ε-Lukasiewicz Fuzzy UP (BCC)-Subalgebras of UP (BCC)-Algebras

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Abstract. The idea of Lukasiewicz t-norm is used to construct the concept of ε-Lukasiewicz fuzzy sets based on a given fuzzy set. The ε-Lukasiewicz fuzzy sets are applied to UP (BCC)-algebras. Moreover, the notion of ε-Lukasiewicz fuzzy UP (BCC)-subalgebras is introduced, and its various properties are investigated. Three subsets, so-called ε-set, q-set, and O-set, are constructed, and the conditions under which they can be UP (BCC)-subalgebras are explored.

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1. Introduction

Zadeh [12] first proposed the idea of fuzzy sets. The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After introducing the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches, such as soft sets and rough sets, has been discussed in [1–3]. The new technology allows very complex inferences about variations on a theme to be anticipated and fixed in a program. Lukasiewicz logic, which is the logic of the Lukasiewicz t-norm, is a non-classical and many-valued logic. It was originally defined in the early 20th century by Lukasiewicz as a three-valued logic. Iampan [7] introduced a new algebraic structure called UP-algebra. Somjanta et al. [11] and Guntasow et al. [5] applied fuzzy set theory in UP-algebras. Dokkhamdang et al. [4] introduced the notion of fuzzy UP-subalgebras.

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with thresholds of UP-algebras. The concepts of UP-algebras (see [7]) and BCC-algebras (see [9]) are the same concept, as shown by Jun et al. [8] in 2022. In this publication and following investigations, our research team will refer to it as BCC rather than UP out of respect for Komori, who first characterized it in 1984.

In this paper, using the idea of Lukasiewicz $t$-norm, we construct the concept of $\varepsilon$-Lukasiewicz fuzzy sets based on a given fuzzy set and apply it to BCC-algebras. We define the concepts of $\varepsilon$-Lukasiewicz fuzzy BCC-subalgebras and investigate several properties. We provide conditions for an $\varepsilon$-Lukasiewicz fuzzy set to be an $\varepsilon$-Lukasiewicz fuzzy BCC-subalgebra. We discuss the characterizations of $\varepsilon$-Lukasiewicz fuzzy BCC-subalgebras. We construct three kinds of subsets, so-called $\varepsilon$-set, $q$-set, and $O$-set, and we find the conditions under which they can be BCC-subalgebras.

2. Preliminaries

The concept of BCC-algebras (see [9]) can be redefined without the condition (2.6) as follows:

An algebra $X = (X, \ast, 0)$ of type $(2, 0)$ is called a BCC-algebra (see [6]) if it satisfies the following conditions:

$$
(\forall x, y, z \in X)(y \ast z) \ast ((x \ast y) \ast (x \ast z)) = 0) \quad (2.1)
$$

$$
(\forall x \in X)(0 \ast x = x) \quad (2.2)
$$

$$
(\forall x \in X)(x \ast 0 = 0) \quad (2.3)
$$

$$
(\forall x, y \in X)(x \ast y = 0 \Rightarrow y \ast x \Rightarrow x = y) \quad (2.4)
$$

After this, we assign $X$ instead of a BCC-algebra $(X, \ast, 0)$ until otherwise specified. We define a binary relation $\leq$ on $X$ as follows:

$$
(\forall x, y \in X)(x \leq y \iff x \ast y = 0) \quad (2.5)
$$

In $X$, the following assertions are valid (see [7]).

$$
(\forall x \in X)(x \leq x) \quad (2.6)
$$

$$
(\forall x, y, z \in X)(x \leq y, y \leq z \Rightarrow x \leq z) \quad (2.7)
$$

$$
(\forall x, y, z \in X)(x \leq y \Rightarrow z \ast x \leq z \ast y) \quad (2.8)
$$

$$
(\forall x, y, z \in X)(x \leq y \Rightarrow y \ast z \leq x \ast z) \quad (2.9)
$$

$$
(\forall x, y \in X)(y \ast x \leq x \iff x = y \ast x) \quad (2.10)
$$

$$
(\forall x, y \in X)(y \ast x \leq x \Leftrightarrow x = y \ast x) \quad (2.11)
$$

$$
(\forall a, x, y, z \in X)(x \ast (y \ast z) \leq x \ast ((a \ast y) \ast (a \ast z))) \quad (2.12)
$$

$$
(\forall a, x, y, z \in X)(((a \ast x) \ast (a \ast y)) \ast z \leq (x \ast y) \ast z) \quad (2.13)
$$

$$
(\forall x, y, z \in X)((x \ast y) \ast z \leq y \ast z) \quad (2.14)
$$

$$
(\forall x, y, z \in X)((x \ast y) \ast z \leq y \ast z) \quad (2.15)
$$
\[ \forall x, y, z \in X \Rightarrow x \leq z \ast y \] (2.16)

\[ \forall x, y, z \in X \Rightarrow (x \ast y) \ast z \leq x \ast (y \ast z) \] (2.17)

\[ \forall a, x, y, z \in X \Rightarrow ((x \ast y) \ast z \leq y \ast (a \ast z)) \] (2.18)

**Definition 1.** [7] A nonempty subset \( S \) of \( X \) is called a BCC-subalgebra of \( X \) if it satisfies the following properties:

\[ \forall x, y \in S \Rightarrow x \ast y \in S \] (2.19)

A fuzzy set [12] in a nonempty set \( X \) is defined to be a function \( \mu : X \rightarrow [0, 1] \), where \([0, 1]\) is the unit closed interval of real numbers.

**Definition 2.** [11] A fuzzy set \( \mu \) in \( X \) is called a fuzzy BCC-subalgebra of \( X \) if it satisfies the following property:

\[ \forall x, y \in X \Rightarrow \mu(x \ast y) \geq \min\{\mu(x), \mu(y)\} \] (2.20)

A fuzzy set \( \mu \) in a set \( X \) of the form

\[ \mu(x) = \begin{cases} t \in (0, 1] & \text{if } x = a \\ 0 & \text{if } x \neq a, \end{cases} \]

is said to be a fuzzy point with support \( a \) and value \( t \) and is denoted by \([a/t]\).

For a fuzzy set \( \mu \) in a set \( X \), we say that a fuzzy point \([a/t]\) is

1. contained in \( \mu \), denoted by \([a/t] \in \mu \), (see [10]) if \( \mu(a) \geq t \),
2. quasi-coincident with \( \mu \), denoted by \([a/t] q \mu \), (see [10]) if \( \mu(a) + t > 1 \).

**Proposition 1.** If \( \mu \) is a fuzzy set in a set \( X \) and \( \varepsilon \in (0, 1) \), then its \( \varepsilon \)-Lukasiewicz fuzzy set \( L^\varepsilon \mu \) satisfies the following property:

1. \[ \forall x, y \in X \Rightarrow (\mu(x) \geq \mu(y) \Rightarrow L^\varepsilon \mu(x) \geq L^\varepsilon \mu(y)) \]
2. \[ \forall x \in X \Rightarrow ([x/\varepsilon] \mu \Rightarrow L^\varepsilon \mu(x) = \mu(x) + \varepsilon - 1) \]
3. \[ \forall x \in X, \forall \delta \in (0, 1)](\varepsilon \geq \delta \Rightarrow L^\varepsilon \mu(x) \geq L^\delta \mu(x)) \]

**3. \( \varepsilon \)-Lukasiewicz fuzzy BCC-subalgebra of a BCC-algebra**

In this section, we will recall the definition of \( \varepsilon \)-Lukasiewicz fuzzy sets and introduce a new concept called \( \varepsilon \)-Lukasiewicz fuzzy BCC-subalgebras.

**Definition 3.** Let \( \mu \) be a fuzzy set in a set \( X \) and let \( \varepsilon \in [0, 1] \). A function \( L^\varepsilon \mu : X \rightarrow [0, 1] \) \( x \mapsto \max\{0, \mu(x) + \varepsilon - 1\} \) is called an \( \varepsilon \)-Lukasiewicz fuzzy set of \( \mu \) in \( X \).
Definition 4. Let $\mu$ be a fuzzy set in $X$. Then its $\varepsilon$-Lukasiewicz fuzzy set $L_\mu^\varepsilon$ in $X$ is called an $\varepsilon$-Lukasiewicz fuzzy BCC-subalgebra of $X$ if it satisfies the following property:

$$(\forall x, y \in X, \forall t_a, t_b \in (0, 1])([x/t_a] \in L_\mu^\varepsilon, [y/t_b] \in L_\mu^\varepsilon \Rightarrow [(x * y)/\min\{t_a, t_b\}] \in L_\mu^\varepsilon) \quad (3.1)$$

Theorem 1. If $\mu$ is a fuzzy BCC-subalgebra of $X$, then its $\varepsilon$-Lukasiewicz fuzzy set $L_\mu^\varepsilon$ in $X$ is an $\varepsilon$-Lukasiewicz fuzzy BCC-subalgebra of $X$.

Proof. Assume that $\mu$ is a fuzzy BCC-subalgebra of $X$. Let $x, y \in X$ and $t_a, t_b \in (0, 1]$ be such that $[x/t_a] \in L_\mu^\varepsilon$ and $[y/t_b] \in L_\mu^\varepsilon$. Then $L_\mu^\varepsilon(x) \geq t_a$ and $L_\mu^\varepsilon(y) \geq t_b$. Thus

$$L_\mu^\varepsilon(x * y) = \max\{0, \mu(x * y) + \varepsilon - 1\} \geq \max\{0, \min\{\mu(x), \mu(y)\} + \varepsilon - 1\} = \max\{0, \min\{\mu(x) + \varepsilon - 1, \mu(y) + \varepsilon - 1\}\} = \min\{\max\{0, \mu(x) + \varepsilon - 1\}, \max\{0, \mu(y) + \varepsilon - 1\}\} = \min\{L_\mu^\varepsilon(x), L_\mu^\varepsilon(y)\} \geq \min\{t_a, t_b\}.$$ 

Hence, $[(x * y)/\min\{t_a, t_b\}] \in L_\mu^\varepsilon$. Therefore, $L_\mu^\varepsilon$ is an $\varepsilon$-Lukasiewicz fuzzy BCC-subalgebra of $X$.

The following example shows that the converse of Theorem 1 may not be true.

Example 1. Let $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<tbody>
<tr>
<td>0</td>
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<td>0</td>
<td>4</td>
<td>2</td>
<td>3</td>
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</tr>
</tbody>
</table>

Then $X$ is a BCC-algebra. Define a fuzzy set $\mu$ as follows:

$$\mu(x) = \begin{cases} 
1.0 & \text{if } x = 0 \\
0.4 & \text{if } x = 1 \\
0.2 & \text{if } x = 2 \\
0.3 & \text{if } x = 3 \\
0.6 & \text{if } x = 4. 
\end{cases}$$

Given $\varepsilon = 0.9$, the $\varepsilon$-Lukasiewicz fuzzy set $L_\mu^\varepsilon$ of $\mu$ in $X$ is given as follows:

$$L_\mu^\varepsilon(x) = \begin{cases} 
0.9 & \text{if } x = 0 \\
0.3 & \text{if } x = 1 \\
0.1 & \text{if } x = 2 \\
0.2 & \text{if } x = 3. \\
0.5 & \text{if } x = 4. 
\end{cases}$$

Then $L_\mu^\varepsilon$ is an $\varepsilon$-Lukasiewicz fuzzy BCC-subalgebra of $X$. 

Theorem 2. Let \( \mu \) be a fuzzy set in \( X \). Then its \( \varepsilon \)-Lukasiewicz fuzzy set \( L^\varepsilon_\mu \) in \( X \) is an \( \varepsilon \)-Lukasiewicz fuzzy BCC-subalgebra of \( X \) if and only if it satisfies the following property:

\[
(\forall x, y \in X)(L^\varepsilon_\mu(x * y) \geq \min\{L^\varepsilon_\mu(x), L^\varepsilon_\mu(y)\})
\]  

(3.2)

Proof. Suppose \( L^\varepsilon_\mu \) is an \( \varepsilon \)-Lukasiewicz fuzzy BCC-subalgebra of \( X \). Let \( x, y \in X \). Then \( [x/L^\varepsilon_\mu(x)] \in L^\varepsilon_\mu \) and \( [y/L^\varepsilon_\mu(y)] \in L^\varepsilon_\mu \). Thus, \( [(x * y)/\min\{L^\varepsilon_\mu(x), L^\varepsilon_\mu(y)\}] \in L^\varepsilon_\mu \) by (3.1), which implies that \( L^\varepsilon_\mu(x * y) \geq \min\{L^\varepsilon_\mu(x), L^\varepsilon_\mu(y)\} \).

Conversely, suppose that \( L^\varepsilon_\mu \) satisfies the condition (3.2). Let \( x, y \in X \) and \( t_a, t_b \in (0, 1] \) be such that \( [x/t_a] \in L^\varepsilon_\mu \) and \( [y/t_b] \in L^\varepsilon_\mu \). Then \( L^\varepsilon_\mu(x) \geq t_a \) and \( L^\varepsilon_\mu(y) \geq t_b \), which implies from (3.2) that \( L^\varepsilon_\mu(x * y) \geq \min\{L^\varepsilon_\mu(x), L^\varepsilon_\mu(y)\} \geq \min\{t_a, t_b\} \). Thus, \( [(x * y)/\min\{t_a, t_b\}] \in L^\varepsilon_\mu \). Hence, \( L^\varepsilon_\mu \) is an \( \varepsilon \)-Lukasiewicz fuzzy BCC-subalgebra of \( X \).

Proposition 2. If \( \mu \) is a fuzzy BCC-subalgebra of \( X \), then its \( \varepsilon \)-Lukasiewicz fuzzy set \( L^\varepsilon_\mu \) satisfies the following property:

\[
(\forall x \in X)(L^\varepsilon_\mu(0) \geq L^\varepsilon_\mu(x))
\]  

(3.3)

Proof. If \( \mu \) is a fuzzy BCC-subalgebra of \( X \), then \( \mu(0) = \mu(x * x) \geq \min\{\mu(x), \mu(x)\} = \mu(x) \) for all \( x \in X \). It follows from Proposition 1 (1) that \( L^\varepsilon_\mu(0) \geq L^\varepsilon_\mu(x) \) for all \( x \in X \).

The following example shows that the converse of Proposition 2 is not true in general.

Example 2. [5] Let \( X = \{0, 1, 2, 3\} \) with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<tbody>
<tr>
<td>0</td>
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<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \( X \) is a BCC-algebra. Define a fuzzy set \( \mu \) as follows:

\[
\mu : X \to [0, 1]; x \mapsto \begin{cases} 
1 & \text{if } x = 0 \\
0 & \text{if } x = 1 \\
1 & \text{if } x = 2 \\
1 & \text{if } x = 3 
\end{cases}
\]

Given \( \varepsilon = 0.9 \), the \( \varepsilon \)-Lukasiewicz fuzzy set \( L^\varepsilon_\mu \) of \( \mu \) in \( X \) is given as follows:

\[
L^\varepsilon_\mu : X \to [0, 1]; x \mapsto \begin{cases} 
0.9 & \text{if } x = 0 \\
0 & \text{if } x = 1 \\
0.9 & \text{if } x = 2 \\
0.9 & \text{if } x = 3 
\end{cases}
\]

Then \( L^\varepsilon_\mu(0) \geq L^\varepsilon_\mu(x) \) for all \( x \in X \) but \( \mu \) is not a fuzzy BCC-subalgebra of \( X \) because \( \mu(2 * 3) = \mu(1) = 0 \not\geq 1 = \min\{\mu(2), \mu(3)\} \).
Proposition 3. If \( \mu \) is a fuzzy BCC-subalgebra of \( X \), then its \( \varepsilon \)-Lukasiewicz fuzzy set \( L_\varepsilon^{\mu} \) satisfies the following property:

\[
(\forall x, y \in X) (L_\varepsilon^{\mu}(y) = L_\varepsilon^{\mu}(0) \Leftrightarrow L_\varepsilon^{\mu}(x * y) \geq L_\varepsilon^{\mu}(x))
\]  

(3.4)

Proof. Assume that \( L_\varepsilon^{\mu}(y) = L_\varepsilon^{\mu}(0) \) for all \( y \in X \). Then \( L_\varepsilon^{\mu}(x * y) \geq \min \{L_\varepsilon^{\mu}(x), L_\varepsilon^{\mu}(y)\} = \min \{L_\varepsilon^{\mu}(x), L_\varepsilon^{\mu}(0)\} = L_\varepsilon^{\mu}(x) \) for all \( x, y \in X \) by the combination of Theorem 1 and Proposition 2.

Conversely, suppose that \( L_\varepsilon^{\mu}(x * y) \geq L_\varepsilon^{\mu}(x) \) for all \( x, y \in X \). Using (2.2) induces \( L_\varepsilon^{\mu}(y) = L_\varepsilon^{\mu}(0 * y) \geq L_\varepsilon^{\mu}(0) \). The combination of this and Proposition 2 leads to \( L_\varepsilon^{\mu}(y) = L_\varepsilon^{\mu}(0) \) for all \( y \in X \).

Proposition 4. If \( \mu \) is a fuzzy BCC-subalgebra of \( X \), then its \( \varepsilon \)-Lukasiewicz fuzzy set \( L_\varepsilon^{\mu} \) satisfies the following property:

\[
(\forall x, y \in X, \forall t_a, t_b \in (0, 1]) \left( \frac{[x/t_a]}{[y/t_b]} \in L_\varepsilon^{\mu}, \frac{[y/t_b]}{[x/t_a]} \in L_\varepsilon^{\mu} \Rightarrow \left( \frac{(x * (0 * y))}{\min \{t_a, t_b\}} \right) \in L_\varepsilon^{\mu} \right)
\]  

(3.5)

Proof. Let \( x, y \in X \) and \( t_a, t_b \in (0, 1] \) be such that \( [x/t_a] \in L_\varepsilon^{\mu} \) and \( [y/t_b] \in L_\varepsilon^{\mu} \). Then \( L_\varepsilon^{\mu}(x) \geq t_a \) and \( L_\varepsilon^{\mu}(y) \geq t_b \). Thus

\[
L_\varepsilon^{\mu}(x * (0 * y)) = \max \{0, \mu(x * (0 * y)) + \varepsilon - 1\} \\
\geq \max \{0, \min \{\mu(x), \mu(0 * y)\} + \varepsilon - 1\} \\
\geq \max \{0, \min \{\mu(x), \min \{\mu(0), \mu(y)\} \} + \varepsilon - 1\} \\
= \max \{0, \min \{\mu(x), \mu(0)\} + \varepsilon - 1\} \\
= \max \{0, \min \{\mu(x) + \varepsilon - 1, \mu(y) + \varepsilon - 1\}\} \\
\geq \min \{[L_\varepsilon^{\mu}(x)], [L_\varepsilon^{\mu}(y)]\} \\
\geq \min \{t_a, t_b\}.
\]

Hence, \( \left( (x * (0 * y))/\min \{t_a, t_b\} \right) \in L_\varepsilon^{\mu} \).

We provide conditions for an \( \varepsilon \)-Lukasiewicz fuzzy set to be an \( \varepsilon \)-Lukasiewicz fuzzy BCC-subalgebra.

Theorem 3. Let \( \mu \) be a fuzzy set in \( X \). If its \( \varepsilon \)-Lukasiewicz fuzzy set \( L_\varepsilon^{\mu} \) satisfies the following property:

\[
[y/t_b] \in L_\varepsilon^{\mu}, [z/t_c] \in L_\varepsilon^{\mu} \Rightarrow \left( \frac{[x/y]}{\min \{t_b, t_c\}} \right) \in L_\varepsilon^{\mu}
\]  

(3.6)

for all \( t_b, t_c \in (0, 1] \) and \( x, y, z \in X \) with \( z \leq x \), then \( L_\varepsilon^{\mu} \) is an \( \varepsilon \)-Lukasiewicz fuzzy BCC-subalgebra of \( X \).

Proof. Let \( x, y \in X \) and \( t_a, t_b \in (0, 1] \) be such that \( [x/t_a] \in L_\varepsilon^{\mu} \) and \( [y/t_b] \in L_\varepsilon^{\mu} \). Since \( x \leq x \) for all \( x \in X \), it follows from (3.6) that \( \left( (x * y)/\min \{t_a, t_b\} \right) \in L_\varepsilon^{\mu} \). Hence, \( L_\varepsilon^{\mu} \) is an \( \varepsilon \)-Lukasiewicz fuzzy BCC-subalgebra of \( X \).
Proposition 5. Let \( \mu \) be a fuzzy set in \( X \). Then every \( \varepsilon \)-Lukasiewicz fuzzy BCC-subalgebra \( L_\mu^\varepsilon \) of \( X \) satisfies the following property:

\[
(\forall x, y \in X, t_a, t_b \in (0, 1)) ([x/t_a] \in L_\mu^\varepsilon, [y/t_b] \in L_\mu^\varepsilon \Rightarrow [(x \ast (0 \ast y))/\min\{t_a, t_b\}] \in L_\mu^\varepsilon)
\]

(3.7)

Proof. Let \( x, y \in X \) and \( t_a, t_b \in (0, 1) \) be such that \( [x/t_a] \in L_\mu^\varepsilon \) and \( [y/t_b] \in L_\mu^\varepsilon \). Then \( L_\mu^\varepsilon(x) \geq t_a \) and \( L_\mu^\varepsilon(y) \geq t_b \). It follows from Theorem 2 and Proposition 2 that

\[
L_\mu^\varepsilon(x \ast (0 \ast y)) \geq \min\{L_\mu^\varepsilon(x), L_\mu^\varepsilon(0 \ast y)\} \\
\geq \min\{L_\mu^\varepsilon(x), \min\{L_\mu^\varepsilon(0), L_\mu^\varepsilon(y)\}\} \\
= \min\{L_\mu^\varepsilon(x), L_\mu^\varepsilon(y)\} \\
\geq \min\{t_a, t_b\}.
\]

Hence, \( [(x \ast (0 \ast y))/\min\{t_a, t_b\}] \in L_\mu^\varepsilon \).

Let \( \mu \) be a fuzzy set in \( X \). For an \( \varepsilon \)-Lukasiewicz fuzzy set \( L_\mu^\varepsilon \) of \( \mu \) in \( X \) and \( t \in (0, 1) \), consider the sets

\[
(L_\mu^\varepsilon, t)_E = \{x \in X : [x/t] \in L_\mu^\varepsilon\},
\]

\[
(L_\mu^\varepsilon, t)_q = \{x \in X : [x/t]qL_\mu^\varepsilon\},
\]

which are called the \( \varepsilon \)-set and \( q \)-set, respectively, of \( L_\mu^\varepsilon \) (with value \( t \)).

We explore the conditions under which the \( \varepsilon \)-set and \( q \)-set of \( \varepsilon \)-Lukasiewicz fuzzy sets can be BCC-subalgebras.

Theorem 4. Let \( L_\mu^\varepsilon \) be an \( \varepsilon \)-Lukasiewicz fuzzy set of a fuzzy set \( \mu \) in \( X \). Then the \( \varepsilon \)-set \( (L_\mu^\varepsilon, t)_E \) of \( L_\mu^\varepsilon \) with value \( t \in (0, 1) \) is a BCC-subalgebra of \( X \) if and only if the following assertion is valid:

\[
(\forall x, y \in X)(\min\{L_\mu^\varepsilon(x), L_\mu^\varepsilon(y)\} \leq \max\{L_\mu^\varepsilon(x \ast y), 0.5\})
\]

(3.8)

Proof. Assume that the \( \varepsilon \)-set \( (L_\mu^\varepsilon, t)_E \) of \( L_\mu^\varepsilon \) with value \( t \in (0, 1) \) is a BCC-subalgebra of \( X \). If the condition (3.8) is not valid, then there exist \( a, b \in X \) such that \( \min\{L_\mu^\varepsilon(a), L_\mu^\varepsilon(b)\} > \max\{L_\mu^\varepsilon(a \ast b), 0.5\} \). If we take \( s = \min\{L_\mu^\varepsilon(a), L_\mu^\varepsilon(b)\} \), then \( s \in (0, 1) \) and \( [a/s], [b/s] \in L_\mu^\varepsilon \), that is, \( a, b \in (L_\mu^\varepsilon, s)_E \). Since \( (L_\mu^\varepsilon, s)_E \) is a BCC-subalgebra of \( X \), we have \( a \ast b \in (L_\mu^\varepsilon, s)_E \). But \( [a \ast b/s] \notin L_\mu^\varepsilon \) implies \( a \ast b \notin (L_\mu^\varepsilon, s)_E \), a contradiction. Thus, \( \min\{L_\mu^\varepsilon(x), L_\mu^\varepsilon(y)\} \leq \max\{L_\mu^\varepsilon(x \ast y), 0.5\} \) for all \( x, y \in X \).

Conversely, suppose that \( L_\mu^\varepsilon \) satisfies the condition (3.8). Let \( t \in (0, 1) \) and \( x, y \in X \) be such that \( x \in (L_\mu^\varepsilon, t)_E \) and \( y \in (L_\mu^\varepsilon, t)_E \). Then \( L_\mu^\varepsilon(x) \geq t \) and \( L_\mu^\varepsilon(y) \geq t \), which imply from (3.8) that \( 0.5 < t \leq \min\{L_\mu^\varepsilon(x), L_\mu^\varepsilon(y)\} \leq \max\{L_\mu^\varepsilon(x \ast y), 0.5\} \). Thus, \( [(x \ast y)/t] \in L_\mu^\varepsilon \), that is, \( x \ast y \in (L_\mu^\varepsilon, t)_E \). So, \( (L_\mu^\varepsilon, t)_E \) is a BCC-subalgebra of \( X \) for \( t \in (0, 1) \).

Theorem 5. Let \( L_\mu^\varepsilon \) be an \( \varepsilon \)-Lukasiewicz fuzzy set of a fuzzy set \( \mu \) in \( X \). If \( \mu \) is a fuzzy BCC-subalgebra of \( X \), then the nonempty \( q \)-set \( (L_\mu^\varepsilon, t)_q \) of \( L_\mu^\varepsilon \) with value \( t \in (0, 1) \) is a BCC-subalgebra of \( X \).
Theorem 7. Let \( L^\varepsilon_\mu \) be an \( \varepsilon \)-Lukasiewicz fuzzy set of a fuzzy set \( \mu \) in \( X \), if \( \mu \) is a fuzzy \( \varepsilon \)-Lukasiewicz fuzzy \( \mu \)-subalgebra of \( X \) and then the nonempty \( \mu \)-set \( O(L^\varepsilon_\mu) \) is a \( \varepsilon \)-Lukasiewicz fuzzy \( \mu \)-subalgebra of \( X \) by Theorem 1. It follows from Theorem 2 that \( L^\varepsilon_\mu(x + t) \geq \min \{ L^\varepsilon_\mu(x), L^\varepsilon_\mu(y) \} = \min \{ \mu(x) + \varepsilon - 1, \mu(y) + \varepsilon - 1 \} > 0 \). Thus, \( x + y \in O(L^\varepsilon_\mu) \). Hence, \( O(L^\varepsilon_\mu) \) is a \( \varepsilon \)-Lukasiewicz fuzzy \( \mu \)-subalgebra of \( X \).

Proof. Let \( t \in (0, 1] \) and \( x, y \in (L^\varepsilon_\mu)_q \). Then \([t/\varepsilon_\mu]qL^\varepsilon_\mu \) and \([\varepsilon_\mu/\varepsilon_\mu]qL^\varepsilon_\mu \), that is, \( L^\varepsilon_\mu(x + \varepsilon_\mu t) \geq \min \{ L^\varepsilon_\mu(x), L^\varepsilon_\mu(y) \} + t = \min \{ L^\varepsilon_\mu(x) + t, L^\varepsilon_\mu(y) + t \} > 1 \). Thus, \([x \times y]/\varepsilon_\mu]qL^\varepsilon_\mu \). So, \( x \times y \in (L^\varepsilon_\mu, t)_q \). Hence, \( (L^\varepsilon_\mu, t)_q \) is a \( \varepsilon \)-Lukasiewicz fuzzy \( \mu \)-subalgebra of \( X \).

The following example shows that the converse of Theorem 5 is not true in general.

Example 3. From the \( \varepsilon \)-Lukasiewicz fuzzy set \( L^\varepsilon_\mu \) of \( \mu \) in \( X \), then \( \forall t \in (0, 1] \) and \( x, y \in (L^\varepsilon_\mu)_q \) be such that \([x/\varepsilon_\mu]qL^\varepsilon_\mu \) and \([\varepsilon_\mu/\varepsilon_\mu]qL^\varepsilon_\mu \). Then \( x \times y \in (L^\varepsilon_\mu, t)_q \). Hence, \( (L^\varepsilon_\mu, t)_q \) is a \( \varepsilon \)-Lukasiewicz fuzzy \( \mu \)-subalgebra of \( X \) because \( \mu(2 \times 3) = \mu(1) = 0 \neq 0.1 = \min \{ \mu(2), \mu(3) \} \).

Theorem 6. Let \( \mu \) be a fuzzy set in \( X \). For an \( \varepsilon \)-Lukasiewicz fuzzy set \( L^\varepsilon_\mu \) of \( \mu \) in \( X \), if the \( \varepsilon \)-Lukasiewicz fuzzy \( \mu \)-subalgebra of \( X \) because \( \mu(2 \times 3) = \mu(1) = 0 \neq 0.1 = \min \{ \mu(2), \mu(3) \} \).

Proof. Let \( x, y \in X \) and \( t_a, t_b \in (0, 0.5] \) be such that \([x/t_a]qL^\varepsilon_\mu \) and \([y/t_b]qL^\varepsilon_\mu \). Then \( x \in (L^\varepsilon_\mu, t_a)_q \subseteq (L^\varepsilon_\mu, \max \{ t_a, t_b \})_q \) and \( y \in (L^\varepsilon_\mu, t_b)_q \subseteq (L^\varepsilon_\mu, \max \{ t_a, t_b \})_q \). Thus, \( x \cdot y \in (L^\varepsilon_\mu, \max \{ t_a, t_b \})_q \). Since \( \max \{ t_a, t_b \} \leq 0.5 \), we have \( L^\varepsilon_\mu(x \cdot y) > 1 - \max \{ t_a, t_b \} \geq \max \{ t_a, t_b \} \). Hence, \([x \cdot y]/\varepsilon_\mu \in (L^\varepsilon_\mu, t_a)_q \). Let \( \mu \) be a fuzzy set in \( X \). For an \( \varepsilon \)-Lukasiewicz fuzzy set \( L^\varepsilon_\mu \) of \( \mu \) in \( X \), consider the set \( O(L^\varepsilon_\mu) = \{ x \in X : \mu(x) + \varepsilon - 1 > 0 \} \), which is called the \( \varepsilon \)-Lukasiewicz fuzzy \( \mu \)-set. It is observed that \( O(L^\varepsilon_\mu) \) is a \( \varepsilon \)-Lukasiewicz fuzzy \( \mu \)-subalgebra of \( X \).
Theorem 8. Let $\mu$ be a fuzzy set in $X$. If an $\varepsilon$-Lukasiewicz fuzzy set $L^\varepsilon_\mu$ of $\mu$ in $X$ satisfies the following property:

$$[x/t_a] \in L^\varepsilon_\mu, [y/t_b] \in L^\varepsilon_\mu \Rightarrow [(x \ast y)/\max\{t_a, t_b\}]qL^\varepsilon_\mu$$

(3.10)

for all $x, y \in X$ and $t_a, t_b \in (0, 1]$, then the nonempty $O$-set $O(L^\varepsilon_\mu)$ of $L^\varepsilon_\mu$ is a BCC-subalgebra of $X$.

Proof. Assume that $L^\varepsilon_\mu$ satisfies the condition (3.10) for all $x, y \in X$ and $t_a, t_b \in (0, 1]$. Let $x, y \in O(L^\varepsilon_\mu)$. Then $\mu(x) + \varepsilon - 1 > 0$ and $\mu(y) + \varepsilon - 1 > 0$. Since $[x/L^\varepsilon_\mu(x)] \in L^\varepsilon_\mu$ and $[y/L^\varepsilon_\mu(y)] \in L^\varepsilon_\mu$, it follows from (3.10) that

$$[(x \ast y)/\max\{L^\varepsilon_\mu(x \ast (y \ast z)), L^\varepsilon_\mu(y)\}]qL^\varepsilon_\mu.$$  

(3.11)

If $x \ast y \notin O(L^\varepsilon_\mu)$, then $L^\varepsilon_\mu(x \ast y) = 0$. Thus,

$$L^\varepsilon_\mu(x \ast y) + \max\{L^\varepsilon_\mu(x), L^\varepsilon_\mu(y)\}
= \max\{L^\varepsilon_\mu(x), L^\varepsilon_\mu(y)\}
= \max\{\max\{0, \mu(x) + \varepsilon - 1\}, \max\{0, \mu(y) + \varepsilon - 1\}\}
= \max\{\mu(x) + \varepsilon - 1, \mu(y) + \varepsilon - 1\}
= \max\{\mu(x), \mu(y)\} + \varepsilon - 1
\leq 1 + \varepsilon - 1
= \varepsilon
\leq 1,$$

which shows that (3.11) is not valid. This is a contradiction. Hence, $x \ast y \in O(L^\varepsilon_\mu)$. Therefore, $O(L^\varepsilon_\mu)$ is a BCC-subalgebra of $X$.

Theorem 9. Let $\mu$ be a fuzzy set in $X$. If an $\varepsilon$-Lukasiewicz fuzzy set $L^\varepsilon_\mu$ of $\mu$ in $X$ satisfies the condition (3.9) for all $x, y \in X$ and $t_a, t_b \in (0, 1]$, then the nonempty $O$-set $O(L^\varepsilon_\mu)$ of $L^\varepsilon_\mu$ is a BCC-subalgebra of $X$.

Proof. Let $x, y \in O(L^\varepsilon_\mu)$. Then $\mu(x) + \varepsilon - 1 > 0$ and $\mu(y) + \varepsilon - 1 > 0$. Thus, $L^\varepsilon_\mu(x) + 1 = \max\{0, \mu(x) + \varepsilon - 1\} + 1 = \mu(x) + \varepsilon - 1 + 1 = \mu(x) + \varepsilon > 1$ and $L^\varepsilon_\mu(y) + 1 = \max\{0, \mu(y) + \varepsilon - 1\} + 1 = \mu(y) + \varepsilon - 1 + 1 = \mu(y) + \varepsilon > 1$, that is, $[x/1]qL^\varepsilon_\mu$ and $[y/1]qL^\varepsilon_\mu$. It follows from (3.9) that

$$[(x \ast y)/1] = [(x \ast y)/\max\{1, 1\}] \in L^\varepsilon_\mu.$$  

(3.12)

If $x \ast y \notin O(L^\varepsilon_\mu)$, then $L^\varepsilon_\mu(x \ast y) = 0 < 1$ and so (3.12) is not valid. This is a contradiction. Thus, $x \ast y \in O(L^\varepsilon_\mu)$. Hence, $O(L^\varepsilon_\mu)$ is a BCC-subalgebra of $X$. 
4. Conclusions

The concept of $\varepsilon$-Lukasiewicz fuzzy sets using Lukasiewicz $t$-norm was introduced by Jun [8]. In this paper, the $\varepsilon$-Lukasiewicz fuzzy set has been applied to BCC-subalgebras in BCC-algebras, introducing the concept of $\varepsilon$-Lukasiewicz fuzzy BCC-subalgebras, and examining several properties. We discussed the characterization of $\varepsilon$-Lukasiewicz fuzzy BCC-subalgebras and considered the relationship between fuzzy BCC-subalgebras and $\varepsilon$-Lukasiewicz fuzzy BCC-subalgebras. We provided conditions under which $\varepsilon$-Lukasiewicz fuzzy sets can be $\varepsilon$-Lukasiewicz fuzzy BCC-subalgebras and further explored conditions under which three subsets: $\varepsilon$-set, $q$-set, and $O$-set, will be BCC-subalgebras. The ideas and results obtained in this paper will be applied to the relevant algebraic systems in the future, further examining their usability as a mathematical tool applicable to decision theory, medical diagnosis systems, automation systems, etc.

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References


