c-(τ₁, τ₂)-Continuity for Multifunctions

Jeeranunt Khampakdee¹, Supannee Sompong², Chawalit Boonpok¹,*

¹ Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand
² Department of Mathematics and Statistics, Faculty of Science and Technology, Sakon Nakhon Rajbhat University, Sakon Nakhon, 47000, Thailand

Abstract. This paper is concerned with the concepts of upper and lower c-(τ₁, τ₂)-continuous multifunctions. Moreover, several characterizations of upper and lower c-(τ₁, τ₂)-continuous multifunctions are investigated.

2020 Mathematics Subject Classifications: 54C08, 54C60, 54E55

Key Words and Phrases: Upper c-(τ₁, τ₂)-continuous multifunction, lower c-(τ₁, τ₂)-continuous multifunction

1. Introduction

The field of the mathematical science which goes under the name of topology is concerned with all questions directly or indirectly related to continuity. Semi-open sets, preopen sets, α-open sets and β-open sets play an important role in topological spaces. Using these sets, many authors introduced and studied various types of generalizations of continuity for functions and multifunctions. In 1970, Gentry and Hoyle III [23] introduced and studied the concept of c-continuous functions. Furthermore, some characterizations of c-continuous functions were investigated in [28], [29] and [32], respectively. Duangphuie et al. [22] introduced and studied the notion of (µ, µ′)(m,n)-continuous functions. Thongmoon and Boonpok [38] introduced and investigated the notion of strongly θ(Λ, p)-continuous functions. Moreover, several characterizations of almost (Λ, p)-continuous functions, almost strongly θ(Λ, p)-continuous functions, θ(Λ, p)-continuous functions, weakly (Λ, b)-continuous functions, θ(·)-precontinuous functions, *-continuous functions, θ-continuous functions, almost (g, m)-continuous functions, (Λ, sp)-continuous functions, δp(Λ, s)-continuous functions, (Λ, p(·))-continuous functions, pairwise almost M-continuous functions, (τ₁, τ₂)-continuous functions, almost (τ₁, τ₂)-continuous functions and weakly...
(τ₁, τ₂)-continuous functions were presented in [36], [12], [34], [17], [11], [10], [5], [2], [40], [37], [9], [3], [18], [16] and [13], respectively.

In 1975, Popa [33] introduced and studied the notion of quasi-continuous multifunctions. Neubrunn [30] and Holá et al. [24] extended the concept of c-continuous functions to the setting of multifunctions. Lipski [27] introduced the notion of c-continuous multifunctions as a generalization of c-continuous multifunctions [30] and quasi-continuous multifunctions [33]. Noiri and Popa [31] introduced and investigated the notion of C₁m-continuous multifunctions. Viriyapong and Boonpok [41] introduced and studied the concept of weakly quasi (Λ, sp)-continuous multifunctions. In [7], the present author introduced and investigated the notions of almost quasi ⋆-continuous multifunctions and weakly quasi ⋆-continuous multifunctions. Laprom et al. [26] introduced and studied the notion of β(τ₁, τ₂)-continuous multifunctions. Additionally, some characterizations of (τ₁, τ₂)δ-semicontinuous multifunctions, almost weakly ⋆-continuous multifunctions, weakly ⋆-continuous multifunctions, weakly α-⋆-continuous multifunctions, c*(τ₁, τ₂)-continuous multifunctions, β(⋆)-continuous multifunctions, almost weakly (τ₁, τ₂)-continuous multifunctions, almost (τ₁, τ₂)-continuous multifunctions and (τ₁, τ₂)α-continuous multifunctions were established in [6], [19], [4], [15], [14], [8], [20], [25] and [39], respectively. Pue-on et al. [35] introduce and investigated the notions of upper and lower (τ₁, τ₂)-continuous multifunctions. In this paper, we introduce the concepts of upper and lower c-(τ₁, τ₂)-continuous multifunctions. In particular, several characterizations of upper and lower c-(τ₁, τ₂)-continuous multifunctions are discussed.

2. Preliminaries

Throughout the present paper, spaces (X, τ₁, τ₂) and (Y, σ₁, σ₂) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ₁, τ₂). The closure of A and the interior of A with respect to τᵢ are denoted by τᵢ-Cl(A) and τᵢ-Int(A), respectively, for i = 1, 2. A subset A of a bitopological space (X, τ₁, τ₂) is called τ₁τ₂-closed [21] if A = τ₁-Cl(τ₂-Cl(A)). The complement of a τ₁τ₂-closed set is called τ₁τ₂-open. Let A be a subset of a bitopological space (X, τ₁, τ₂). The intersection of all τ₁τ₂-closed sets of X containing A is called the τ₁τ₂-closure [21] of A and is denoted by τ₁τ₂-Cl(A). The union of all τ₁τ₂-open sets of X contained in A is called the τ₁τ₂-interior [21] of A and is denoted by τ₁τ₂-Int(A).

Lemma 1. [21] Let A and B be subsets of a bitopological space (X, τ₁, τ₂). For the τ₁τ₂-closure, the following properties hold:

(1) A ⊆ τ₁τ₂-Cl(A) and τ₁τ₂-Cl(τ₁τ₂-Cl(A)) = τ₁τ₂-Cl(A).

(2) If A ⊆ B, then τ₁τ₂-Cl(A) ⊆ τ₁τ₂-Cl(B).

(3) τ₁τ₂-Cl(A) is τ₁τ₂-closed.

(4) A is τ₁τ₂-closed if and only if A = τ₁τ₂-Cl(A).
In particular, \( F \) is called \( (\tau_1, \tau_2)\)-continuous if every cover of \( X \) by \( \tau_1, \tau_2 \)-open sets of \( X \) has a finite subcover. A subset \( A \) of a bitopological space \((X, \tau_1, \tau_2)\) is called \((\tau_1, \tau_2)\)-open [39] (resp. \((\tau_1, \tau_2)\)s-open [6], \((\tau_1, \tau_2)\)p-open [6], \((\tau_1, \tau_2)\)\(\beta\)-open [6], \(\alpha(\tau_1, \tau_2)\)-open) [42] if \( A = \tau_1, \tau_2\)-Int(\(\tau_1, \tau_2\)-Cl(\(\alpha(\tau_1, \tau_2)\)-compact)) (resp. \( A \subseteq \tau_1, \tau_2\)-Cl(\(\tau_1, \tau_2\)-Int(\(\alpha(\tau_1, \tau_2)\)-open)), \( A \subseteq \tau_1, \tau_2\)-Int(\(\tau_1, \tau_2\)-Cl(\(\alpha(\tau_1, \tau_2)\)-open))). The complement of a \((\tau_1, \tau_2)\)-open set is called \((\tau_1, \tau_2)\)-closed (resp. \((\tau_1, \tau_2)\)s-closed, \((\tau_1, \tau_2)\)p-closed, \((\tau_1, \tau_2)\)\(\beta\)-closed, \(\alpha(\tau_1, \tau_2)\)-closed).

By a multifunction \( F : X \to Y \), we mean a point-to-set correspondence from \( X \) into \( Y \), and we always assume that \( F(x) \neq \emptyset \) for all \( x \in X \). For a multifunction \( F : X \to Y \), following [1] we shall denote the upper and lower inverse of a set \( B \) of \( Y \) by \( F^+(B) \) and \( F^-(B) \), respectively, that is, \( F^+(B) = \{ x \in X \mid F(x) \subseteq B \} \) and \( F^-(B) = \{ x \in X \mid F(x) \cap B \neq \emptyset \} \).

In particular, \( F^-(y) = \{ x \in X \mid y \in F(x) \} \) for each point \( y \in Y \). For each \( A \subseteq X \), \( F(A) = \bigcup_{x \in A} F(x) \).

### 3. Upper and lower \( c(\tau_1, \tau_2)\)-continuous multifunctions

In this section, we introduce the notions of upper and lower \( c(\tau_1, \tau_2)\)-continuous multifunctions. Moreover, we investigate some characterizations of upper and lower \( c(\tau_1, \tau_2)\)-continuous multifunctions.

**Definition 1.** A multifunction \( F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is said to be upper \( c(\tau_1, \tau_2)\)-continuous at a point \( x \in X \) if for each \( \sigma_1\sigma_2\)-open set \( V \) of \( Y \) containing \( F(x) \) and having \( \sigma_1\sigma_2\)-compact complement, there exists a \( \tau_1, \tau_2\)-open set \( U \) of \( X \) containing \( x \) such that \( F(U) \subseteq V \). A multifunction \( F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is said to be upper \( c(\tau_1, \tau_2)\)-continuous if \( F \) has this property at every point of \( X \).

**Theorem 1.** For a multifunction \( F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \), the following properties are equivalent:

1. \( F \) is upper \( c(\tau_1, \tau_2)\)-continuous at \( x \in X \);

2. \( x \in \tau_1, \tau_2\)-Int(\(F^+(V)\)) for each \( \sigma_1\sigma_2\)-open set \( V \) containing \( F(x) \) and having \( \sigma_1\sigma_2\)-compact complement;

3. \( x \in F^-(\sigma_1, \sigma_2\)-Cl(B)) for each subset \( B \) of \( Y \) having the \( \sigma_1\sigma_2\)-compact \( \sigma_1\sigma_2\)-closure such that \( x \in \tau_1, \tau_2\)-Cl(\(F^-(B)\));

4. \( x \in \tau_1, \tau_2\)-Int(\(F^+(B)\)) for each subset \( B \) of \( Y \) such that \( Y - \sigma_1, \sigma_2\)-Int(B) is \( \sigma_1\sigma_2\)-compact and \( x \in F^+(\sigma_1, \sigma_2\)-Int(B)).
Proof. (1) ⇒ (2): Let $V$ be any $\sigma_1\sigma_2$-open set of $Y$ containing $F(x)$ and having $\sigma_1\sigma_2$-compact complement and $x \in F^+(V)$. By (1), there exists a $\tau_1\tau_2$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq V$. Thus, $x \in U \subseteq F^+(V)$. Since $U$ is $\tau_1\tau_2$-open, we have $x \in \tau_1\tau_2$-Int($F^+(V)$).

(2) ⇒ (3): Suppose that $B$ is any subset of $Y$ having the $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closure. Then, $\sigma_1\sigma_2$-Cl($B$) is $\sigma_1\sigma_2$-closed and $Y - \sigma_1\sigma_2$-Cl($B$) is a $\sigma_1\sigma_2$-open set having $\sigma_1\sigma_2$-compact complement. Let $x \notin F^-(\sigma_1\sigma_2$-Cl($B$)). Thus,

$$x \in X - F^-(\sigma_1\sigma_2$-Cl($B$)) = F^+(Y - \sigma_1\sigma_2$-Cl($B$)).$$

This implies $F(x) \subseteq Y - \sigma_1\sigma_2$-Cl($B$). Since $Y - \sigma_1\sigma_2$-Cl($B$) is a $\sigma_1\sigma_2$-open set having $\sigma_1\sigma_2$-compact complement, by (2) we have

$$x \in \tau_1\tau_2$-Int($F^+(Y - \sigma_1\sigma_2$-Cl($B$))) = \tau_1\tau_2$-Int($X - F^-(\sigma_1\sigma_2$-Cl($B$)))$$

$$= X - \tau_1\tau_2$-Cl($F^-(\sigma_1\sigma_2$-Cl($B$)))$$

$$\subseteq X - \tau_1\tau_2$-Cl($F^-(B)$).$$

Therefore, $x \notin \tau_1\tau_2$-Cl($F^-(B)$).

(3) ⇒ (4): Let $B$ be any subset of $Y$ such that $Y - \sigma_1\sigma_2$-Int($B$) is $\sigma_1\sigma_2$-compact and let $x \notin \tau_1\tau_2$-Int($F^+(B)$). Then, we have

$$x \in X - \tau_1\tau_2$-Int($F^+(B)$) = \tau_1\tau_2$-Cl($X - F^+(B)$)$$

$$= \tau_1\tau_2$-Cl($F^-(Y - B)$)$$

and by (3),

$$x \in F^-(\sigma_1\sigma_2$-Cl($Y - B$)) = F^-(Y - \sigma_1\sigma_2$-Int($B$))$$

$$= X - F^+(\sigma_1\sigma_2$-Int($B$)).$$

Thus, $x \notin F^+(\sigma_1\sigma_2$-Int($B$)).

(4) ⇒ (1): Let $V$ be any $\sigma_1\sigma_2$-open set of $Y$ containing $F(x)$ and having $\sigma_1\sigma_2$-compact complement. We have $F^+(V) = F^+(\sigma_1\sigma_2$-Int($V$)). Then, $Y - \sigma_1\sigma_2$-Int($V$) = $Y - V$ which is $\sigma_1\sigma_2$-compact and by (4), $x \in \tau_1\tau_2$-Int($F^+(V)$). Therefore, there exists a $\tau_1\tau_2$-open set $U$ of $X$ containing $x$ such that $x \in U \subseteq F^+(V)$. Thus, $F(U) \subseteq V$. This shows that $F$ is upper $\mathcal{C}$-($\tau_1, \tau_2$)-continuous at $x$.

**Definition 2.** A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower $\mathcal{C}$-($\tau_1, \tau_2$)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$ and having $\sigma_1\sigma_2$-compact complement, there exists a $\tau_1\tau_2$-open set $U$ of $X$ containing $x$ such that $F(z) \cap V \neq \emptyset$ for each $z \in U$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower $\mathcal{C}$-($\tau_1, \tau_2$)-continuous if $F$ has this property at every point of $X$.

**Theorem 2.** For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:
Theorem 3. For a multifunction $F$ equivalent:

(1) $F$ is lower $c-(\tau_1, \tau_2)$-continuous at $x \in X$;

(2) $x \in \tau_1 \tau_2$-$\text{Int}(F^-(V))$ for each $\sigma_1 \sigma_2$-open set $V$ of $Y$ containing $F(x)$ and having $\sigma_1 \sigma_2$-compact complement;

(3) $x \in F^+(\sigma_1 \sigma_2$-$\text{Cl}(B))$ for each subset $B$ of $Y$ having the $\sigma_1 \sigma_2$-compact $\sigma_1 \sigma_2$-closure such that $x \in \tau_1 \tau_2$-$\text{Cl}(F^+(B))$;

(4) $x \in \tau_1 \tau_2$-$\text{Int}(F^-(B))$ for each subset $B$ of $Y$ such that $Y - \sigma_1 \sigma_2$-$\text{Int}(B)$ is $\sigma_1 \sigma_2$-compact and $x \in F^-(\sigma_1 \sigma_2$-$\text{Int}(B))$.

Proof. The proof is similar to that of Theorem 1.

**Definition 3.** A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be $c-(\tau_1, \tau_2)$-continuous at a point $x \in X$ if for each $\sigma_1 \sigma_2$-open set $V$ of $Y$ containing $f(x)$ and having $\sigma_1 \sigma_2$-compact complement, there exists a $\tau_1 \tau_2$-open set $U$ of $X$ containing $x$ such that $f(U) \subseteq V$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be $c-(\tau_1, \tau_2)$-continuous if $f$ has this property at every point of $X$.

**Corollary 1.** For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) $f$ is $c-(\tau_1, \tau_2)$-continuous at $x \in X$;

(2) $x \in \tau_1 \tau_2$-$\text{Int}(f^{-1}(V))$ for each $\sigma_1 \sigma_2$-open set $V$ of $Y$ containing $f(x)$ and having $\sigma_1 \sigma_2$-compact complement;

(3) $x \in f^{-1}(\sigma_1 \sigma_2$-$\text{Cl}(B))$ for each subset $B$ of $Y$ having the $\sigma_1 \sigma_2$-compact $\sigma_1 \sigma_2$-closure such that $x \in \tau_1 \tau_2$-$\text{Cl}(f^{-1}(B))$;

(4) $x \in \tau_1 \tau_2$-$\text{Int}(f^{-1}(B))$ for each subset $B$ of $Y$ such that $Y - \sigma_1 \sigma_2$-$\text{Int}(B)$ is $\sigma_1 \sigma_2$-compact and $x \in f^{-1}(\sigma_1 \sigma_2$-$\text{Int}(B))$.

**Theorem 3.** For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) $F$ is upper $c-(\tau_1, \tau_2)$-continuous;

(2) $F^+(V)$ is $\tau_1 \tau_2$-open in $X$ for each $\sigma_1 \sigma_2$-open set $V$ of $Y$ having $\sigma_1 \sigma_2$-compact complement;

(3) $F^-(K)$ is $\tau_1 \tau_2$-closed in $X$ for every $\sigma_1 \sigma_2$-compact $\sigma_1 \sigma_2$-closed set $K$ of $Y$;

(4) $\tau_1 \tau_2$-$\text{Cl}(F^-(B)) \subseteq F^-(\sigma_1 \sigma_2$-$\text{Cl}(B))$ for every subset $B$ of $Y$ having the $\sigma_1 \sigma_2$-compact $\sigma_1 \sigma_2$-closure;

(5) $F^+(\sigma_1 \sigma_2$-$\text{Int}(B)) \subseteq \tau_1 \tau_2$-$\text{Int}(F^+(B))$ for every subset $B$ of $Y$ such that $Y - \sigma_1 \sigma_2$-$\text{Int}(B)$ is $\sigma_1 \sigma_2$-compact.
Proof. (1) ⇒ (2): Let $V$ be any $\sigma_1\sigma_2$-open set of $Y$ containing $F(x)$ and having $\sigma_1\sigma_2$-compact complement and $x \in F^+(V)$. Then, we have $F(x) \subseteq V$. By Theorem 1, $x \in \tau_{1\tau_2}\text{-Int}(F^+(V))$. Thus, $F^+(V) \subseteq \tau_{1\tau_2}\text{-Int}(F^+(V))$ and hence $F^+(V)$ is $\tau_{1\tau_2}$-open in $X$.

(2) ⇒ (3): The proof follows immediately from the fact that $F^+(Y-B) = Y - F^-(B)$ for every subset $B$ of $Y$.

(3) ⇒ (4): Let $B$ be any subset of $Y$ having the $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closure. Then, $\sigma_1\sigma_2\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed and by (3), $F^-(\sigma_1\sigma_2\text{-Cl}(B))$ is $\tau_{1\tau_2}$-closed in $X$. Thus,

$$F^-(B) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B)) = \tau_{1\tau_2}\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(B)))$$

and hence $\tau_{1\tau_2}\text{-Cl}(F^-(B)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B))$.

(4) ⇒ (5): Let $B$ be any subset of $Y$ such that $Y - \sigma_1\sigma_2\text{-Int}(B)$ is $\sigma_1\sigma_2$-compact. By (4), we have

$$X - \tau_{1\tau_2}\text{-Int}(F^+(B)) = \tau_{1\tau_2}\text{-Cl}(X - F^+(B)) = \tau_{1\tau_2}\text{-Cl}(F^-(Y - B)) \subseteq \tau_{1\tau_2}\text{-Cl}(F^-(Y - \sigma_1\sigma_2\text{-Int}(B))) \subseteq F^-(Y - \sigma_1\sigma_2\text{-Int}(B)) = X - F^+(\sigma_1\sigma_2\text{-Int}(B)).$$

Thus, $F^+(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \tau_{1\tau_2}\text{-Int}(F^+(B))$.

(5) ⇒ (1): Let $x \in X$ and $V$ be any $\sigma_1\sigma_2$-open set of $Y$ containing $F(x)$ and having $\sigma_1\sigma_2$-compact complement. Then, $x \in F^+(V) = F^+(\sigma_1\sigma_2\text{-Int}(V)) \subseteq \tau_{1\tau_2}\text{-Int}(F^+(V))$. By Theorem 1, $F$ is upper $c(\tau_1, \tau_2)$-continuous at $x$. This shows that $F$ is upper $c(\tau_1, \tau_2)$-continuous.

Theorem 4. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) $F$ is lower $c(\tau_1, \tau_2)$-continuous;

(2) $F^-(V)$ is $\tau_{1\tau_2}$-open in $X$ for each $\sigma_1\sigma_2$-open set $V$ of $Y$ having $\sigma_1\sigma_2$-compact complement;

(3) $F^+(K)$ is $\tau_{1\tau_2}$-open in $X$ for every $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closed set $K$ of $Y$;

(4) $\tau_{1\tau_2}\text{-Cl}(F^+(B)) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(B))$ for every subset $B$ of $Y$ having the $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closure;

(5) $F^-(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \tau_{1\tau_2}\text{-Int}(F^-(B))$ for every subset $B$ of $Y$ such that $Y - \sigma_1\sigma_2\text{-Int}(B)$ is $\sigma_1\sigma_2$-compact.

Proof. The proof is similar to that of Theorem 3.
Corollary 2. For a function \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \), the following properties are equivalent:

1. \( f \) is \((\tau_1, \tau_2)\)-continuous;
2. \( f^{-1}(V) \) is \( \tau_1\tau_2 \)-open in \( X \) for each \( \sigma_1\sigma_2 \)-open set \( V \) of \( Y \) having \( \sigma_1\sigma_2 \)-compact complement;
3. \( f^{-1}(K) \) is \( \tau_1\tau_2 \)-open in \( X \) for every \( \sigma_1\sigma_2 \)-compact \( \sigma_1\sigma_2 \)-closed set \( K \) of \( Y \);
4. \( \tau_1\tau_2 - \text{Cl}(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2 - \text{Cl}(B)) \) for every subset \( B \) of \( Y \) having the \( \sigma_1\sigma_2 \)-compact \( \sigma_1\sigma_2 \)-closure;
5. \( f^{-1}(\sigma_1\sigma_2 - \text{Int}(B)) \subseteq \tau_1\tau_2 - \text{Int}(f^{-1}(B)) \) for every subset \( B \) of \( Y \) such that \( Y - \sigma_1\sigma_2 - \text{Int}(B) \) is \( \sigma_1\sigma_2 \)-compact.

4. Some characterizations

The \( \tau_1\tau_2 \)-frontier [21] of a subset \( A \) of a bitopological space \( (X, \tau_1, \tau_2) \), denoted by \( \tau_1\tau_2 - \text{fr}(A) \), is defined by

\[
\tau_1\tau_2 - \text{fr}(A) = \tau_1\tau_2 - \text{Cl}(A) \cap \tau_1\tau_2 - \text{Cl}(X - A) = \tau_1\tau_2 - \text{Cl}(A) - \tau_1\tau_2 - \text{Int}(A).
\]

Theorem 5. The set of all points \( x \in X \) at which a multifunction

\[
F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)
\]

is not upper \((\tau_1, \tau_2)\)-continuous is identical with the union of the \( \tau_1\tau_2 \)-frontier of the upper inverse images of the \( \sigma_1\sigma_2 \)-closures of \( \sigma_1\sigma_2 \)-open sets containing \( F(x) \) and having \( \sigma_1\sigma_2 \)-compact complement.

Proof. Suppose that \( F \) is not upper \((\tau_1, \tau_2)\)-continuous at \( x \in X \). Then, there exists a \( \sigma_1\sigma_2 \)-open set \( V \) of \( Y \) containing \( F(x) \) and having \( \sigma_1\sigma_2 \)-compact complement such that \( U \cap (X - F^+(V)) \neq \emptyset \) for every \( \tau_1\tau_2 \)-open set \( U \) of \( X \) containing \( x \). Then, we have \( x \in \tau_1\tau_2 - \text{Cl}(X - F^+(V)) \). On the other hand, we have \( x \in F^+(V) \subseteq \tau_1\tau_2 - \text{Cl}(F^+(V)) \) and hence \( x \in \tau_1\tau_2 - \text{fr}(F^+(V)) \).

Conversely, suppose that \( V \) is a \( \sigma_1\sigma_2 \)-open set of \( Y \) containing \( F(x) \) and having \( \sigma_1\sigma_2 \)-compact complement such that \( x \in \tau_1\tau_2 - \text{fr}(F^+(V)) \). If \( F \) is upper \((\tau_1, \tau_2)\)-continuous at \( x \in X \), there exists a \( \tau_1\tau_2 \)-open set \( U \) of \( X \) containing \( x \) such that \( U \subseteq F^+(V) \) and hence \( x \in \tau_1\tau_2 - \text{Int}(F^+(V)) \). This is a contradiction and so \( F \) is not upper \((\tau_1, \tau_2)\)-continuous at \( x \).

Theorem 6. The set of all points \( x \in X \) at which a multifunction

\[
F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)
\]

is not lower \((\tau_1, \tau_2)\)-continuous is identical with the union of the \( \tau_1\tau_2 \)-frontier of the lower inverse images of the \( \sigma_1\sigma_2 \)-closures of \( \sigma_1\sigma_2 \)-open sets meeting \( F(x) \) and having \( \sigma_1\sigma_2 \)-compact complement.
Lemma 3. Let \( V \) be paracompact and \( \tau \) regular and for each \( x \in X \), we have
\[
[21] \text{shows that } F \text{ is } \tau \text{ regular if for each } x \in X, \text{ then } Cl(F(x)) \text{ for each } x \in X.
\]

Theorem 8. For a multifunction \( F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \), we denote a multifunction defined as follows: \( Cl\sigma(x) = \sigma_1\sigma_2-Cl(F(x)) \) for each \( x \in X \).

Definition 4. [21] A subset \( A \) of a bitopological space \((X, \tau_1, \tau_2)\) is said to be:

1. \( \tau_1\tau_2 \)-paracompact if every cover of \( A \) by \( \tau_1\tau_2 \)-open sets of \( X \) is refined by a cover of \( A \) which consists of \( \tau_1\tau_2 \)-open sets of \( X \) and is \( \tau_1\tau_2 \)-locally finite in \( X \);

2. \( \tau_1\tau_2 \)-regular if for each \( x \in A \) and each \( \tau_1\tau_2 \)-open set \( U \) of \( X \) containing \( x \), there exists a \( \tau_1\tau_2 \)-open set \( V \) of \( X \) such that \( x \in V \subseteq \tau_1\tau_2 \)-Cl\( V \) \( \subseteq U \).

Lemma 2. [21] If \( A \) is a \( \tau_1\tau_2 \)-regular \( \tau_1\tau_2 \)-paracompact set of a bitopological space \((X, \tau_1, \tau_2)\) and \( U \) is a \( \tau_1\tau_2 \)-open neighbourhood of \( A \), then there exists a \( \tau_1\tau_2 \)-open set \( V \) of \( X \) such that \( A \subseteq V \subseteq \tau_1\tau_2 \)-Cl\( V \) \( \subseteq U \).

Lemma 3. [21] If \( F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is a multifunction such that \( F(x) \) is \( \tau_1\tau_2 \)-regular and \( \tau_1\tau_2 \)-paracompact for each \( x \in X \), then \( Cl\sigma_2^+(V) = F^+(V) \) for each \( \sigma_1\sigma_2 \)-open set \( V \) of \( Y \).

Theorem 7. Let \( F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a multifunction such that \( F(x) \) is \( \sigma_1\sigma_2 \)-paracompact and \( \sigma_1\sigma_2 \)-regular for each \( x \in X \). Then, the following properties are equivalent:

1. \( F \) is upper \( c-(\tau_1, \tau_2) \)-continuous;

2. \( Cl\sigma_2 \) is upper \( c-(\tau_1, \tau_2) \)-continuous.

Proof. We put \( G = Cl\sigma_2 \). Suppose that \( F \) is upper \( c-(\tau_1, \tau_2) \)-continuous. Let \( x \in X \) and \( V \) be any \( \sigma_1\sigma_2 \)-open set of \( Y \) containing \( G(x) \) and having \( \sigma_1\sigma_2 \)-compact complement. By Lemma 3, we have \( x \in G^+(V) = F^+(V) \) and hence there exists a \( \tau_1\tau_2 \)-open set \( U \) of \( X \) containing \( x \) such that \( F(U) \subseteq V \). Since \( F(z) \) is \( \sigma_1\sigma_2 \)-paracompact and \( \sigma_1\sigma_2 \)-regular for each \( z \in U \), by Lemma 2 there exists a \( \tau_1\tau_2 \)-open set \( W \) of \( X \) such that \( F(z) \subseteq W \subseteq \sigma_1\sigma_2-Cl(W) \subseteq V \); hence \( G(z) \subseteq \sigma_1\sigma_2-Cl(W) \subseteq V \) for each \( z \in U \). Thus, \( G(U) \subseteq V \) and hence \( G \) is upper \( c-(\tau_1, \tau_2) \)-continuous.

Conversely, suppose that \( G \) is upper \( c-(\tau_1, \tau_2) \)-continuous. Let \( x \in X \) and \( V \) be any \( \sigma_1\sigma_2 \)-open set of \( Y \) containing \( F(x) \) and having \( \sigma_1\sigma_2 \)-compact complement. By Lemma 3, we have \( x \in F^+(V) = G^+(V) \) and hence \( G(x) \subseteq V \). There exists a \( \tau_1\tau_2 \)-open set \( U \) of \( X \) containing \( x \) such that \( G(U) \subseteq V \). Thus, \( U \subseteq G^+(V) = F^+(V) \) and so \( F(U) \subseteq V \). This shows that \( F \) is upper \( c-(\tau_1, \tau_2) \)-continuous.

Lemma 4. [21] For a multifunction \( F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \), \( Cl\sigma_2^-(V) = F^-(V) \) for each \( \sigma_1\sigma_2 \)-open set \( V \) of \( Y \).

Theorem 8. For a multifunction \( F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \), the following properties are equivalent:
(1) $F$ is lower $c_{(\tau_1, \tau_2)}$-continuous;

(2) $ClF_{\otimes}$ is lower $c_{(\tau_1, \tau_2)}$-continuous.

Proof. By using Lemma 4 this can be shown similarly to that of Theorem 7.

Acknowledgements

This research project was financially supported by Mahasarakham University.

References


REFERENCES


