On Minimal Geodetic Hop Domination In Graphs

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Abstract. Let $G$ be a nontrivial connected graph with vertex set $V(G)$ and the edge set $E(G)$. A set $S \subseteq V(G)$ is a geodetic hop dominating set of $G$ if the following two conditions hold for each $x \in V(G) \setminus S$: (1) $x$ lies in some $u$-$v$ geodesic in $G$ with $u, v \in S$, and (2) $x$ is of distance 2 from a vertex in $S$. The minimum cardinality $\gamma_{hg}(G)$ of a geodetic hop dominating set of $G$ is the geodetic hop domination number of $G$.

A geodetic hop dominating set $S$ is a minimal geodetic hop dominating set if $S$ does not contain a proper subset that is itself a geodetic hop dominating set. The maximum cardinality of a minimal geodetic hop dominating set in $G$ is the upper geodetic hop domination number of $G$, and is denoted by $\gamma_{hg}^+(G)$.

This paper initiates the study of the minimal geodetic hop dominating set and the corresponding upper geodetic hop domination number of nontrivial connected graphs. Interestingly, every pair of positive integers $a$ and $b$ with $2 \leq a \leq b$ is realizable as the geodetic domination number and the upper geodetic hop domination number, respectively, of some graph. Furthermore, this paper investigates the concept in the join, corona and lexicographic product of graphs.

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1. Introduction

In 2021, D. Anusha et al. [17] introduced the concept of geodetic hop domination in graphs, and initially investigated the concept in the complementary prism of graphs. Further investigation of the concept was done by D. Anusha [3] in 2022, by C.J. Saromines and S.R. Canoy Jr. [26, 27] in 2023, and just recently by D. Anusha et al. [4].

This present paper introduces and initiates the study of the concept of minimal geodetic hop domination, a natural variation of the geodetic hop domination.

Throughout this paper, all graphs considered are simple and undirected. Common graph terminologies and notations used here are adapted from [6, 11].

Given two graphs $G$ and $H$ with disjoint vertex sets, the join of $G$ and $H$ is the graph $G + H$ with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The corona of $G$ and $H$ is the graph $G \circ H$ obtained by taking $|V(H)|$ copies of $H$ in every vertex of $V(G)$ and then joining the $i$th vertex of $V(G)$ to every vertex of the $i$th copy of $H$.

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The lexicographic product of $G$ and $H$ is the graph $G[H]$ with $V(G[H]) = V(G) \times V(H)$ and $(u, v)(w, z) \in E(G[H])$ if and only if $uw \in E(G)$ or $u = w$ and $vz \in E(H)$.

Let $G$ be a connected graph. For any two distinct $u, v \in E(G)$, a shortest path joining $u$ and $v$ is called a $u$-$v$ geodesic. The length of a $u$-$v$ geodesic is called the distance between $u$ and $v$, which is denoted by $d_G(u, v)$. The diameter of the graph $G$, denoted $diam(G)$, is the maximum distance between any pair of vertices in $G$. If $diam(G) = 1$, then $G$ is a complete graph. A clique of a graph $G$ is any complete subgraph of $G$. The clique number of $G$ is the maximum cardinality $\omega(G)$ of a clique of $G$.

By the (open) neighborhood $N_G(v)$ of a vertex $v$ is meant the set of vertices that are adjacent to $v$, that is, $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The closed neighborhood of a vertex $v$ is $N_G[v] = N_G(v) \cup \{v\}$. For $S \subseteq V(G)$, $N_G(S) = \cup_{v \in S} N_G(v)$, while $N_G[S] = N_G(S) \cup S$. A vertex $v$ is an extreme vertex if the induced subgraph $\langle N_G(v) \rangle$ is a complete graph. The symbol $Ext(G)$ denotes the set of all extreme vertices in $G$.

For $u, v \in V(G)$, the set $I_G(u, v)$ refers to the set consisting of all the vertices lying in any $u - v$ geodesic of $G$ and $I_G[u, v] = I_G(u, v) \cup \{u, v\}$. For a subset $S \subseteq V(G)$, the geodetic closure $I_G[S]$ is defined by $I_G[S] = \cup\{I_G[u, v] : u, v \in S\}$. A geodetic set is any set $S \subseteq V(G)$ with $I_G[S] = V(G)$. The minimum cardinality $g(G)$ of a geodetic set is the geodetic number of $G$. A geodetic set of cardinality $g(G)$ is called a geodetic basis. A geodetic set $G$ in $G$ is a minimal geodetic set if $S$ does not have a proper subset that is itself a geodetic set in $G$. The maximum cardinality of a minimal geodetic set in $G$ is denoted by $g^+(G)$. Geodetic sets and geodetic numbers of graphs are, in fact, among the very well-studied concepts in graph theory (see $1, 2, 7, 8, 10, 12, 15, 16, 18, 19, 28$).

For $S \subseteq V(G)$, the 2-path closure of $S$, denoted by $P_2[S]_G$, is the set

$$P_2[S]_G = S \cup \{w \in V(G) : w \in I_G[u, v] \text{ for } u, v \in S \text{ with } d_G(u, v) = 2\}$$

A set $S$ is called a $2$-path closure absorbing if $P_2[S]_G = V(G)$. The minimum cardinality of a $2$-path closure absorbing set in $G$ is denoted by $\rho_2^+(G)$. A $2$-path closure absorbing set $S$ is a minimal $2$-path closure absorbing set if $S$ does not contain a proper subset that is itself a $2$-path closure absorbing. The maximum cardinality of a minimal $2$-path closure absorbing set in $G$ is denoted by $\rho^+_2(G)$. The concept of $2$-path closure absorbing sets was introduced in $[15, 18, 19]$.

The (open) hop neighborhood of a vertex $v$ refers to the set $N^2_G(v) = \{u \in V(G) : d_G(u, v) = 2\}$. The closed hop neighborhood of a vertex $v$ is $N^2_G[v] = N^2_G(v) \cup \{v\}$. For $S \subseteq V(G)$, the (open) hop neighborhood $S$ is the set $N^2_G(S) = \cup_{v \in S} N^2_G(v)$. The closed hop neighborhood of the set $S$ is $N^2_G[S] = N^2_G(S) \cup S$.

A set $S \subseteq V(G)$ is called a hop dominating set if $N^1_G[S] = V(G)$. The hop domination number of $G$, denoted by $\gamma_h(G)$, is the minimum cardinality among all hop dominating sets in $G$. A hop dominating set with cardinality equal to $\gamma_h(G)$ is called a $\gamma_h$-set of $G$. The authors refer to $[4, 5, 9, 13, 14, 20–25]$ for the definitions and results on hop domination which are essential in this study.

A subset $S \subseteq V(G)$ is a geodetic hop dominating set if $S$ is both a geodetic and a hop dominating set of $G$. The geodetic hop domination number $\gamma_{hg}(G)$ of $G$ is the minimum cardinality among all geodetic hop dominating sets in $G$. Any geodetic hop dominating set of $G$ with cardinality $\gamma_{hg}(G)$ is called a $\gamma_{hg}$-set.
2. Main Results

2.1. The minimal geodetic hop domination in graphs

Let $G$ be a graph. A geodetic hop dominating set $S$ is a **minimal geodetic hop dominating set** if $S$ does not contain a proper subset that is itself a geodetic hop dominating set. The maximum cardinality of a minimal geodetic hop dominating set of $G$ is the **upper geodetic hop domination number** of $G$ denoted by $\gamma_{hg}^+(G)$. A minimal geodetic hop dominating set with cardinality $\gamma_{hg}^+(G)$ is called a $\gamma_{hg}^+$-set.

Every dominating vertex and every extreme vertex in $G$ is included in any minimal geodetic hop dominating set of $G$.

**Proposition 1.** Let $G$ be a nontrivial connected graph of order $n$. Then

$$2 \leq \gamma_{hg}^+(G) \leq \gamma_{hg}^+(G) \leq n$$

In particular, $\gamma_{hg}^+(G) < n$ if $G$ does not have any dominating vertex.

**Proof.** Since a $\gamma_{hg}$-set is a minimal geodetic hop dominating set, $\gamma_{hg}(G) \leq \gamma_{hg}^+(G)$. Suppose that $\gamma_{hg}(G) = n$ and $G$ has no dominating vertex. Since $G$ is not a complete graph, $G$ contains a geodesic of the form $[u, w, v]$. Since $w \in I_G(u, v)$, $S = V(G) \setminus \{w\}$ is a geodetic set of $G$. Since $w$ is a non-dominating vertex, there exists $z \in V(G)$ such that $d_G(w, z) = 2$. Because $z \in S$, $S$ is a hop dominating set of $G$. Thus, $S$ is a geodetic hop dominating set of $G$. Since $V(G)$ is a minimal geodetic hop dominating set, this is impossible. □

The contrapositive of the second statement in Proposition 1 gives the following:

**Corollary 1.** If $\gamma_{hg}^+(G) = n$ for a nontrivial connected graph, then $G$ contains at least one dominating vertex.

**Theorem 1.** Let $G$ be a nontrivial connected graph of order $n$. Then $\gamma_{hg}^+(G) = n$ if and only if one of the following holds:

(i.) $G = K_n$

(ii.) $G \neq K_n$ and $V(G) \setminus S$ induces a disconnected graph of complete components where $S \subseteq V(G)$ is the set of all dominating vertices of $G$.

**Proof.** Clearly, if $G = K_n$, then $\gamma_{hg}^+(G) = n$. Suppose $G \neq K_n$ with a nonempty set $S$ of dominating vertices of $G$ and each of the components of $(V(G) \setminus S)$ is complete. Note first that $V(G)$ is a geodetic hop dominating set of $G$. Suppose $T \subseteq V(G)$ is a geodetic hop dominating set of $G$. By the above remark, $S \subseteq T$. Let $C$ be a component of $(V(G) \setminus S)$. We claim that $V(C) \subseteq Ext(G)$. Let $x \in V(C)$, and let $u, v \in N_G(x)$ with $u \neq v$. Then $u, v \in S \cup V(C)$. If $u, v \in V(C)$, then since $C$ is complete, $uv \in E(G)$. If $u \in S$ or $v \in S$, then $uv \in E(G)$. Accordingly, $x \in Ext(G)$. Since $x$ is arbitrary, $V(C) \subseteq Ext(G)$. Thus, $V(C) \subseteq T$. Since $C$ is arbitrary, $V(G) \setminus S \subseteq T$. Therefore, $T = V(G)$. This means $G$ does not have a proper subset that is itself a geodetic hop dominating set. In other words, $V(G)$ is a minimal geodetic hop dominating set of $G$. Therefore, $\gamma_{hg}^+(G) = n$.

Conversely, suppose $\gamma_{hg}^+(G) = n$. If $G = K_n$, then we are done. Now, suppose $G \neq K_n$. By Corollary 1, the set $S \subseteq V(G)$ consisting of the dominating vertices of $G$ is nonempty. Suppose that $(V(G) \setminus S)$ has a component $C$ that is not complete. Then $C$ contains a geodesic of the form $[u, w, v]$. We claim that $W = V(G) \setminus \{w\}$ is a geodetic hop dominating set of $G$. Since
Let \( G \) be a nontrivial graph connected graph. Then \( \gamma_{hg}^+(G) = 2 \) if and only if \( G \) satisfies the following graphs:

(i) \( G = P_2 \);

(ii) \( G = C_6 \);

(iii) \( G \) has a geodetic set \( S = \{u, v\} \) such that \( d_G(u, v) = 3 \) and \( u, v \in Ext(G) \).

Proof. It is easy to verify that if \( G = P_2 \) or \( G = C_6 \), then \( \gamma_{hg}^+(G) = 2 \). Suppose that \( G \) satisfies condition (iii). Let \( w \in V(G) \setminus S \). Then \( w \) lies on a \( u\)-\( v \) geodesic. Thus, either \( d_G(u, w) = 2 \) or \( d_G(v, w) = 2 \). Since \( w \) is arbitrary, \( S \) is a hop dominating set of \( G \). Therefore, \( S \) is a minimal geodetic hop dominating set of \( G \). Now, \( S \subseteq T \) for every geodetic hop dominating set \( T \) of \( G \). Hence, \( S \) is a minimal \( \gamma_{hg}^+ \)-set of \( G \). Therefore, \( \gamma_{hg}^+(G) = 2 \).

Conversely, assume \( \gamma_{hg}^+(G) = 2 \). Suppose that \( G \notin \{P_2, C_6\} \). Let \( S = \{u, v\} \) be a \( \gamma_{hg}^+ \)-set of \( G \). Since \( G \neq P_2 \), \( uv \notin E(G) \). For each \( a \in V(G) \setminus S \), \( a \) lies on a \( u\)-\( v \) geodesic. Choose \( a \in V(G) \setminus S \) such that \( wa \in E(G) \). Since \( S \) is a hop dominating set of \( G \), \( d_G(a, v) = 2 \). Necessarily, \( d_G(u, v) = 3 \). Suppose that \( u \notin Ext(G) \). Then \( G \) contains a geodesic \([x, u, y]\) containing \( u \). Let \([u, y, z, v]\) be a \( u\)-\( v \) geodesic containing \( y \). We consider two cases:

Case 1: Suppose \( deg_G(u) \geq 3 \), and let \( w \in N_G(u) \setminus \{x, y\} \). If \( xz \in E(G) \), then a minimal geodetic hop dominating set of \( G \) can be constructed containing \( x, y, z \). Clearly, in this case, \( |T| \geq 3 \). On the other hand, if \( xz \notin E(G) \), a minimal geodetic hop dominating set \( T \) can be constructed containing \( x, z \). Since \( w \notin N_G(x, z) \), \( |T| \geq 3 \). Either subcase yields a contradiction.

Case 2: Suppose that \( deg_G(u) = 2 \). If \( xz \in E(G) \), then \( T = \{x, y, z, v\} \) makes a minimal geodetic hop dominating set of \( G \). Suppose \( xz \notin E(G) \). Then there exists a \( u\)-\( v \) geodesic in \( G \) of the form \([u, x, w, v] \) with \( w \neq z \). Consequently, \( |V(G)| \geq 6 \). Suppose that \( |V(G)| \geq 7 \). Then a minimal geodetic hop dominating set can be constructed containing \( x \) and \( y \). In this case, \( |T| \geq 3 \). Finally, suppose that \( |V(G)| = 6 \). Since \( G \neq C_6 \), \( G \) is obtained from \( C_6 \) by adding at least one edge to join a pair of nonadjacent vertices. In this case, a minimal geodetic hop dominating set can be constructed with \( |T| \geq 3 \). All possibilities yield to contradiction.

The above cases imply that \( u \in Ext(G) \). Similarly, \( v \in Ext(G) \).

Proposition 2. For the complete graph \( K_n \), path \( P_n \), cycle \( C_n \) and Petersen graph \( P \), and for \( k \geq 1 \),

(i) \( \gamma_{hg}^+(K_n) = n \);

(ii) \( \gamma_{hg}^+(P_n) = \begin{cases} n & \text{if } n = 1, 2, 3 \\ 2k + 1 & \text{if } n = 4k + 1 \\ 2 \left\lfloor \frac{n}{4} \right\rfloor & \text{otherwise} \end{cases} \)}
\[ \gamma^+_{hg}(C_n) = \begin{cases} 
3 & \text{if } n = 3, 4, 5 \\
2 & \text{if } n = 6 \\
2k + 1 & \text{if } n = 4k + 3 \\
2 \left\lfloor \frac{n}{4} \right\rfloor & \text{otherwise} 
\end{cases} \]

(iii) \[ \gamma^+_{hg}(P_n) = 6. \]

**Proof.** Since \( \gamma_{hg}(K_n) = n \), Proposition 1 yields \( \gamma^+_{hg}(K_n) = n \).

Let \( P_n = \{x_1, x_2, \ldots, x_n\} \). The case where \( n = 1, 2, 3 \) is trivial. Suppose \( n = 4k + 1 \), where \( k \geq 1 \). Since \( S = \{x_1, x_2, x_3, x_4, \ldots, x_n\} \) is a minimal geodetic hop dominating set, \( 2k + 1 = |S| \leq \gamma^+_{hg}(G) \). Conversely, let \( S \) be a \( \gamma^+_{hg} \)-set of \( G \). Being a geodetic set, \( x_1, x_n \in S \). For every \( 1 \leq j \leq n - 3, S \) contains at most two vertices in \( x_j, x_{j+1}, x_{j+2}, x_{j+3} \). Thus, \( |S| \leq 2k + 1 \). Hence, \( \gamma^+_{hg}(P_n) = 2k + 1 \).

For the third case, assume \( n \neq 4k + 1 \). If \( n = 4k \), then assume \( S = \{x_1, x_2, x_5, x_6, \ldots, x_{n-3}, x_n\} \).

By the same argument provided above, we have \( S \) to be a minimal geodetic hop dominating set so that \( S \) is a \( \gamma^+_{hg} \)-set. Hence, \( |S| = \frac{4k}{2} = 2k = 2 \left\lfloor \frac{n}{4} \right\rfloor \). For \( n > 4k + 1 \), choose the set \( S = \{x_1, x_2, x_5, x_6, \ldots, x_{n-1}, x_n\} \) to be minimal geodetic hop dominating set. By the same argument, we have \( S \) to be a \( \gamma^+_{hg} \)-set with \( |S| = 2 \left\lfloor \frac{n}{4} \right\rfloor \). Therefore, \( \gamma^+_{hg}(G) = 2 \left\lfloor \frac{n}{4} \right\rfloor \).

Let \( G = C_n = \{v_1, v_2, \ldots, v_n, v_1\} \). The first and second case is trivial. Let \( k \geq 1 \). For the third case, assume \( n = 4k + 3 \). Suppose \( S = \{v_1, v_2, v_5, v_6, \ldots, v_{n-2}\} \). Note that for every \( v_i, v_{i+1} \notin S \) where \( i \geq 3 \), there exist \( v_i, v_{i+1} \in S \) such that \( v_i, v_{i+1} \in I_G(v_{i-1}, v_{i+2}) \) with \( v_i \in N_G(v_{i+2}) \) and \( v_{i+1} \in N_G(v_{i-1}) \) so that \( S \) is both a geodetic set and hop dominating set. Thus, \( S \) is a minimal geodetic hop dominating set so that \( |S| = \frac{4k}{2} + 1 = 2k + 1 \leq \gamma^+_{hg}(G) \). Conversely, by the same argument provided in above, \( S \) is a minimal geodetic hop dominating set and since \( S \) is arbitrary we have \( \gamma^+_{hg}(G) \leq 2k + 1 \). Therefore, \( \gamma^+_{hg}(G) = 2k + 1 \). For the fourth case, assume \( n < 4k + 3 \) with \( k \geq 2 \). Suppose \( S = \{v_1, v_2, v_5, v_6, \ldots, v_{n-3}, v_{n-2}\} \). The same argument would have \( S \) to be a minimal geodetic hop dominating set so that \( S \) is a \( \gamma^+_{hg} \)-set. It follows that \( |S| = \frac{4k}{2} = 2k = 2 \left\lfloor \frac{n}{4} \right\rfloor \).

Let \( G \) be the Petersen graph shown in Figure 1.

![Figure 1: A Petersen graph](image)

Then \( S = \{a, c, d, f, g, j\} \) is a \( \gamma^+_{hg} \)-set. Hence, \( \gamma^+_{hg}(P) = 6. \)

**Proposition 3.** Let \( G = K_{m,n} \) with partite sets \( U \) and \( W \) with \( |U| = m \geq 2 \) and \( |W| = n \geq 2 \). Then \( S \subseteq V(G) \) is a minimal geodetic hop dominating set of \( G \) if and only if \( S \) is one of the following:

(i) \( S = U \cup \{w\} \) where \( w \in W \)
(ii) $S = W \cup \{u\}$ where $u \in U$

(iii) $S = \{u, v, w, z\}$ where $u, v \in U$ and $w, z \in W$, in case where $m, n \geq 3$.

Proof. It is easy to verify that if $S$ is any of the sets described in (i), (ii) and (iii), then $S$ is a minimal geodetic hop dominating set of $G$.

Conversely, suppose $S$ is a minimal geodetic hop dominating set of $G$. Being a hop dominating set, $S \cap U \neq \emptyset$ and $S \cap W \neq \emptyset$. Suppose that $U \subseteq S$. Note that if $w \in S \cap W$, then $U \cup \{w\}$ is a geodetic hop dominating set of $G$. By the minimality of $S$, $S = U \cup \{w\}$. Similarly, if $W \subseteq S$, then $S = W \cup \{u\}$, where $u \in U$. Now, suppose $U \setminus S \neq \emptyset$ and $W \setminus S \neq \emptyset$. Since $S$ is geodetic, $|S \cap U| \geq 2$ and $|S \cap W| \geq 2$. Pick $u, v \in S \cap U$ and $w, z \in S \cap W$. Then, $\{u, v, w, z\}$ is a minimal geodetic hop dominating set of $G$. Thus, $S = \{u, v, w, z\}$.

Corollary 2. Let $G = K_{m,n}$ where $m, n \geq 2$. Then $\gamma_{hg}^+(G) = \max\{m, n\} + 1$.

Proof. If $n = m = 2$, then $\gamma_{hg}^+(G) = 3 = 1 + \max\{m, n\}$. If $n \geq 3$ or $m \geq 3$, then $\gamma_{hg}^+(G) \geq 4$. Thus, $\gamma_{hg}^+(G) = 1 + \max\{m, n\}$.

3. Realization problem

Theorem 3. For every pair of positive integers $a$ and $b$ with $2 \leq a \leq b$, there exists a connected graph $G$ such that $\gamma_{hg}(G) = a$ and $\gamma_{hg}^+(G) = b$.

Proof. If $a = b$, then take $G = K_a$. For this graph, $\gamma_{hg}(G) = a = b = \gamma_{hg}^+(G)$

Assume $a < b$. Write $b = a + k$ for some $k \geq 1$. If $a = 2$, then we consider the graph $G = G_1$ in Figure 2 obtained by constructing $k + 2$ copies of $P_4$ with common end-vertices.

![Figure 2: $G_1$: A connected graph complying with the specifications of Theorem 3 when $a = 2$](image)

Let $S = \{u, v\}$ and

$$T = \begin{cases} \{w_{2i-1}, z_{2i} : i = 1, 2, 3, \ldots, \lfloor k/2 \rfloor + 1\}, & k \text{ is even;} \\
\{w_1, w_{2i+1}, z_{2i} : i = 1, 2, 3, \ldots, \lfloor (k+1)/2 \rfloor\}, & k \text{ is odd.} \end{cases}$$

Then $S$ and $T$ are $\gamma_{hg}$-set and $\gamma_{hg}^+$-set of $G$, respectively. Suppose that $a \geq 3$. Obtain the graph $G = G_2$ as in Figure 3 from $G_1$ by adding to $G_1 (a - 2)$ pendant edges $vx_j$, $j = 1, 2, \ldots, a - 2$. 
Put $S = \{u, v, x_i : i = 1, 2, \ldots, a - 2\}$ and

$$T = \begin{cases} 
\{w_{2i-1}, z_{2i}, x_j : i = 1, 2, 3, \ldots, \frac{k}{2} + 1; j = 1, 2, \ldots, a - 2\}, & k \text{ is even;} \\
\{w_{1}, w_{2i+1}, z_{2i}, x_j : i = 1, 2, 3, \ldots, \lfloor \frac{k+1}{2} \rfloor; j = 1, 2, \ldots, a - 2\}, & k \text{ is odd.}
\end{cases}$$

Then $S$ and $T$ are $\gamma_{hg}$-set and $\gamma^+_{hg}$-set of $G$, respectively. In any case,

$$\gamma_{hg}(G) = a \quad \text{and} \quad \gamma^+_{hg}(G) = a + k = b$$

**Corollary 3.** The difference between $\gamma^+_{hg}(G)$ and $\gamma_{hg}(G)$ can be made arbitrarily large.

### 4. In the join of graphs

For the purposes of the remaining sections, we define the following variations of pointwise non-dominating sets.

A pointwise non-dominating set $S \subseteq V(G)$ is a **minimal pointwise non-dominating set** of $G$ if $S$ does not contain a proper subset which is itself a pointwise non-dominating set in $G$.

A set $S \subseteq V(G)$ is a 2-path closure absorbing pointwise non-dominating set if $S$ is both a 2-path closure absorbing set and a pointwise non-dominating set in $G$. A 2-path closure absorbing pointwise non-dominating set $S$ is said to be a **minimal 2-path closure absorbing pointwise non-dominating set** whenever $S$ does not contain a proper subset which is itself a 2-path closure absorbing pointwise non-dominating set. We denote by $\rho^+_{2\text{nd}}(G)$ the maximum cardinality of a minimal 2-path closure absorbing pointwise non-dominating set in $G$. A minimal 2-path closure absorbing pointwise non-dominating set $S$ is called a $\rho^+_{2\text{nd}}$-set if $|S| = \rho^+_{2\text{nd}}(G)$.

**Theorem 4.** [26] Let $G$ and $H$ be any two graphs. A set $S \subseteq V(G+H)$ is a geodetic hop dominating set of $G+H$ if and only if $S = S_G \cup S_H$, where $S_G$ and $S_H$ are pointwise non-dominating sets of $G$ and $H$, respectively, such that

i) $S_G$ is a 2-path closure absorbing set in $G$ whenever $\langle S_H \rangle$ is a complete subgraph of $H$ and

ii) $S_H$ is a 2-path closure absorbing set in $H$ whenever $\langle S_G \rangle$ is a complete subgraph of $G$.

**Proposition 4.** Let $G$ and $H$ be connected graphs, and let $S \subseteq G + H$. If $S$ is a minimal geodetic hop dominating set of $G + H$, then $S = S_G \cup S_H$, where $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$ are pointwise non-dominating sets of $G$ and $H$, respectively, such that

i) $S_G$ is a minimal 2-path closure absorbing pointwise non-dominating set in $G$ whenever $\langle S_H \rangle$ is a complete subgraph of $H$;
(ii) \( S_H \) is a minimal 2-path closure absorbing pointwise non-dominating set in \( H \) whenever \( \langle S_G \rangle \) is a complete subgraph of \( G \).

Proof. Let \( S \) be a minimal geodetic hop dominating set of \( G + H \). By Theorem 4, \( S = S_G \cup S_H \), where \( S_G \) and \( S_H \) are pointwise non-dominating sets in \( G \) and \( H \), respectively. Suppose that \( \langle S_H \rangle \) is a complete subgraph of \( H \). By Theorem 4, \( S_G \) is a 2-path closure absorbing set in \( G \). Let \( S_G^* \subseteq V(G) \) be a 2-path closure absorbing set in \( G \) with \( S_G^* \subseteq S_G \). By Theorem 4, \( S_G \cup S_H \) is a geodetic hop dominating set of \( G + H \). Since \( S_G^* \cup S_H \subseteq S \), the minimality of \( S \) implies that \( S_G = S_G^* \). Thus, \( S_G \) is a minimal 2-path closure absorbing pointwise non-dominating set in \( G \) and (i) holds. Similarly, if \( \langle S_G \rangle \) is complete, then (ii) holds. \( \square \)

Lemma 1. [1] Let \( G \) be a connected noncomplete graph and \( S \subseteq V(G) \). If \( S \) is a 2-path closure absorbing set in \( G \), then \( \langle S \rangle \) is not complete.

Proposition 5. Let \( G \) and \( H \) be connected graphs, and let \( S \subseteq G + H \). Suppose that \( S = S_G \cup S_H \), where \( S_G \subseteq V(G) \) and \( S_H \subseteq V(H) \) are pointwise non-dominating sets of \( G \) and \( H \), respectively, such that one of the following holds:

(i) \( \langle S_H \rangle \) is a complete subgraph of \( H \) and \( S_G \) is a minimal 2-path closure absorbing pointwise non-dominating set in \( G \).

(ii) \( \langle S_G \rangle \) is a complete subgraph of \( G \) and \( S_H \) is a minimal 2-path closure absorbing pointwise non-dominating set in \( H \).

Then \( S \) is a minimal geodetic hop dominating set of \( G + H \).

Proof. Assume that \( S_G \) and \( S_H \) are pointwise non-dominating sets of \( G \) and \( H \), respectively. Then \( S \) is a hop dominating set in \( G + H \). Assume further that (i) holds for \( S \). Let \( x \in V(G + H) \setminus S \). Suppose that \( x \in V(G) \setminus S \). Since \( S_G \) is a 2-path closure absorbing set in \( G \), there exists \( u, v \in S_G \) for which \( d_G(u, v) = 2 \) and \( x \in I_G(u, v) \). Observe that \( d_{G + H}(u, v) = d_G(u, v) = 2 \) and \( I_{G + H}(u, v) \subseteq I_{G + H}(u, v) \). Suppose that \( x \in V(H) \setminus S \). By Lemma 1, \( \langle S_G \rangle \) is noncomplete. Thus, there exists \( u, v \in S_G \) such that \( d_G(u, v) = 2 \). Then \( d_{G + H}(u, v) = 2 \) and \( x \in I_{G + H}(u, v) \). Since \( x \) is arbitrary, \( S \) is a geodetic set in \( G + H \). Therefore, \( S \) is a geodetic hop dominating set in \( G + H \).

Now, let \( T \subseteq V(G + H) \) be a geodetic hop dominating set in \( G + H \) with \( T \subseteq S \). Write \( T = T_G \cup T_H \), where \( T_G = T \cap V(G) \) and \( T_H = T \cap V(H) \). Then \( T_G \) and \( T_H \) are pointwise non-dominating sets in \( G \) and \( H \), respectively, by Theorem 4. Note that \( T_G \subseteq S_G \) and \( T_H \subseteq S_H \). Since \( \langle S_H \rangle \) is complete, \( \langle T_H \rangle \) is a complete subgraph of \( H \). If \( S_H \setminus T_H \neq \emptyset \) and \( x \in S_H \setminus T_H \), then \( xy \in E(G + H) \) for all \( y \in T_H \), a contradiction since \( T_H \) is a pointwise non-dominating set in \( H \). Thus, \( S_H = T_H \). By Theorem 4, \( T_G \) is a 2-path closure absorbing set in \( G \). Since \( T_G \subseteq S_G \), the minimality of \( S_G \) implies that \( S_G = T_G \). Therefore, \( S = T \) and \( S \) is a minimal geodetic hop dominating set in \( G + H \).

Similarly, if condition (ii) holds, then \( S \) is a minimal geodetic hop domiinating set in \( G + H \). \( \square \)

Corollary 4. Let \( G \) be a nontrivial connected graphs and \( p \geq 1 \), and let \( S \subseteq V(G + K_p) \). Then \( S \) is a minimal geodetic hop dominating set of \( G + K_p \) if and only if \( S = V(K_p) \cup S_G \) where \( S_G \subseteq V(G) \) is a minimal 2-path closure absorbing pointwise non-dominating set of \( G \). More precisely,

\[ \gamma_{hg}^+(G + K_p) = p + \rho_{2 mod}(G). \]
Similarly, \( \gamma \) by Proposition 4, \( S \) is a minimal geodetic hop dominating set in \( G + K_p \). Note that \( \langle S_H \rangle \) is complete and \( S_H = V(K_p) \) is a pointwise non-dominating set in \( H \). Thus, \( S_G \) is a minimal 2-path closure absorbing pointwise non-dominating set in \( G \) by Proposition 4.

Conversely, suppose that \( S_G \) is a minimal 2-path closure absorbing pointwise non-dominating set in \( G \). By Proposition 5, \( S = S_G \cup S_H \) is a minimal geodetic hop dominating set in \( G + H \).

\[ \square \]

**Proposition 6.** Let \( G \) and \( H \) be connected noncomplete graphs. Then

\[ \gamma_{hg}^+(G + H) \geq \max\{\rho_{2pmd}^+(G) + \omega(H), \rho_{2pmd}^+(H) + \omega(G)\}. \]

**Proof.** Let \( S_G \subseteq V(G) \) be a \( \rho_{2pmd}^+ \)-set of \( G \) and \( S_H \subseteq V(H) \) be \( \omega \)-set of \( H \). Because \( H \) is noncomplete, \( V(H) \neq S_H \). Let \( x \in V(H) \setminus S_H \). Since \( \langle S_H \cup \{x\} \rangle \) is a not complete, there exists \( y \in S_H \) for which \( xy \notin E(H) \). Since \( x \) is arbitrary, \( S_H \) is a pointwise non-dominating set in \( H \).

By Proposition 4, \( S = S_G \cup S_H \) is a minimal geodetic hop dominating set in \( G + H \). Thus,

\[ \gamma_{hg}^+(G + H) \geq |S| = \rho_{hg}^+(G) + \omega(H). \]

Similarly, \( \gamma_{hg}^+(G + H) \geq |S| = \rho_{hg}^+(H) + \omega(G) \).

Since \( \gamma_{hg}^+(P_3 + C_3) = 5 = \rho_{2pmd}^+(P_3) + \omega(P_3) \), the lower bound in Proposition 6 is sharp.

5. In the corona of graphs

**Theorem 5.** [27] Let \( G \) and \( H \) be any two graphs. A set \( S \subseteq V(G \circ H) \) is a geodetic hop dominating set of \( G \circ H \) if and only if

\[ S = A \cup (\cup_{v \in V(G)} S_v), \]

where \( A \subseteq V(G) \) and \( S_v \subseteq V(H^v) \) for each \( v \in V(G) \), and satisfies the following conditions:

(i) \( S_v \) is a pointwise non-dominating set in \( H^v \) for each \( v \in V(G) \setminus N_G(A) \);

(ii) For each \( w \in V(G) \setminus A \), one of the following holds:

(a) \( \exists a, b \in S_w \) with \( d_{H^w}(a, b) \neq 1 \);

(b) \( \exists x, y \in V(G) \) with \( w \in I_G(x, y) \);

(c) \( \exists s \in S_w \) and \( t \in A \).

(iii) \( S_v \) is a 2-path closure absorbing set in \( H^v \) for all \( v \in V(G) \).

Observe that if \( G \) is a nontrivial connected graph, then condition (ii) in Theorem 5 may be removed.

**Corollary 5.** Let \( G \) and \( H \) be two graphs, where \( G \) is connected and nontrivial. A set \( S \subseteq V(G \circ H) \) is a geodetic hop dominating set of \( G \circ H \) if and only if

\[ S = A \cup (\cup_{v \in V(G)} S_v), \]

where \( A \subseteq V(G) \) and \( S_v \subseteq V(H^v) \) for each \( v \in V(G) \), and satisfies the following conditions:
Following the same proof as one given in [9] for Theorem 5, \( S_v \) is a pointwise non-dominating set in \( H^v \) for each \( v \in V(G) \setminus N_G(A) \);

(ii) \( S_v \) is a 2-path closure absorbing set in \( H^v \) for all \( v \in V(G) \).

Proof. The necessity part follows from Theorem 5. Suppose that (i) and (ii) hold for \( S \). Following the same proof as one given in [9] for Theorem 5, \( S \) is a hop dominating set in \( G \circ H \) and for every \( w \in V(H^w) \setminus S_w \), there exist \( u, v \in S \) such that \( w \in I_{G+H}(u, v) \). Now, suppose that \( w \in V(G) \setminus A \). Since \( G \) is nontrivial and connected, \( N_G(w) \neq \emptyset \). By (ii), \( S_z \neq \emptyset \) and \( S_w \neq \emptyset \). Pick \( u \in S_z \) and \( v \in S_w \). Then \( w \in I_{G\circ H}(u, v) \).

\[ \square \]

Proposition 7. Let \( G \) and \( H \) be two graphs, where \( G \) is connected of order \( n \geq 2 \). Then
\[ \gamma_{hg}^+(G \circ H) \geq n \cdot \rho_{2pnd}^+(H), \]
and this bound is sharp.

Proof. For each \( v \in V(G) \), let \( S_v \subseteq V(H^v) \) be a \( \rho_{2pnd}^+ \)-set of \( H^v \). By Corollary 5, \( \cup_{v \in V(G)} S_v \) is a geodetic hop dominating set in \( G \circ H \). Let \( T \subseteq V(G \circ H) \) be a geodetic hop dominating set in \( G \circ H \) with \( T \subseteq S \). By Corollary 5 and since \( T \subseteq S \), \( T = \cup_{v \in V(G)} T_v \), where \( T_v \subseteq S_v \) is a 2-path closure absorbing pointwise non-dominating set in \( H^v \) for each \( v \in V(G) \). By the minimality of \( S_v \), \( T_v = S_v \) for each \( v \in V(G) \). Therefore, \( S = T \) and \( S \) is a minimal geodetic hop dominating set in \( G \circ H \). Thus,
\[ \gamma_{hg}^+(G \circ H) \geq |S| = n \cdot \rho_{2pnd}^+(H). \]

Further, since \( \gamma_{hg}^+(P_2 \circ P_2) = 4 = 2 \rho_{2pnd}^+(P_2) \), the given bound is sharp. \( \square \)

Proposition 8. Let \( G \) and \( H \) be any two graphs, where \( G \) is connected and nontrivial. If \( S \subseteq V(G \circ H) \) is a minimal geodetic hop dominating set in \( G \circ H \), then
\[ S = A \cup \left( \cup_{v \in V(G)} S_v \right), \]
where \( A \subseteq V(G) \) and \( S_v \subseteq V(H^v) \) for each \( v \in V(G) \), and satisfies the following conditions:

(i) \( S_v \) is a minimal 2-path closure absorbing pointwise non-dominating set in \( H^v \) for each \( v \in V(G) \setminus N_G(A) \);

(ii) \( S_v \) is a minimal 2-path closure absorbing set in \( H^v \) for all \( v \in V(G) \cap N_G(A) \).

Proof. In view of Corollary 5, we are left to work only on the minimality part. Let \( S \) be a minimal geodetic hop dominating set in \( G \circ H \). Let \( v \in V(G) \setminus N_G(A) \). Let \( D \subseteq V(H^v) \) be a 2-path closure absorbing pointwise non-dominating set in \( H^v \) with \( D \subseteq S_v \). Then \( T = A \cup \left( \cup_{v \in V(G)} S_v \right) \cup D \) is a geodetic hop dominating set of \( G \circ H \). Since \( T \subseteq S \), the minimality of \( S \) implies that \( T = S \). Necessarily, \( S_v = D \), showing that \( S_v \) is a minimal 2-path closure absorbing pointwise non-dominating set in \( H^v \), and (i) holds. Similarly, (ii) holds. \( \square \)

Corollary 6. Let \( G \) and \( H \) be two graphs, where \( G \) is connected of order \( n \geq 2 \) and \( \rho_{2pnd}^+(H) = \rho_2^+(H) \). Then
\[ \gamma_{hg}^+(G \circ H) = n \cdot \rho_{2pnd}^+(H). \]

In particular, for \( p \geq 1 \),
\[ \gamma_{hg}^+(G \circ K_p) = np. \]
Consequently, the minimality of $S$. For nontrivial connected graphs $G$ all $x$ for each $v \in V(G)$ such that $S_x$ is a minimal 2-path closure absorbing pointwise non-dominating set in $H^v$ for each $v \in V(G) \setminus N_G(A)$ and $S_v$ is a minimal 2-path closure absorbing set in $H^v$ for all $v \in V(G) \cap N_G(A)$. Since $S$ is a $\gamma_{hg}^+$-set, $|S_v| = \rho_{2pnd}(H)$ for all $v \in V(G)$. Consequently, the minimality of $S$ implies that $A = \emptyset$. Thus,

$$\gamma_{hg}^+(G \circ H) = |S| = |V(G)| \cdot \rho_{2pnd}(H).$$

\[\Box\]

6. In the lexicographic product of graphs

**Theorem 6.** [27] Let $G$ and $H$ be connected nontrivial graphs. A subset $C = \bigcup_{x \in S} \{\{x\} \times S_x\} \subseteq V(G[H])$, where $S \subseteq V(G)$ and $S_x \subseteq V(H)$, is a geodetic hop dominating set in $G[H]$ if and only if the following conditions hold:

(i) $S$ is a geodetic hop dominating set in $G$;

(ii) $S_x$ is a pointwise non-dominating set in $H$ for each $x \in S \setminus N^2_G(S)$;

(iii) $S_x$ is a 2-path closure absorbing set in $H$ for each $x \in S \setminus I_G(S)$.

The following follows from Theorem 6.

**Corollary 7.** Let $G$ and $H$ be connected nontrivial graphs. Let $C = \bigcup_{x \in S} \{\{x\} \times S_x\} \subseteq V(G[H])$, where $S \subseteq V(G)$ and $S_x \subseteq V(H)$, such that the following conditions hold:

(i) $S$ is a minimal geodetic hop dominating set in $G$;

(ii) $S_x$ is a minimal pointwise non-dominating set in $H$ for each $x \in S \setminus N^2_G(S)$;

(iii) $S_x$ is a minimal 2-path closure absorbing set in $H$ for each $x \in S \setminus I_G(S)$.

Then $C$ is minimal geodetic hop dominating set in $G[H]$.

**Proof.** By Theorem 6, $C$ is a geodetic hop dominating set in $G[H]$. Let $C^* \subseteq C$ be a geodetic hop dominating set in $G[H]$. By Theorem 6, there exist $T \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in T$ such that

$$C^* = \bigcup_{x \in T} \{\{x\} \times T_x\}.$$  

Moreover, $T$ is a geodetic hop dominating set in $G$, $T_x$ is a pointwise non-dominating set in $H$ for each $x \in T \setminus N^2_G(T)$, and $T_x$ is a 2-path closure absorbing set in $H$ for each $x \in T \setminus I_G(T)$.

Since $C^* \subseteq C$, $T \subseteq S$ and $T_x \subseteq S_x$ for each $x \in T$. The minimality of $S$ implies that $T = S$. Consequently, the minimality of $S_x$ implies that $S_x = T_x$ for all $x \in T \setminus N^2_G(T)$ and $S_x = T_x$ for all $x \in T \setminus I_G(T)$. Hence, $C = C^*$, showing that $C$ is a minimal geodetic hop dominating set in $G[H]$.

**Corollary 8.** For nontrivial connected graphs $G$ and $p \geq 2$,

(i) $\gamma_{hg}^+(G[K_p]) = p \cdot \gamma_{hg}^+(G)$.

(ii) $\gamma_{hg}^+(K_p[G]) = p \cdot \rho_{2pnd}(H)$.
Proof. Let $S \subseteq V(G)$ be a $\gamma_{hg}^+$-set of $G$, and let $D = V(K_p)$. Then $C = \bigcup_{x \in S} \{x\} \times D$ is a minimal geodetic hop dominating set in $G[K_p]$ by Corollary 7. Thus,

$$\gamma_{hg}^+(G[K_p]) \geq |C| = |S| \cdot |D| = p \cdot \gamma_{hg}(G).$$

To get the other inequality, let $C = \bigcup_{x \in S} \{x\} \times S_x \subseteq V(G[H])$ be a $\gamma_{hg}^-$-set of $G[K_p]$. By Theorem 6 and the minimality of $C$, $S$ is a minimal geodetic hop dominating set in $G$ and $S_x = V(K_p)$ for all $x \in S \setminus N^2_G(S)$ and for all $x \in S \setminus I_G(S)$. Thus,

$$\gamma_{hg}^+(G[K_p]) = |C| = \sum_{x \in S} |S_x| \leq \sum_{x \in S} |V(K_p)| = p \cdot |S| \leq p \cdot \gamma_{hg}^+(G).$$

This proves (i).

To prove (ii), let $C = \bigcup_{x \in S} \{x\} \times T_x \subseteq V(G[K_p])$ be a $\gamma_{hg}^-$-set of $K_p[G]$. By Theorem 6, $S = V(K_p)$ and $S_x$ is a 2-path closure absorbing pointwise non-dominating set in $H$. Moreover, by the minimality of $C$, $S_x$ is a minimal 2-path closure absorbing pointwise non-dominating set in $H$. Thus,

$$\gamma_{hg}^+(K_p[G]) = |C| = \sum_{x \in V(G)} |S_x| \leq p \cdot \rho_{2pnd}^+(H).$$

To get the other inequality, $D \subseteq V(H)$ be a $\rho_{2pnd}^+$-set of $H$. Then

$$C = \bigcup_{v \in V(K_p)} \{x\} \times D = V(K_p) \times D$$

is a minimal geodetic hop dominating set in $K_p[G]$. Therefore,

$$\gamma_{hg}^+(K_p[G]) \geq |C| = p \cdot \rho_{2pnd}^+(H).$$

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