$. Then $S^0_G = W_0$, $S^1_G = W_1$, and $S^2_G = W_2$. Hence, $S^1_G \cup S^2_G = W_1 \cup W_2$ is a convex dominating set in G and $S^1_H \cup S^2_H = V(H)$. Hence, by Theorem 13, $f = (V_0, V_1, V_2)$ is a CvRDF on $G\Box H$ and

$$\begin{aligned} \gamma_{CvR}(G\Box H) &\le \omega^{CvR}_{G\Box H}(f) \\ &= |V_1| + 2|V_2| \\ &= |W_1 \times V(H)| + 2|W_2 \times V(H)| \\ &= |V(H)| \times (|W_1| + 2|W_2|) \\ &= n(|W_1| + 2|W_2|) \\ &= n \cdot \gamma_{CvR}(G). \end{aligned}$$

A similar argument is used to show that $\gamma_{CvR}(G\Box H) \le m \cdot \gamma_{CvR}(H)$. Hence, $\gamma_{CvR}(G\Box H) \le \min\{n \cdot \gamma_{CvR}(G), m \cdot \gamma_{CvR}(H)\}$. $\Box$

Remark 2. *The bound given in Corollary 6 is sharp. It can be verified that for any connected graph H of order m, $\gamma_{CvR}(K_n\Box H) = 2m = \gamma_{CvR}(K_n) \cdot m$.*

4. Conclusion

The concept of convex Roman domination in a graph has been investigated further in this study. Specifically, convex Roman dominating functions on graphs resulting from the corona, edge corona, complementary prism, lexicographic, and Cartesian product of graphs have been characterized. These characterizations have been utilized to derive bounds or exact values for the convex Roman domination number of each of these graphs. Interested researchers may investigate this concept for other graphs not considered in this paper. Moreover, it may be interesting to investigate the complexity of the convex Roman domination problem and explore some relationships, if any, of this newly defined parameter with other existing and related parameters to it.

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