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Enhanced Results in Common and Coincidence Fixed Point Theory with Applications to Simulation Mappings

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Abstract. In this manuscript, we apply the simulation mappings to present and verify several original results of common and coincidence fixed point in complete $\mathcal{S}$-metric spaces. Moreover, using $\mathcal{S}$-metric to expand and generalized diverse results in the literature involving simulation mappings. On the other hand, we apply our major results to derive several common and coincidence fixed point theorems for right monotone simulation map in complete $\mathcal{S}$-metric. As implementations, various related outcomes of fixed-point theory via specific simulation mappings are obtained in complete $\mathcal{S}$-metric spaces. Additionally, illustrative examples and some applications to solve an integral equation are introduced to support our major results.

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1. Introduction And Preliminaries

Most significant results in fixed point theory is Banach contraction principle $\{$Assume $(\mathcal{X},\ d)$ is a complete metric space. $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is contraction map if $\exists\ q \in (0,\ 1)$ $(s.t)$, $d(\mathcal{T}(x),\ \mathcal{T}(y)) \leq qd(x,\ y),\ \forall\ x, y \in \mathcal{X}\}$. In that case, Banach fixed point theorem (B. F. P. Th) illustrates that $\mathcal{T}$ permanently has unique fixed point. After witnessing the applications of (B. F. P. Th) in giving the existence and uniqueness solutions for a lot of differential and integral equations, diverse extensions of (B. F. P. Th) were completed. Due to applications of Banach contraction principle in all branch of pure and applied mathematics as well in other various sciences, numerous researchers have expanded it in nonlinear analysis (see [2, 13, 17]).

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Newly, Khojasteh et al, in [18], offered the concept of simulation mappings to express diverse contractivity situations in unified method. This class of contraction simplifies the Banach contraction and verifies some kinds of non-linear contractions. In the same year, the authors in [4, 5], independently improved the idea of simulation maps and established various common and coincidence fixed point results involving the most recent type of simulation mappings. Recently, many results involving fixed point, common and coincidence fixed point are established endowed with various kinds of binary relations ([6, 12, 23]).

In 2010, Imdad and Soliman [14], as well Soliman et al. [24] expanded some outcomes in the literature to symmetric spaces using the notion of weakly compatible pair maps together with common (E.A) property (idea due to Liu et al. [20]), for additional information on advance of common fixed point theory in symmetric spaces refer authors to [1, 8, 10, 11, 16].

Recent, B. Alqahtani et al. [3], scrutinized the existence and uniqueness of fixed point in Δ-symmetric quasi-metric spaces utilizing simulation mappings. After that, M. Kumar et al. [19] offered and established various common and coincidence fixed-point results in symmetrical G-metric utilizing the simulation mappings. Soon, T. Hamaizia and P.P. Murthy [9] verified some common fixed-point results for two pairs of maps under the extended Z-contraction with related to the idea of simulation mappings in b-metric spaces.

S.Shaban, et al. [22] introduced the idea of $\mathcal{D}^*$-metric-Sps. In [15] A. AL. Jumaili, employed the idea of $\mathcal{D}^*$- metric-Sps and proved various coincidence fixed point results for mappings satisfying contractive conditions relating to nondecreasing φ-maps in partially ordered $\mathcal{D}^*$-metric. In this article we introduce another extension Al-Argoubi outcomes in [4] utilizing the idea of $\mathcal{S}$-metric spaces. Several basic definitions and essential conclusions under the idea of $\mathcal{S}$-metric spaces and simulation mappings have been presented in the beginning. A novel concept of extended metric spaces introduced via S. Shaban, et al. [22] as follows:

Definition 1. *[22] Suppose $\mathcal{X} \neq \emptyset$. A $\mathcal{D}^*$-metric is a mapping, $\mathcal{D}^* : \mathcal{X}^3 \to [0, +\infty)$, that satisfies the next statements $\forall\, x, y, \mathfrak{z}, \mathfrak{b} \in \mathcal{X}$:*

$(\mathcal{D}_1^*)$ $\mathcal{D}^*(x,y,\mathfrak{z}) \geq 0, \ \forall\, x, y, \mathfrak{z} \in \mathcal{X}$;
$(\mathcal{D}_2^*)$ $\mathcal{D}^*(x,y,\mathfrak{z}) = 0 \iff x = y = \mathfrak{z}$;
$(\mathcal{D}_3^*)$ $\mathcal{D}^*(x,y,\mathfrak{z}) = \mathcal{D}^*(\mathcal{P}\{x,y,\mathfrak{z}\})$, *(Symmetry) where P is permutation mapping*;
$(\mathcal{D}_4^*)$ $\mathcal{D}^*(x,y,\mathfrak{z}) \leq \mathcal{D}^*(x,y,\mathfrak{b}) + \mathcal{D}^*(\mathfrak{b},\mathfrak{z},\mathfrak{z})$.

In that case the map $\mathcal{D}^*$ is called $\mathcal{D}^*$-metric and $(\mathcal{X}, \mathcal{D}^*)$ is namely, $\mathcal{D}^*$-metric.

Example 1. *[22]*

(i) Let $(\mathcal{X}, d)$ be a metric space, then $(\mathcal{X}, \mathcal{D}^*)$, with a mapping $\mathcal{D}^* : \mathcal{X}^3 \to [0, +\infty)$ be defined as follows:

(a) $\mathcal{D}^*(x,y,\mathfrak{z}) = d(x,y) + d(y,\mathfrak{z}) + d(\mathfrak{z},x)$.

(b) $\mathcal{D}^*(x,y,z) = max\ \{d(x,y), d(y,z), d(z,x)\}$.

$\forall\ x, y,\ z \in \mathcal{X}$, *is* $\mathcal{D}^*$*-metric.*

(ii) If $\mathcal{X} = \mathbb{R}$, *then we define:*

$$\mathcal{D}^*(x,y,z) = \begin{cases} 0, & if\ x = y = z \\ max\ \{x,y,z\}, & otherwise. \end{cases}$$

For all $x,\ y,\ z \in \mathcal{X}, (\mathcal{X},\ \mathcal{D}^*)$ *is* $\mathcal{D}^*$*-metric.*

In 2012, S. Shaban, et al. [21] categorizing symmetry condition as common weakness of G-metric & $\mathcal{D}^*$-metric. Therefore, S. Shaban, et al. [21] studied and established a novel meaning of generalized metric space namely, ($\mathcal{S}$-metric space) which as potential improvement of $\mathcal{D}^*$-metric which was studied via S. Shaban,et al. [22] and gave some of their properties.

Definition 2. *[21] Assume* $\mathcal{X} \neq \emptyset.A\mathcal{S}$ *–metric on* $\mathcal{X}$ *is* $\mathcal{S}:\ \mathcal{X}^3 \longrightarrow [0,\ +\infty)$, *satisfies the next statements* $\forall\ x,\ y,\ z,\ a \in \mathcal{X}$,

$$\begin{aligned} &(\mathcal{S}_1)\ \ \mathcal{S}(x,y,z) \geq 0; \\ &(\mathcal{S}_2)\, \mathcal{S}(x,y,z) = 0 \iff x = y = z; \\ &(\mathcal{S}_3)\, \mathcal{S}(x,y,z) \leq \mathcal{S}(x,x,a) + \mathcal{S}(y,y,a) + \mathcal{S}(z,z,a). \end{aligned}$$

Then, $(\mathcal{X},\ \mathcal{S})$ *is said* $\mathcal{S}$ *–metric space.*

Example 2. *[21] Direct examples of such* $\mathcal{S}$*-metric spaces are:*

(i) Assume $\mathcal{X} = \mathbb{R}^n$ *with* $\|.\|$ *is a norm , consequently* $\mathcal{S} = (x,y,z) = \|x - z\| + \|y - z\|$ *is* $\mathcal{S}$*-metric.*

(ii) Assume $\mathcal{X} \neq \emptyset, d$ *is ordinary metric , so* $\mathcal{S}(x,y,z) = d(x,z) + d(y,z)$ *is* $\mathcal{S}$*-metric.*

Remark 1. *[21] It's clear to see each D*-metric-sp is* $\mathcal{S}$*-metric, the converse not true in general, as shown in the example below.*

Example 3. *[21] Suppose* $\mathcal{X} = \mathbb{R}^n$ *with* $\|.\|$ *a norm on* $\mathcal{X}$, *consequently* $\mathcal{S} = (x,y,z) = \|y + z - 2x\| + \|y - z\|$ *is* $\mathcal{S}$*-metric, but it isn't* $\mathcal{D}^*$*-metric since it isn't symmetric.*

Example 4. *[21] Suppose* $\mathcal{X} = \mathbb{R}^2$, *and d is ordinary metric , consequently,* $\mathcal{S}(x,y,z) = d(x,y) + d(x,z) + d(y,z)$ *is* $\mathcal{S}$*-metric on* $\mathcal{X}$. *If connect the points* x, y, z *via a line, we have triangle and select a point mediating this triangle in that case* $\mathcal{S}(x,y,z) \leq \mathcal{S}(x,x,a) + \mathcal{S}(y,y,a) + \mathcal{S}(z,z,a)$ *holds. In fact*

$$\begin{aligned} &\mathcal{S}(x,y,z) = d(x,y) + d(x,z) + d(y,z) \\ &\leq d(x,a) + d(a,y) + d(x,a) + d(a,z) + d(y,a) + d(z,a) \\ &= \mathcal{S}(x,x,a) + \mathcal{S}(y,y,a) + \mathcal{S}(z,z,a). \end{aligned}$$

Lemma 1. *[21] In $\mathcal{S}$-metric-space,* $\mathcal{S}(x,x,y) = \mathcal{S}(y,y,x), \ \forall\ x,\ y \in \mathcal{X}$.

Definition 3. *[21] Suppose* $(\mathcal{X}, \mathcal{S})$ *is* $\mathcal{S}$*-metric and* $\mathcal{A} \subseteq \mathcal{X}$.

(i) $\mathcal{A}$ *is said to* $\mathcal{S}$*-bounded if* $\exists\ \mathfrak{r} > 0(s.t), \mathcal{S}(x,x,y) < \mathfrak{r},\ \forall\ x,y \in \mathcal{A}$.

(ii) A sequence $\{x_s\}$ *in* $\mathcal{X}$ *is called* $\mathcal{S}$*-converges to* $x \in \mathcal{X} \Longleftrightarrow \mathcal{S}(x_s,\ x_s,\ x) \to 0$ *since* $s \to +\infty$. *(i.e)* $\forall\ \varepsilon > 0, \exists\ s_0 \in \mathbb{N}$ *(s.t),* $\forall\ s \geq s_0 \Longrightarrow \mathcal{S}(x_s,\ x_s,\ x) < \varepsilon$ *, and denote this via* $\lim_{s\to+\infty} x_s = x$.

(iii) $\{x_s\}$ *in* $\mathcal{X}$ *is called* $\mathcal{S}$*-Cauchy sequence if* $\forall\ \varepsilon > 0, \exists\ s_0 \in \mathbb{N}$*(s. t),* $\mathcal{S}(x_s,\ x_s,\ x_{\mathfrak{r}}) < \varepsilon,\ \forall\ s,\ \mathfrak{r} \geq s_0$.

(iv) $\mathcal{X}$ *is complete if each* $\mathcal{S}$*-Cauchy sequence of* $(\mathcal{X},\ \mathcal{S})$ *is convergent.*

Now, we mention the concept of simulation mappings which presented via Khojasteh et al [18], as follows:

Definition 4. $\xi : [0,\ +\infty) \times [0,\ +\infty) \to \mathbb{R}$ *is said to be simulation map if it's satisfying the next statements:*

$$(\xi_1)\ \ \xi(0,\ 0) = 0,$$
$$(\xi_2)\ \ \xi(t,\ \omega) < w - t,\ \forall\ , \omega, t > 0,$$
$$(\xi_3)\ \ If\ \{t_n\}\ \ \&\ \{\omega_n\} \subseteq (0,\ +\infty)$$

satisfying

$$\lim_{n\longrightarrow+\infty}\{t_n\} = \lim_{n\longrightarrow+\infty}\{\omega_n\} = \mathcal{L} \in (0,\ +\infty),$$

so

$$\lim_{n\longrightarrow+\infty} sup\xi(t_n,\ \omega_n) < 0.$$

Remark 2. *The authors in [5] modified the condition* (ξ_3) *of simulation mappings as:* (ξ_3^*) *If* $\{t_n\}\ \&\ \{\omega_n\} \subseteq (0,\ \infty)$ *satisfying* $\lim_{n\longrightarrow+\infty}\{t_n\} = \lim_{n\longrightarrow+\infty}\{\omega_n\} = \mathcal{L} \in (0,\ +\infty)$, *and*

$$t_n < \omega_n,\ \forall\ n \in \mathbb{N},\ so\ \lim_{n\longrightarrow+\infty} \sup\xi(t_n,\ \omega_n) < 0.$$

To see various examples of simulation mappings we refer the authors to [5, 7, 18].

The main objective of this article is to investigate and verify another original common and coincidence fixed point theorems in symmetrical complete $\mathcal{S}$-metric spaces involving simulation mappings. Furthermore, by applying our outcomes to derive various common & coincidence fixed point theorems for right monotone simulation maps in generalized metric. Additionally, suitable examples, and some implementations to solve an integral equation are given to support our major results.

2. Some Common and Coincidence Fixed Point Results by Means of Simulation Mappings

In this part, we introduce and investigate several common and coincidence fixed point outcomes utilizing simulation mappings in complete $\mathcal{S}$-metric. In the beginning, generalized several fundamental propositions in the literature, which are needed throughout this work.

Proposition 1. *If $\mathcal{F}, \mathcal{T} : (\mathcal{X}, \mathcal{S}) \to (\mathcal{X}, \mathcal{S})$ are self mappings and $\mathcal{F}$ is $\mathcal{T}$-non-decreasing in a $\mathcal{S}$-metric $(\mathcal{X}, \mathcal{S})$ and satisfies the next cases:*

(i) If $\mathcal{T}(\mathcal{X})$ closed and $\mathcal{F}(\mathcal{X}) \subset \mathcal{T}(\mathcal{X})$, then $\exists\ x^* \in \mathcal{X}$ with $\mathcal{T}x^* \leq \mathcal{F}x^*$. Furthermore, if $\{\mathcal{T}x_s\} \subset \mathcal{X}$ is non-decreasing sequence; (w. r. t. $\leq$) with $\mathcal{T}x_s \to \mathcal{T}\mathfrak{z}$ of $\mathcal{T}(\mathcal{X})$, so $\mathcal{T}p \leq \mathcal{T}(\mathcal{T}p)$ & $\mathcal{T}x_s \leq (\mathcal{T}p)$, $\forall\ s \in \mathbb{N}$.

(ii) If there exists simulation mapping ξ; (s.t), $\forall\ (x,\ y) \in \mathcal{X} \times \mathcal{X}$ & $\mathcal{T}x \leq \mathcal{T}y$, we have

$$\xi\left(\mathcal{S}\left(\mathcal{F}x,\ \mathcal{F}y,\ \mathcal{F}\mathfrak{z}\right),\ \mathcal{M}_1\left(\mathcal{F},\ \mathcal{T},\ x,\ y,\ \mathfrak{z}\right)\right) \geq 0, \tag{1}$$

Where

$$(\xi_1)\ \ \xi(0,\ 0) = 0,$$
$$(\xi_2)\ \ \xi(t,\ w) < w - t,\ \forall\ , w, t > 0,$$
$$(\xi_3)\ \ If\ \{t_n\}\ \ \&\ \{w_n\} \subseteq (0,\ +\infty)$$

$$\mathcal{M}_1(\mathcal{F}, \mathcal{T}, x,\ y,\ \mathfrak{z}) = \max\big\{\mathcal{S}(\mathcal{T}x,\ \mathcal{T}y,\ \mathcal{T}\mathfrak{z}), \mathcal{S}(\mathcal{T}x, \mathcal{F}y,\ \mathcal{T}\mathfrak{z}), \mathcal{S}(\mathcal{T}y,\ \mathcal{F}x,\ \mathcal{T}\mathfrak{z}),\ \mathcal{S}(\mathcal{T}x,\ \mathcal{F}x,\ \mathcal{T}\mathfrak{z}),\ \mathcal{S}(\mathcal{T}y,\ \mathcal{F}y,\ \mathcal{T}\mathfrak{z})\big\}$$

Let x_s be sequence in $\mathcal{X}$ (s. t), $\mathcal{T}x_{s+1} = \mathcal{F}x_s$, $\forall\ s \in \mathbb{N}$. If $\mathcal{T}x_s \neq \mathcal{T}x_{s+1} \forall\ s \in \mathbb{N}$, so

$$\lim_{s \to +\infty} \mathcal{S}(\mathcal{T}x_s, \mathcal{T}x_{s+1}, \mathcal{T}x_{s+1}) = 0.$$

Proof. At the beginning, observe that from the hypothesis, we have $\mathcal{T}x_0 \leq \mathcal{T}x_1 \leq \mathcal{T}x_2 \leq \ldots \leq \mathcal{T}x_s \leq \mathcal{T}x_{s+1}$. It follows from part (vi) that for all $s \geq 1$, we have

$$0 \ \leq\ \xi(\mathcal{S}(\mathcal{F}x_{s-1},\ \mathcal{F}x_s,\ \mathcal{F}x_s),\ \mathcal{M}_1(\mathcal{F},\ \mathcal{T},\ x_{s-1},\ x_s,\ x_s)),$$

That is,

$$0\ \leq \xi\left(\mathcal{S}\left(\mathcal{T}x_s, \mathcal{T}x_{s-1}, \mathcal{T}x_{s+1}\right), \mathcal{M}_1\left(\mathcal{F}, \mathcal{T}, x_{s-1}, x_s, x_s\right)\right)$$

Where,

$$\mathcal{M}_1(\mathcal{F}, \mathcal{T}, x_{s-1}, x_s,\ x_s) = \max\big\{\mathcal{S}(\mathcal{T}x_{s-1}, \mathcal{T}x_s, \mathcal{T}x_s), \mathcal{S}(\mathcal{T}x_{s-1}, \mathcal{F}x_{s-1}, \mathcal{T}x_s), \mathcal{S}(\mathcal{T}x_{s-1}, \mathcal{F}x_{s-1}, \mathcal{T}x_s), \mathcal{S}(\mathcal{T}x_{s-1}, \mathcal{F}x_{s-1}, \mathcal{T}x_s), \mathcal{S}(\mathcal{T}x_{s-1}, \mathcal{F}x_{s-1}, \mathcal{T}x_s)\big\}.$$

Furthermore, by utilizing the assumptions, we have

$$\begin{aligned}\mathcal{M}_1(\mathcal{F},\mathcal{T},x_{s-1},\ x_s,\ x_s) = \max\big\{&\mathcal{S}(\mathcal{T}x_{s-1},\mathcal{T}x_s,\mathcal{T}x_s),\mathcal{S}(\mathcal{T}x_{s-1},\mathcal{T}x_s,\mathcal{T}x_s),\\ &\mathcal{S}(\mathcal{T}x_{s-1},\mathcal{T}x_s,\mathcal{T}x_s),\mathcal{S}(\mathcal{T}x_{s-1},\mathcal{T}x_s,\mathcal{T}x_s),\mathcal{S}(\mathcal{T}x_{s-1},\mathcal{T}x_s,\mathcal{T}x_s)\big\}.\\ &= \max\{\mathcal{S}(\mathcal{T}x_{s-1},\mathcal{T}x_{s-1},\mathcal{T}x_s),\mathcal{S}(\mathcal{T}x_{s-1},\mathcal{T}x_s,\mathcal{T}x_s)\}.\end{aligned}$$

Since $(\mathcal{X},\mathcal{S})$ is $\mathcal{S}$-metric, so we get

$$\mathcal{M}_1(\mathcal{F},\mathcal{T},x_{s-1},\ x_s,\ x_s) = \mathcal{S}(\mathcal{T}x_{s-1},\mathcal{T}x_s,\mathcal{T}x_s).$$

From the condition (ξ_2) of simulation mapping we obtain:

$$\begin{aligned}0 &\le \xi(\mathcal{S}(\mathcal{T}x_s,\mathcal{T}x_{s+1},\mathcal{T}x_{s+1}),\mathcal{S}(\mathcal{T}x_{s-1},\mathcal{T}x_s,\mathcal{T}x_s))\\ &< S(\mathcal{T}x_{s-1},\mathcal{T}x_s,\mathcal{T}x_s) - \mathcal{S}(\mathcal{T}x_s,\mathcal{T}x_{s+1},\mathcal{T}x_{s+1})\end{aligned}$$

Thus,

$$\mathcal{S}(\mathcal{T}x_s,\mathcal{T}x_{s+1},\mathcal{T}x_{s+1}) < S(\mathcal{T}x_{s-1},\mathcal{T}x_s,\mathcal{T}x_s).$$

Which implies that $\{\mathcal{S}(\mathcal{T}x_{s-1},\mathcal{T}x_s,\mathcal{T}x_s)\}$ monotonically decreasing of non-negative real numbers and thus it should be convergent. Therefore, $\exists\ \rho \ge 0$ where:

$$\lim_{s\to+\infty}\mathcal{S}(\mathcal{T}x_s,\mathcal{T}x_{s+1},\mathcal{T}x_{s+1}) = \rho.$$

Suppose that $\rho > 0$. Utilizing the condition (ξ_3) we get

$$0 \le \sup \xi(\mathcal{S}(\mathcal{T}x_s,\mathcal{T}x_{s+1},\mathcal{T}x_{s+1}),\mathcal{S}(\mathcal{T}x_{s-1},\mathcal{T}x_s,\mathcal{T}x_s)) < 0,$$

This is a contraction. Then, we conclude that $\rho = 0$. Therefore,

$$\lim_{s\to+\infty}\mathcal{S}(\mathcal{T}x_s,\mathcal{T}x_{s+1},\mathcal{T}x_{s+1}) = 0,$$

Proposition 2. *If $\mathcal{F},\ \mathcal{T} : (\mathcal{X},\mathcal{S}) \to (\mathcal{X},\mathcal{S})$ are self maps and $\mathcal{F}$ is $\mathcal{T}$-non-decreasing in $\mathcal{S}$-metric $(\mathcal{X},\mathcal{S})$ and satisfies the cases $((i)\ \&\ (ii))$ of Proposition 1. If x_s is sequence (s. t) $\mathcal{T}x_{s+1} \ne \mathcal{T}x_s, \forall\ s \in \mathbb{N}$. Then, $\mathcal{T}x_s$ is bounded sequence.*

Proof. Suppose $\mathcal{T}x_s$ is not bounded. In that case there exists subsequence $\{x_{s_\hbar}\}$ of x_s (s. t) $s_1 = 1$ and $\forall\ j \in \mathbb{N}, s_{\hbar+1}$ is minimum integer satisfying

$$\mathcal{S}(\mathcal{T}x_{s_{\hbar+1}},\mathcal{T}x_{s_{\hbar+1}},\mathcal{T}x_{s_\hbar}) > 1,$$

And

$$\mathcal{S}(\mathcal{T}x_j,\mathcal{T}x_j,\mathcal{T}x_{s_\hbar}) \le 1,$$

For $s_\hbar \le j \le s_{\hbar+1} - 1$. Utilizing triangle inequality, get

$$1 < \mathcal{S}(\mathcal{T}x_{s_\hbar+1},\mathcal{T}x_{s_\hbar+1},\mathcal{T}x_{s_\hbar})$$

$$\leq 2\mathcal{S}\left(\mathcal{T}x_{s_{k}+1},\mathcal{T}x_{s_{k}+1},\mathcal{T}x_{s_{k+1}-1}\right)+\mathcal{S}(\mathcal{T}x_{s_{k}},\mathcal{T}x_{s_{k}},\mathcal{T}x_{s_{k+1}-1})$$
$$\leq 2\mathcal{S}\left(\mathcal{T}x_{s_{k}+1},\mathcal{T}x_{s_{k}+1},\mathcal{T}x_{s_{k+1}-1}\right)+1.$$

Letting $k \to \infty$ in the above inequality and utilizing Proposition 1, we get

$$\mathcal{S}\left(\mathcal{T}x_{s_{k}+1},\mathcal{T}x_{s_{k}+1},\mathcal{T}x_{s_{k}}\right)=1.$$

Utilizing the triangle inequality, we obtain

$$\begin{aligned}1 &< \mathcal{S}\left(\mathcal{T}x_{s_{k}+1},\mathcal{T}x_{s_{k}+1},\mathcal{T}x_{s_{k}}\right)\\ &\leq \mathcal{S}\left(\mathcal{T}x_{s_{k+1}-1},\mathcal{T}x_{s_{k+1}-1},\mathcal{T}x_{s_{k}-1}\right)\\ &\leq 2\mathcal{S}\left(\mathcal{T}x_{s_{k+1}-1},\mathcal{T}x_{s_{k+1}-1},\mathcal{T}x_{s_{k}}\right)+\mathcal{S}(\mathcal{T}x_{s_{k}-1},\mathcal{T}x_{s_{k}-1},\mathcal{T}x_{s_{k}})\\ &\leq 2+\mathcal{S}(\mathcal{T}x_{s_{k}-1},\mathcal{T}x_{s_{k}-1},\mathcal{T}x_{s_{k}}).\end{aligned}$$

Letting $k \to +\infty$ in the above inequality and utilizing Proposition 1, we obtain

$$\lim_{k\to+\infty}\mathcal{S}\left(\mathcal{T}x_{s_{k+1}-1},\mathcal{T}x_{s_{k+1}-1},\mathcal{T}x_{s_{k}-1}\right)=1. \tag{2}$$

Again, due to the triangle inequality yields

$$\left|\mathcal{S}\left(\mathcal{T}x_{s_{k+1}-1},\mathcal{T}x_{s_{k+1}-1},\mathcal{T}x_{s_{k}}\right)-\mathcal{S}\left(\mathcal{T}x_{s_{k}},\mathcal{T}x_{s_{k}},\mathcal{T}x_{s_{k}+1}\right)\right|\leq \mathcal{S}\left(\mathcal{T}x_{s_{k+1}-1},\mathcal{T}x_{s_{k+1}-1},\mathcal{T}x_{s_{k+1}}\right).$$

Permitting $k \to +\infty$ in the above inequality and utilizing Proposition 1, we obtain

$$\lim_{k\to+\infty}\mathcal{S}\left(\mathcal{T}x_{s_{k+1}-1},\mathcal{T}x_{s_{k+1}-1},\mathcal{T}x_{s_{k}}\right)=1. \tag{3}$$

Via similar method, we get

$$\left|\mathcal{S}\left(\mathcal{T}x_{s_{k}-1},\mathcal{T}x_{s_{k}-1},\mathcal{T}x_{s_{k+1}}\right)-\mathcal{S}\left(\mathcal{T}x_{s_{k}-1},\mathcal{T}x_{s_{k}-1},\mathcal{T}x_{s_{k+1}-1}\right)\right|\leq \mathcal{S}\left(\mathcal{T}x_{s_{k+1}},\mathcal{T}x_{s_{k+1}},\mathcal{T}x_{s_{k+1}-1}\right).$$

Permitting $k \to \infty$ in the above inequality and utilizing Proposition 1, we get

$$\lim_{k\to+\infty}\mathcal{S}\left(\mathcal{T}x_{s_{k}-1},\mathcal{T}x_{s_{k}-1},\mathcal{T}x_{s_{k+1}}\right)=1. \tag{4}$$

Now, utilizing Equations 2, 3, 4 and proposition 1, we obtain

$$\mathcal{M}_1\left(\mathcal{F},\mathcal{T},x_{s_{k+1}-1},x_{s_{k+1}-1},x_{s_{k}-1}\right)=1. \tag{5}$$

Using Equations 1, 2, 3, 4, 5 and the condition (ξ_3) of Definition 4, we obtain

$$0\leq \lim_{k\to+\infty}\sup\xi(\mathcal{S}\left(\mathcal{T}x_{s_{k}+1},\mathcal{T}x_{s_{k}+1},\mathcal{T}x_{s_{k}}\right),\mathcal{M}_1(\mathcal{F},\mathcal{T},x_{s_{k+1}-1},x_{s_{k+1}-1},x_{s_{k}-1}))<0,$$

This is contradiction. This completes the evidence.

Proposition 3. *If $\mathcal{F},\mathcal{T}:(\mathcal{X},\mathcal{S})\to(\mathcal{X},\mathcal{S})$ are self mappings and $\mathcal{F}$ is $\mathcal{T}$-non-decreasing in$\mathcal{S}$-metric $(\mathcal{X},\mathcal{S})$ and satisfies the cases ((i) & (ii)) of Proposition 1. If x_s is sequence (s. t), $\mathcal{T}x_{s+1}=\mathcal{F}x_s$, & $\mathcal{T}x_{s+1}\neq\mathcal{T}x_s$, $\forall\ s\in\mathbb{N}$, then $\mathcal{T}x_s$ is Cauchy sequence.*

Proof. Presume $\mathcal{A}_s = sup\left\{\mathcal{S}\left(\mathcal{T}x_u, \mathcal{T}x_u, \mathcal{T}x_v\right) : u, v \geq s\right\}$.

From Proposition 2, we know that the sequence $\mathcal{T}x_s$ is bounded. Hence, $\mathcal{A}_s < \infty$, for all $s \in \mathbb{N}$ which implies that, $\mathcal{A}_s$ is bounded and monotonically decreasing sequence, thus is convergent. Consequently, $\exists\ \mathcal{A} \geq 0$ were

$$\lim_{s\to+\infty}\mathcal{A}_s = \mathcal{A}.$$

Next establish $\mathcal{A} = 0$, to show that $\mathcal{T}x_s$ is a Cauchy sequence. Assume $\mathcal{A} > 0$. Via definition of $\mathcal{A}_s$, $\forall\ k \in \mathbb{N}\ \exists\ s_k, r_k \in \mathbb{N}, (s.t), r_k > s_k \geq k and\ \frac{k\ \mathcal{A}_k - 1}{k} < \mathcal{S}\left(\mathcal{T}x_{r_k}, \mathcal{T}x_{r_k}, \mathcal{T}x_{s_k}\right) \leq \mathcal{A}_k$. Therefore,

$$\lim_{k\to+\infty} \mathcal{S}\left(\mathcal{T}x_{r_k}, \mathcal{T}x_{r_k}, \mathcal{T}x_{s_k}\right) = \mathcal{A}. \tag{6}$$

Utilizing triangle inequality, Lemma 1 and Proposition 2, we obtain:

$$\begin{aligned}\mathcal{S}\left(\mathcal{T}x_{r_k}, \mathcal{T}x_{r_k}, \mathcal{T}x_{s_k}\right) &\leq \mathcal{S}\left(\mathcal{T}x_{r_{k-1}}, \mathcal{T}x_{r_{k-1}}, \mathcal{T}x_{s_{k-1}}\right) \\ &\leq 2\mathcal{S}\left(\mathcal{T}x_{r_{k-1}}, \mathcal{T}x_{r_{k-1}}, \mathcal{T}x_{r_k}\right) + 2\mathcal{S}\left(\mathcal{T}x_{s_k}, \mathcal{T}x_{s_k}, \mathcal{T}x_{s_{k-1}}\right) \\ &+ \mathcal{S}\left(\mathcal{T}x_{r_k}, \mathcal{T}x_{r_k}, \mathcal{T}x_{s_k}\right).\end{aligned}$$

Utilizing Proposition 2, and Equation 6 and letting $k \to +\infty$, we get

$$\lim_{k\to+\infty} \mathcal{S}\left(\mathcal{T}x_{r_{k-1}}, \mathcal{T}x_{r_{k-1}}, \mathcal{T}x_{s_{k-1}}\right) = \mathcal{A}. \tag{7}$$

Similarly, we can prove that

$$\lim_{k\to+\infty} \mathcal{S}\left(\mathcal{T}x_{r_{k-1}}, \mathcal{T}x_{r_{k-1}}, \mathcal{T}x_{s_k}\right) = \mathcal{A}. \tag{8}$$

And

$$\lim_{k\to+\infty} \mathcal{S}\left(\mathcal{T}x_{s_{k-1}}, \mathcal{T}x_{s_{k-1}}, \mathcal{T}x_{r_k}\right) = \mathcal{A}. \tag{9}$$

Utilizing Propositions 2 and Equations 7, 8, 9, we obtain

$$\lim_{k\to+\infty} \mathcal{M}_1\left(\mathcal{F}, \mathcal{T}, x_{r_k-1}, x_{r_k-1}, x_{s_k-1}\right) = \mathcal{A}. \tag{10}$$

Utilizing the condition of simulation mapping (ξ_3) and 1,6,10, we get

$$0 \leq \lim_{k\to+\infty} sup\xi(\mathcal{S}\left(\mathcal{T}x_{r_k}, \mathcal{T}x_{r_k}, \mathcal{T}x_{s_k}\right), \mathcal{M}_1(\mathcal{F}, \mathcal{T}, x_{r_k-1}, x_{r_k-1}, x_{s_k-1})) < 0.$$

This is a contradiction. Therefore, we have $\mathcal{A} = 0$, that is, $\lim_{s\to+\infty} \mathcal{A}_s = \mathcal{A}$. Thus, this proves $\mathcal{T}x_s$ is Cauchy sequence. Next, introduce the first major outcome in our article.

Theorem 1. *If $\mathcal{F}, \mathcal{T} : (\mathcal{X}, \mathcal{S}) \to (\mathcal{X}, \mathcal{S})$ are self maps and $\mathcal{F}$ is $\mathcal{T}$-non-decreasing in complete $\mathcal{S}$-metric $(\mathcal{X}, \mathcal{S})$ and satisfies each cases of Proposition 1. if there exists simulation map; (s. t), $\forall\ (x,\ y) \in \mathcal{X} \times \mathcal{X}$ and $\mathcal{T}x \leq \mathcal{T}y$, we have*

$$\xi\left(\mathcal{S}\left(\mathcal{F}x,\ \mathcal{F}y,\ \mathcal{F}z\right),\ \mathcal{M}_1\left(\mathcal{F},\ \mathcal{T},\ x,\ y,\ z\right)\right) \geq 0.$$

Where,

$$\mathcal{M}_1(\mathcal{F},\mathcal{T},x,\ y,\ z)=\max\big\{\mathcal{S}(\mathcal{T}x,\ \mathcal{T}y,\ \mathcal{T}z),\mathcal{S}(\mathcal{T}x,\ \mathcal{F}y,\ \mathcal{T}z),\ \mathcal{S}(\mathcal{T}y,\ \mathcal{F}x,\ \mathcal{T}z),\\ \mathcal{S}(\mathcal{T}x,\ \mathcal{F}x,\ \mathcal{T}z),\ \mathcal{S}(\mathcal{T}y,\ \mathcal{F}y,\ \mathcal{T}z)\big\}.$$

So, $\mathcal{F}$ & $\mathcal{T}$ have coincidence point. Additional, if $\mathcal{F}$ & $\mathcal{T}$ commute, in that case $\mathcal{F}$ & $\mathcal{T}$ have common fixed point.

Proof. Via Proposition 3, we have $\mathcal{T}x_s$ is Cauchy and via the completeness of $\mathcal{X}$ $\exists$ some $p\in\mathcal{X}$ satisfying,

$$\mathcal{T}x_s\to\mathcal{T}p,\ when\ s\to+\infty. \tag{11}$$

Now, explain p is coincidence point of $\mathcal{F}$ & $\mathcal{T}$. Presume $\mathcal{S}(\mathcal{F}p,\mathcal{F}p,\mathcal{T}p)>0$. Letting $s\to+\infty$, and utilizing 11) we get:

$$\begin{aligned}\mathcal{M}_1(\mathcal{F},\mathcal{T},x_s,x_s,p)=&\max\big\{\mathcal{S}(\mathcal{F}x_s,\mathcal{F}x_s,\mathcal{F}p),\mathcal{S}(\mathcal{F}x_s,\mathcal{T}p,\mathcal{F}p),\\ &\mathcal{S}(\mathcal{F}p,\mathcal{T}x_s,\mathcal{F}p),\mathcal{S}(\mathcal{F}x_s,\mathcal{T}x_s,\mathcal{F}p),\mathcal{S}(\mathcal{F}p,\mathcal{T}p,\mathcal{F}p)\big\}\\ &=\max\big\{\mathcal{S}(\mathcal{F}p,\mathcal{F}p,\mathcal{F}p),\mathcal{S}(\mathcal{F}p,\mathcal{T}p,\mathcal{F}p),\mathcal{S}(\mathcal{F}p,\mathcal{T}p,\mathcal{F}p),\\ &\mathcal{S}(\mathcal{F}p,\mathcal{T}p,\mathcal{F}p),\mathcal{S}(\mathcal{F}p,\mathcal{T}p,\mathcal{F}p)\big\}\\ &=\mathcal{S}(\mathcal{F}p,\mathcal{F}p,\mathcal{T}p)>0.\end{aligned}$$

On the other hand, utilizing 1, 11 and the case (ξ_3), obtain:

$$0\leq\lim_{k\to+\infty}sup\xi(\mathcal{S}(\mathcal{F}p,\mathcal{F}p,\mathcal{T}x_{s+1}),\mathcal{M}_1(\mathcal{F},\mathcal{T},x_s,x_s,p))<0.$$

This is a contradiction. Hence, we have $\mathcal{S}(\mathcal{F}p,\mathcal{F}p,\mathcal{T}p)=0$. Thus, p is coincident point of $\mathcal{F}$ & $\mathcal{T}$. Now, assume $\mathcal{F}$ & $\mathcal{T}$ commute at their coincident point p. Put $q=\mathcal{T}p=\mathcal{F}p$. Then, $\mathcal{F}q=\mathcal{F}(\mathcal{T}p)=\mathcal{T}(\mathcal{F}p)=\mathcal{T}q$. Via part (v), we have $\mathcal{T}p\leq\mathcal{T}(\mathcal{T}p)=\mathcal{T}q$.

$$\begin{aligned}\mathcal{M}_1(\mathcal{F},\mathcal{T},q,q,p)&=\max\big\{\mathcal{S}(\mathcal{T}q,\mathcal{T}q,\mathcal{T}p),\mathcal{S}(\mathcal{T}q,\mathcal{F}q,\mathcal{T}p),\mathcal{S}(\mathcal{T}p,\mathcal{F}q,\mathcal{T}p),\\ &\mathcal{S}(\mathcal{T}q,\mathcal{F}q,\mathcal{T}p),\mathcal{S}(\mathcal{T}p,\mathcal{F}p,\mathcal{T}p)\big\}\\ &=\max\mathcal{S}(\mathcal{T}q,\mathcal{T}q,q),\mathcal{S}(\mathcal{T}q,\mathcal{T}q,q),\mathcal{S}(q,\mathcal{F}q,q),\mathcal{S}(\mathcal{T}q,\mathcal{F}q,q),\mathcal{S}(q,q,q)\end{aligned}$$

Since, $(\mathcal{X},\mathcal{S})$ is $\mathcal{S}$-metric. Thus, $\mathcal{M}_1(\mathcal{F},\mathcal{T},q,q,p)=\mathcal{S}(\mathcal{T}q,\mathcal{T}q,\ q)$.

Via 1 and the condition (ξ_3), we obtain

$$\begin{aligned}0&\leq\lim_{k\to+\infty}\sup\xi(\mathcal{S}(\mathcal{F}q,\mathcal{F}q,\mathcal{F}p),\mathcal{M}_1(\mathcal{F},\mathcal{T},q,q,p))\\ &=\lim_{k\to+\infty}\sup\xi(\mathcal{S}(\mathcal{F}q,\mathcal{F}q,q),\mathcal{S}(\mathcal{F}q,\mathcal{F}q,q))<0,\end{aligned}$$

This is a contradiction. Thus, we have $\mathcal{S}(\mathcal{T}q,\mathcal{T}q,q)=0,\Longrightarrow\mathcal{T}q=\mathcal{F}q=q$. In that case, the common fixed point of $\mathcal{F}$ & $\mathcal{T}isq$.

Now, via means of simulation mapping and Theorem 1, can present numerous outcomes of common and coincidence fixed point.

Corollary 1. *If $\mathcal{F}$, $\mathcal{T} : (\mathcal{X}, \mathcal{S}) \to (\mathcal{X}, \mathcal{S})$ are self mappings and $\mathcal{F}$ is $\mathcal{T}$-non-decreasing in complete $\mathcal{S}$-metric $(\mathcal{X}, \mathcal{S})$ and satisfies the cases ((i) & (ii)) of Proposition 1. If there exists monotone simulation mapping ξ; (s. t) $\forall$ $(x, y) \in \mathcal{X} \times \mathcal{X}$ & $\mathcal{T}x \leq \mathcal{T}y$, we have*

$$\xi\left(\mathcal{S}\left(\mathcal{F}x,\ \mathcal{F}y,\ \mathcal{F}z\right),\ \mathcal{S}\left(\mathcal{T}x,\ \mathcal{T}y,\ \mathcal{T}z\right)\right) \geq 0.$$

So, $\mathcal{F}$ & $\mathcal{T}$ have coincidence point. Additional, if $\mathcal{F}$ & $\mathcal{T}$ commute, so $\mathcal{F}$ & $\mathcal{T}$ have a common fixed point.

Proof. For all $x, y, z \in \mathcal{X}$

$$\mathcal{S}\left(\mathcal{T}x,\ \mathcal{T}y,\ \mathcal{T}z\right) \leq \mathcal{M}_1\left(\mathcal{F}, \mathcal{T}, x,\ y, z\right). \tag{12}$$

Suppose that $\xi : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is described as: $\xi\left(t, w\right) = \lambda w - t$,

For $\lambda \in [0, 1)$. Utilizing the given supposing, we obtain

$$\begin{aligned} 0 &\leq \xi(\mathcal{S}(\mathcal{F}x, \mathcal{F}y, \mathcal{F}z),\ \mathcal{S}(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) \\ &< \mathcal{S}\left(\mathcal{T}x,\ \mathcal{T}y,\ \mathcal{T}z\right) - \mathcal{S}\left(\mathcal{F}x, \mathcal{F}y, \mathcal{F}z\right) \end{aligned}$$

By using 12, we get

$$\begin{aligned} \mathcal{S}\left(\mathcal{F}x, \mathcal{F}y, \mathcal{F}z\right) &< \mathcal{S}\left(\mathcal{T}x,\ \mathcal{T}y,\ \mathcal{T}z\right) \\ &\leq \mathcal{M}_1\left(\mathcal{F}, \mathcal{T}, x, y, z\right), \end{aligned}$$

This implies that

$$\xi(\mathcal{S}(\mathcal{F}x, \mathcal{F}y, \mathcal{F}z)\,, \mathcal{M}_1\left(\mathcal{F}, \mathcal{T}, x, y, z\right) \geq 0.$$

Consequently, via Theorem 1 $\mathcal{F}$ & $\mathcal{T}$ have common point and coincidence fixed point.

Corollary 2. *Assume $\mathcal{F} : (\mathcal{X}, \mathcal{S}) \to (\mathcal{X}, \mathcal{S})$ is self map in complete $\mathcal{S}-metric\,(\mathcal{X}, \mathcal{S})$, so exists $x^* \in \mathcal{X}$ (s. t), $x^* \leq \mathcal{F}x^*$; if $(x, y) \in \mathcal{X} \times \mathcal{X},\ x \leq y \Longrightarrow \mathcal{F}x \leq \mathcal{F}y$; Additionally, if $\{x_s\} \subset \mathcal{X}$ is nondecreasing; (w. r. t. $\leq$) with $x_s \to p$, $\forall s \in \mathbb{N}$ and there exists monotone simulation map ξ; (s. t), $\forall$ $(x, y) \in \mathcal{X} \times \mathcal{X}$ & $x \leq y$, we get*

$$\xi\left(\mathcal{S}\left(\mathcal{F}x,\ \mathcal{F}y,\ \mathcal{F}z\right),\ \mathcal{M}_2\left(\mathcal{F}, x, y, z\right)\right) \geq 0$$

where

$$\mathcal{M}_2\left(\mathcal{F}, x, y, z\right) = \max\left\{\mathcal{S}(x, y, z)\,, \mathcal{S}(x, \mathcal{F}y, z), \mathcal{S}\left(y, \mathcal{F}x, z\right), \mathcal{S}\left(x, \mathcal{F}x, z\right), \mathcal{S}\left(y, \mathcal{F}y, z\right)\right\}.$$

Then, $\{\mathcal{F}^s x_0\}$ converges to fixed point in $\mathcal{F}$.

Proof. Consequence immediately of Theorem 1 via choosing $\mathcal{T}$ as the identity map. Now, introduce instructive example, which displays the interest of Theorem 1.

Example 5 (23). *Assume that* $\mathcal{X} = [0,\ 1]$ *with* $\mathcal{S}$*-metric described via* $\mathcal{S}(x,y,z) = \max\{|z-x|, |x-y|, |y-z|\}$ *for each* $x,y,z \in \mathcal{X}$. *Suppose,* $z \leq y \leq x$. *Consequently,* $\mathcal{S}(x,y,z) = |x-z|$. *Define the mappings* $\mathcal{F}, \mathcal{T} : (\mathcal{X}, \mathcal{S}) \to (\mathcal{X}, \mathcal{S})$ *by* $\mathcal{F}x = \frac{1}{25}x$ & $\mathcal{T}x = \frac{1}{5}x,\ \forall\, x \in \mathcal{X}$. *Obviously, the cases* (i) *to* (v) *of Theorem 1 are satisfied at* $x^* = 0$. *Presume* $\xi :\ \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ *given via* $\xi(t, w) = \lambda w - t$, *For,* $\lambda \in [0,1)$. *Certainly* $\forall\ x \neq y \neq z$, *get*

$$\xi(\mathcal{S}(\mathcal{F}x, \mathcal{F}y, \mathcal{F}z), \mathcal{M}_1(\mathcal{F}, \mathcal{T}, x, y, z)) = \lambda \mathcal{M}_1(\mathcal{F}, \mathcal{T}, x, y, z) - \mathcal{S}(\mathcal{F}x, \mathcal{F}y, \mathcal{F}z).$$

In particular if we select, $\lambda = \frac{1}{3}$, *we find*

$$\xi(\mathcal{S}(\mathcal{F}x, \mathcal{F}y, \mathcal{F}z), \mathcal{M}_1(\mathcal{F}, \mathcal{T}, x, y, z)) = \frac{1}{3}\mathcal{M}_1(\mathcal{F}, \mathcal{T}, x, y, z) - \mathcal{S}(\mathcal{F}x, \mathcal{F}y, \mathcal{F}z). \tag{13}$$

$\forall\ x, y, z \in \mathcal{X}$. *we have*

$$\begin{aligned} \mathcal{S}(\mathcal{F}x, \mathcal{F}y, \mathcal{F}z) &= \left|\frac{1}{25}x - \frac{1}{25}z\right| \ \leq \frac{1}{4}\ \left|\frac{1}{5}x - \frac{1}{5}z\right| \\ &= \frac{1}{4}\mathcal{S}(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) \leq \frac{1}{3}\mathcal{M}_1(\mathcal{F}, \mathcal{T}, x, y, z) \end{aligned}$$

This means,

$$\frac{1}{3}\mathcal{M}_1(\mathcal{F}, \mathcal{T}, x, y, z) - \mathcal{S}(\mathcal{F}x, \mathcal{F}y, \mathcal{F}z) \geq 0. \tag{14}$$

Utilizing of 13 and 14, we get

$$\xi(\mathcal{S}(\mathcal{F}x, \mathcal{F}y, \mathcal{F}z), \mathcal{M}_1(\mathcal{F}, \mathcal{T}, x, y, z)) \geq 0.$$

Then, all suppositions of Theorem 1 satisfied. Therefore, $\mathcal{F}$ & $\mathcal{T}$ *have coincident* $(\ 0 \in \mathcal{X})$. *Moreover,* $\mathcal{F}$ & $\mathcal{T}$ *commute at point 0, which illustrate 0 unique common fixed point of a maps* $\mathcal{F}$ & $\mathcal{T}$.

3. Various Common and Coincidence Fixed Point Results Utilize Right Monotone Simulation Maps

In this section, utilize right monotone simulation mappings to deduce various common and coincidence fixed point outcomes in symmetrical complete $\mathcal{S}$-metric.

Definition 5. *[4]* $\xi :\ [0,+\infty) \times\ [0,+\infty) \to \mathbb{R}$, *is called right-monotone simulation mapping, if it's a simulation mapping that satisfies for each* $t,\ w_1, w_2 \geq 0, If\, w_1 \leq w_2$, *then* $\xi(t, w_1) \leq \xi(t, w_2)$,

Example 6. *Suppose that* $\xi :\ [0,+\infty) \times\ [0,+\infty) \to \mathbb{R}$ *be a mapping described as follows:*

$$\xi\ (t, w) = w - \frac{t+2}{t+1}\ t, \forall\ t,\ w \geq 0.$$

So, ξ *is right-monotone simulation mapping.*

Remark 3. *It's clear that each right-monotone simulation mapping is simulation mapping; the converse needn't to be true in general.*

Example 7. *Suppose* $\xi : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$, *is mapping described as:*

$$\xi(t,\omega) = |sint| - \omega, \ \forall\, t,\omega \geq 0.$$

In that case, ξ *is simulation mapping, but it isn't right monotone simulation mapping.*

Theorem 2. *If* $\mathcal{F},\ \mathcal{T} : (\mathcal{X},\mathcal{S}) \to (\mathcal{X},\mathcal{S})$ *are self maps and* $\mathcal{F}$ *is* $\mathcal{T}$*-non-decreasing in* $\mathcal{S}$*-metric* $(\mathcal{X},\mathcal{S})$ *and satisfies the case* (**i**) *of Proposition 1. If there exists right monotone simulation mapping* ξ*; (s. t),* $\forall\ (x,y) \in \mathcal{X} \times \mathcal{X}$*, and* $\mathcal{T}x \leq \mathcal{T}y$*, we have* $\xi\big(\mathcal{S}\big(\mathcal{F}x,\ \mathcal{F}y,\ \mathcal{F}z\big),\ \mathcal{S}(\mathcal{T}x,\ \mathcal{T}y,\ \mathcal{T}z)\big) \geq 0$*, So,* $\mathcal{F}$ *&* $\mathcal{T}$ *have coincidence point. Additional, if* $\mathcal{F}$ *&* $\mathcal{T}$ *commute, so* $\mathcal{F}$ *&* $\mathcal{T}$ *have common fixed point.*

Proof. Choosing $t = \mathcal{S}(\mathcal{F}x,\ \mathcal{F}y,\ \mathcal{F}z)$, $\omega_1 = \mathcal{S}(\mathcal{T}x,\ \mathcal{T}y,\ \mathcal{T}z)$ and $\omega_2 = \mathcal{M}_1(\mathcal{F},\mathcal{T},x,y,z)$. due to the given supposition, we have

$$\xi\left(\mathcal{S}(\mathcal{F}x,\ \mathcal{F}y,\ \mathcal{F}z),\ \mathcal{S}(\mathcal{T}x,\ \mathcal{T}y,\ \mathcal{T}z)\right) \tag{15}$$

We know that

$$\mathcal{S}(\mathcal{T}x,\ \mathcal{T}y,\ \mathcal{T}z) \leq \mathcal{M}_1(\mathcal{F},\mathcal{T},x,y,z) \tag{16}$$

By utilizing of 16 and the part ξ_4 of right-monotone simulation map of Definition 5, get

$$\xi(\mathcal{S}(\mathcal{F}x,\ \mathcal{F}y,\ \mathcal{F}z),\mathcal{S}(\mathcal{T}x,\ \mathcal{T}y,\ \mathcal{T}z)) \leq \xi(\mathcal{S}(\mathcal{F}x,\ \mathcal{F}y,\ \mathcal{F}z),\mathcal{M}_1(\mathcal{F},\mathcal{T},x,y,z)). \tag{17}$$

By using 15 and 17, we obtain

$$\xi(\mathcal{S}(\mathcal{F}x,\ \mathcal{F}y,\ \mathcal{F}z),\mathcal{M}_1(\mathcal{F},\mathcal{T},x,y,z)) \geq 0.$$

Next, by similar procedure of Theorem 1, acquire common fixed and coincidence point of $\mathcal{F}$ & $\mathcal{T}$.

Corollary 3. *Suppose* $\mathcal{F},\ \mathcal{T} : (\mathcal{X},\mathcal{S}) \to (\mathcal{X},\mathcal{S})$ *are self mappings and* $\mathcal{F}$ *is* $\mathcal{T}$*-non-decreasing in complete* $\mathcal{S}$*-metric* $(\mathcal{X},\mathcal{S})$ *and satisfies the case* (**i**) *of Proposition 1. If there exists a monotone simulation mapping* ξ*; (s. t)* $\forall\ (x,\ y) \in \mathcal{X} \times \mathcal{X}$ & $\mathcal{T}x \leq \mathcal{T}y$*, we have*

$$\xi\left(\mathcal{S}(\mathcal{F}x,\ \mathcal{F}y,\ \mathcal{F}z),\ \mathcal{M}_3(\mathcal{F},\mathcal{T},x,\ y,z)\right) \geq 0,$$

where

$$\mathcal{M}_3(\mathcal{F},\mathcal{T},x,\ y,z) = \max\left\{\mathcal{S}(\mathcal{T}x,\mathcal{F}x,\mathcal{T}z),\mathcal{S}(\mathcal{T}y,\mathcal{F}y,\mathcal{T}z)\right\}.$$

So, $\mathcal{F}$ *&* $\mathcal{T}$ *have coincidence point. Additional, if* $\mathcal{F}$ *&* $\mathcal{T}$ *commute, so* $\mathcal{F}$ *&* $\mathcal{T}$ *have common fixed point.*

Proof. it can be established independently via choosing the following right monotone simulation mapping $\xi : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$, where

$$\xi\left(t, \mathfrak{w}\right) = \mathfrak{w} - \frac{t+2}{t+1}\, t, \quad \forall \;\; t, \mathfrak{w} \geq 0.$$

Corollary 4. *Suppose $\mathcal{F}$, $\mathcal{T} : (\mathcal{X}, \mathcal{S}) \to (\mathcal{X}, \mathcal{S})$ are self mappings and $\mathcal{F}$ is $\mathcal{T}$-non-decreasing in complete $\mathcal{S}$-metric $(\mathcal{X}, \mathcal{S})$ satisfies the case* (**i**) *of Proposition 1. If there exists a monotone simulation mapping ξ; (s. t) $\forall$ $(x, \; y) \in \mathcal{X} \times \mathcal{X}$ & $\mathcal{T}x \leq \mathcal{T}y$, we have*

$$\xi\left(\mathcal{S}\left(\mathcal{F}x, \; \mathcal{F}y, \; \mathcal{F}\mathfrak{z}\right), \; \mathcal{M}_4\left(\mathcal{F}, \mathcal{T}, x, y, \mathfrak{z}\right)\right) \geq 0,$$

where

$$\mathcal{M}_4\left(\mathcal{F}, \mathcal{T}, x, y, \mathfrak{z}\right) = \max\left\{\mathcal{S}(\mathcal{T}x, \mathcal{T}y, \mathcal{T}\mathfrak{z}), \mathcal{S}(\mathcal{T}x, \mathcal{F}y, \mathcal{T}\mathfrak{z}), \mathcal{S}(\mathcal{T}y, \mathcal{F}x, \mathcal{T}\mathfrak{z})\right\}.$$

So, $\mathcal{F}$ & $\mathcal{T}$ have coincidence point. Additional, if $\mathcal{F}$ & $\mathcal{T}$ commute, so $\mathcal{F}$ & $\mathcal{T}$ have common fixed point.

Proof. It can be established independently via choosing the following right monotone simulation mapping $\xi : \; [0, +\infty) \times [0, +\infty) \to \mathbb{R}$, where

$$\xi\left(t, \mathfrak{w}\right) = \mathfrak{w} - \frac{t+2}{t+1}\, t, \; \forall \; t, \mathfrak{w} \geq 0.$$

4. Applications of Integral Equations

This segment devoted to introduce an application to explain the existence and uniqueness problem of the solution to an integral equation of the following structure in $\mathcal{S}$-metric spaces:

$$\delta\left(q\right) = j(q) + \lambda \int_{\mathfrak{r}}^{s} \psi\left(q, p, p\right) \mu\left(p, \eta\left(p\right), \eta\left(p\right)\right) dp \tag{18}$$

Assume $\mathcal{F}(\mathcal{X}, \mathcal{S}) \to (\mathcal{X}, \mathcal{S})$ a self-map defined as follows:

$$\mathcal{F}\delta\left(q\right) = j(q) + \lambda \int_{\mathfrak{r}}^{s} \psi\left(q, p, p\right) \mu\left(p, \delta\left(p\right), \delta\left(p\right)\right) dp$$

Let $\mathcal{X}$ be provided with $\mathcal{S}$-metric which is described as $\mathcal{S}\left(\delta, j, j\right) = 2\sup\left|\delta\left(q\right) - j(q)\right|$.
Where, $q \in [\mathfrak{r}, s]$, and $\eta : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

Theorem 3. *Let the following suppositions hold:*

(i) $\sup \int_{\mathfrak{r}}^{s} \psi\left(q, p, p\right) dp \leq \frac{1}{2(s-\mathfrak{r})}$,

(ii) $\mathcal{S}\left(p, \delta, \delta\right) - \mathcal{S}\left(p, j, j\right) \leq \varphi\left(\left|\delta - j\right|\right)$;

(iii) $|\lambda| \leq 1,$

Where φ is nondecreasing continuous mapping having $\varphi(n) < n,\ \forall\ n > 0$. So, integral Equation 18 has unique solution.

Proof. For $\delta_1,\ \delta_2 \in \mathcal{X}$, we have

$$
\begin{aligned}
&\mathcal{S}\left(\mathcal{F}\delta_1, \mathcal{F}\delta_2, \mathcal{F}\delta_2\right) = 2\sup\left|\mathcal{F}\delta_1\left(q\right) - \mathcal{F}\delta_2(q)\right| \\
&= 2sup\left|j\left(q\right) + \lambda\int_{\imath}^{s}\psi\left(q,p,p\right)\mu\left(p,\delta_1\left(p\right),\delta_1\left(p\right)\right)dp - j\left(q\right) - \lambda\int_{\imath}^{s}\psi\left(q,p,p\right)\mu\left(p,\delta_2\left(p\right),\delta_2\left(p\right)\right)dp\right| \\
&= 2\left|\lambda\right| sup\left|\int_{\imath}^{s}\psi\left(q,p,p\right)\mu\left(p,\delta_1\left(p\right),\delta_1\left(p\right)\right) - \mu\left(p,\delta_2\left(p\right),\delta_2\left(p\right)\right)dp\right| \\
&\leq 2\left|\lambda\right| sup\left[\int_{\imath}^{s}\psi\left(q,p,p\right)dp\int_{\imath}^{s}\left(\mu\left(p,\delta_1\left(p\right),\delta_1\left(p\right)\right) - \mu\left(p,\delta_2\left(p\right),\delta_2\left(p\right)\right)dp\right] \\
&\leq \frac{2\left|\lambda\right|}{2(s-\imath)}\left[\int_{\imath}^{s}\varphi(\left|\delta_1\left(p\right) - \delta_2\left(p\right)\right|)dp\right] \leq \frac{\left|\lambda\right|}{s-\imath}\left[\int_{\imath}^{s}\varphi(\mathcal{S}(\delta_1,\delta_2,\delta_2))dp\right] \\
&= \frac{\left|\lambda\right|}{s-\imath}\ \varphi(\mathcal{S}(\delta_1,\delta_2,\delta_2)) \times s - \imath = \left|\lambda\right|\ \varphi(\mathcal{S}(\delta_1,\delta_2,\delta_2)) \leq \varphi(\mathcal{S}(\delta_1,\delta_2,\delta_2)).
\end{aligned}
$$

Consequently, $\mathcal{F}$ has unique solution, which means that Equation 18 has unique solution in $\mathcal{X}$. Now, introduce an application to explain the existence and uniqueness problem of the solution to an integral equation of the following form in $\mathcal{S}$-metric spaces:

$$
\delta\left(q\right) = j\left(q\right) + \int_{0}^{1}\psi\left(q,p,u\left(p\right)\right)dp,\ q \in [0,1]. \tag{19}
$$

Assume $\mathcal{F} : (\mathcal{X}, \mathcal{S}) \to (\mathcal{X}, \mathcal{S})$ is self-mapping defined as follows:

$$
\mathcal{F}\delta\left(q\right) = j\left(q\right) + \int_{0}^{1}\psi\left(q,p,\delta\left(p\right)\right)dp, \quad q \in [0,1].
$$

resume $X = C([0,1])$ space of real continuous mappings described on [0,1], and let $\mathcal{X}$ equipped with $\mathcal{S}$-metric which is described as follows:

$$
\mathcal{S}\left(\delta,\alpha,\beta\right) = \sup_{q\in[0,1]}\left|\delta\left(q\right) - \alpha(q)\right| + \sup_{q\in[0,1]}\left|\alpha\left(q\right) - \beta(q)\right| + \sup_{q\in[0,1]}\left|\beta\left(q\right) - \delta(q)\right|
$$

is complete $\mathcal{S}$-metric-space.

Theorem 4. *If the following suppositions hold:*

(i) $\psi : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ *and* $j : \mathbb{R} \to \mathbb{R}$ *are continuous mappings,*

(ii) There exists $\Phi : [0,1] \times [0,1] \to [0,\infty)$ *such that* $\frac{1}{3}\mathcal{S}\left(\delta,\mathcal{F}\delta,\mathcal{F}\delta\right) \leq \mathcal{S}\left(\delta,\alpha,\alpha\right)$ *implies that* $\left|\psi\left(q,p,u\right) - \psi\left(q,p,v\right)\right| \leq \Phi\left(q,p\right)\left|u - v\right|$. *For all distinct* $\delta,\alpha \in \mathcal{X}, q,p \in [0,1]$ *with* $u,v \in \mathbb{R}$,

(iii) $\sup_{q\in[0,1]} \int_0^1 \Phi(q,p)\,dp < \varphi,$ *where* $\varphi \in (0,1)$.

So, integral Equation 19 has unique solution.

Proof. For $\delta, \alpha \in \mathcal{X}$, we have

$$\begin{aligned}\mathcal{S}(\mathcal{F}\delta, \mathcal{F}\alpha, \mathcal{F}\alpha) &= \sup q \in [0,1]\,|\mathcal{F}\delta(q) - \mathcal{F}\alpha(q)| \\ &= 2\sup q \in [0,1]\left|\int_0^1 \psi(q,p,\delta(p)) - \psi(q,p,\alpha(p))\,dp\right| \\ &\leq 2\sup q \in [0,1]\int_0^1 |\psi(q,p,\delta(p)) - \psi(q,p,\alpha(p))|\,dp \\ &\leq 2\sup q \in [0,1]\int_0^1 \Phi(q,p)\,|\delta(p) - \alpha(p)|\,dp \\ &\leq 2\sup q \in [0,1]\,|\delta(q) - \alpha(q)|\sup q \in [0,1]\int_0^1 \Phi(q,p)\,dp \leq \varphi(\mathcal{S}(\delta,\alpha,\alpha))\end{aligned}$$

Consequently, $\mathcal{F}$ has unique solution, which means that equation 4.2 has unique solution in $\mathcal{X}$.

5. Conclusion

Fixed point theory plays a significant role in various fields of pure and applied mathematical analysis and scientific implementations, as well provides a technique for solving a variety of pure and applied issues in mathematics, physics, and other sciences and has been expanded and enhanced in various directions. Therefore, in this article several results were concluded, Firstly, It is possible to get coincidence and common fixed point results if the maps used are non-decreasing in generalized complete $\mathcal{S}$-metric spaces, as well as if there are a simulation and right monotone simulation mappings (ξ). Secondly, the relationship between simulation mapping and right monotone simulation mapping was clarified. It was concluded that each right monotone simulation mapping is simulation mapping, but the converse need not be true in general. These conclusions were supported by appropriate examples. Third, various common and coincidence fixed point results in symmetrical complete $\mathcal{S}$-metric have been deduced. On the other hand, Through the applications presented in the fourth section, it was verified existence and uniqueness of the solution for some nonlinear integral equations in generalized $\mathcal{S}$-metric. Finally, the obtained results may be beneficial for further research on extended metric spaces, providing a foundation for practical applications in engineering and various kinds of general dynamical systems.

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