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c-(τ_1, τ_2)-Continuity for Multifunctions

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Abstract. This paper is concerned with the concepts of upper and lower c-(τ_1, τ_2)-continuous multifunctions. Moreover, several characterizations of upper and lower c-(τ_1, τ_2)-continuous multifunctions are investigated.

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1. Introduction

The field of the mathematical science which goes under the name of topology is concerned with all questions directly or indirectly related to continuity. Semi-open sets, preopen sets, α-open sets and β-open sets play an important role in topological spaces. Using these sets, many authors introduced and studied various types of generalizations of continuity for functions and multifunctions. In 1970, Gentry and Hoyle III [23] introduced and studied the concept of c-continuous functions. Furthermore, some characterizations of c-continuous functions were investigated in [28], [29] and [32], respectively. Duangphui et al. [22] introduced and studied the notion of $(\mu, \mu')^{(m,n)}$-continuous functions. Thongmoon and Boonpok [38] introduced and investigated the notion of strongly $\theta(\Lambda, p)$-continuous functions. Moreover, several characterizations of almost (Λ, p)-continuous functions, almost strongly $\theta(\Lambda, p)$-continuous functions, $\theta(\Lambda, p)$-continuous functions, weakly (Λ, b)-continuous functions, $\theta(\star)$-precontinuous functions, $\star$-continuous functions, θ-$\mathscr{I}$-continuous functions, almost (g, m)-continuous functions, (Λ, sp)-continuous functions, $\delta p(\Lambda, s)$-continuous functions, $(\Lambda, p(\star))$-continuous functions, pairwise almost M-continuous functions, (τ_1, τ_2)-continuous functions, almost (τ_1, τ_2)-continuous functions and weakly

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(τ_1, τ_2)-continuous functions were presented in [36], [12], [34], [17], [11], [10], [5], [2], [40], [37], [9], [3], [18], [16] and [13], respectively.

In 1975, Popa [33] introduced and studied the notion of quasi-continuous multifunctions. Neubrunn [30] and Holá et al. [24] extended the concept of c-continuous functions to the setting of multifunctions. Lipski [27] introduced the notion of c-continuous multifunctions as a generalization of c-continuous multifunctions [30] and quasi-continuous multifunctions [33]. Noiri and Popa [31] introduced and investigated the notion of C-m-continuous multifunctions. Viriyapong and Boonpok [41] introduced and studied the concept of weakly quasi (Λ, sp)-continuous multifunctions. In [7], the present author introduced and investigated the notions of almost quasi $\star$-continuous multifunctions and weakly quasi $\star$-continuous multifunctions. Laprom et al. [26] introduced and studied the notion of $\beta(\tau_1, \tau_2)$-continuous multifunctions. Additionally, some characterizations of $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions, almost weakly $\star$-continuous multifunctions, weakly $\star$-continuous multifunctions, weakly α-$\star$-continuous multifunctions, $\imath^\star$-continuous multifunctions, $\beta(\star)$-continuous multifunctions, almost weakly (τ_1, τ_2)-continuous multifunctions, almost (τ_1, τ_2)-continuous multifunctions and $(\tau_1, \tau_2)\alpha$-continuous multifunctions were established in [6], [19], [4], [15], [14], [8], [20], [25] and [39], respectively. Pue-on et al. [35] introduce and investigated the notions of upper and lower (τ_1, τ_2)-continuous multifunctions. In this paper, we introduce the concepts of upper and lower c-(τ_1, τ_2)-continuous multifunctions. In particular, several characterizations of upper and lower c-(τ_1, τ_2)-continuous multifunctions are discussed.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [21] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [21] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [21] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [21] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ and $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.

(3) $\tau_1\tau_2\text{-}Cl(A)$ is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2\text{-}Cl(A)$.

(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

A bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*compact* [21] if every cover of X by $\tau_1\tau_2$-open sets of X has a finite subcover. A subset A of a bitopological space (X, τ_1, τ_2) is called $(\tau_1, \tau_2)r$-*open* [39] (resp. $(\tau_1, \tau_2)s$-*open* [6], $(\tau_1, \tau_2)p$-*open* [6], $(\tau_1, \tau_2)\beta$-*open* [6], $\alpha(\tau_1, \tau_2)$-*open*) [42]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))))$. The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open, $\alpha(\tau_1, \tau_2)$-open) set is called $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*, $\alpha(\tau_1, \tau_2)$-*closed*).

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [1] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and

$$F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3. Upper and lower c-(τ_1, τ_2)-continuous multifunctions

In this section, we introduce the notions of upper and lower c-(τ_1, τ_2)-continuous multifunctions. Moreover, we investigate some characterizations of upper and lower c-(τ_1, τ_2)-continuous multifunctions.

Definition 1. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper c-(τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$ and having $\sigma_1\sigma_2$-compact complement, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq V$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper c-(τ_1, τ_2)-continuous if F has this property at every point of X.*

Theorem 1. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper c-(τ_1, τ_2)-continuous at $x \in X$;

(2) $x \in \tau_1\tau_2\text{-}Int(F^+(V))$ for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$ and having $\sigma_1\sigma_2$-compact complement;

(3) $x \in F^-(\sigma_1\sigma_2\text{-}Cl(B))$ for each subset B of Y having the $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closure such that $x \in \tau_1\tau_2\text{-}Cl(F^-(B))$;

(4) $x \in \tau_1\tau_2\text{-}Int(F^+(B))$ for each subset B of Y such that $Y - \sigma_1\sigma_2\text{-}Int(B)$ is $\sigma_1\sigma_2$-compact and $x \in F^+(\sigma_1\sigma_2\text{-}Int(B))$.

Proof. (1) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$ and having $\sigma_1\sigma_2$-compact complement and $x \in F^+(V)$. By (1), there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq V$. Thus, $x \in U \subseteq F^+(V)$. Since U is $\tau_1\tau_2$-open, we have $x \in \tau_1\tau_2\text{-Int}(F^+(V))$.

(2) $\Rightarrow$ (3): Suppose that B is any subset of Y having the $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closure. Then, $\sigma_1\sigma_2\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed and $Y - \sigma_1\sigma_2\text{-Cl}(B)$ is a $\sigma_1\sigma_2$-open set having $\sigma_1\sigma_2$-compact complement. Let $x \notin F^-(\sigma_1\sigma_2\text{-Cl}(B))$. Thus,

$$x \in X - F^-(\sigma_1\sigma_2\text{-Cl}(B)) = F^+(Y - \sigma_1\sigma_2\text{-Cl}(B)).$$

This implies $F(x) \subseteq Y - \sigma_1\sigma_2\text{-Cl}(B)$. Since $Y - \sigma_1\sigma_2\text{-Cl}(B)$ is a $\sigma_1\sigma_2$-open set having $\sigma_1\sigma_2$-compact complement, by (2) we have

$$\begin{aligned} x \in \tau_1\tau_2\text{-Int}(F^+(Y - \sigma_1\sigma_2\text{-Cl}(B))) &= \tau_1\tau_2\text{-Int}(X - F^-(\sigma_1\sigma_2\text{-Cl}(B))) \\ &= X - \tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(B))) \\ &\subseteq X - \tau_1\tau_2\text{-Cl}(F^-(B)). \end{aligned}$$

Therefore, $x \notin \tau_1\tau_2\text{-Cl}(F^-(B))$.

(3) $\Rightarrow$ (4): Let B be any subset of Y such that $Y - \sigma_1\sigma_2\text{-Int}(B)$ is $\sigma_1\sigma_2$-compact and let $x \notin \tau_1\tau_2\text{-Int}(F^+(B))$. Then, we have

$$\begin{aligned} x \in X - \tau_1\tau_2\text{-Int}(F^+(B)) &= \tau_1\tau_2\text{-Cl}(X - F^+(B)) \\ &= \tau_1\tau_2\text{-Cl}(F^-(Y - B)) \end{aligned}$$

and by (3),

$$\begin{aligned} x \in F^-(\sigma_1\sigma_2\text{-Cl}(Y - B)) &= F^-(Y - \sigma_1\sigma_2\text{-Int}(B)) \\ &= X - F^+(\sigma_1\sigma_2\text{-Int}(B)). \end{aligned}$$

Thus, $x \notin F^+(\sigma_1\sigma_2\text{-Int}(B))$.

(4) $\Rightarrow$ (1): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$ and having $\sigma_1\sigma_2$-compact complement. We have $F^+(V) = F^+(\sigma_1\sigma_2\text{-Int}(V))$. Then, $Y - \sigma_1\sigma_2\text{-Int}(V) = Y - V$ which is $\sigma_1\sigma_2$-compact and by (4), $x \in \tau_1\tau_2\text{-Int}(F^+(V))$. Therefore, there exists a $\tau_1\tau_2$-open set U of X containing x such that $x \in U \subseteq F^+(V)$. Thus, $F(U) \subseteq V$. This shows that F is upper c-(τ_1, τ_2)-continuous at x.

Definition 2. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower c-(τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$ and having $\sigma_1\sigma_2$-compact complement, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for each $z \in U$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower c-(τ_1, τ_2)-continuous if F has this property at every point of X.*

Theorem 2. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower c-(τ_1, τ_2)-continuous at $x \in X$;

(2) $x \in \tau_1\tau_2$-Int$(F^-(V))$ for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$ and having $\sigma_1\sigma_2$-compact complement;

(3) $x \in F^+(\sigma_1\sigma_2$-Cl$(B))$ for each subset B of Y having the $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closure such that $x \in \tau_1\tau_2$-Cl$(F^+(B))$;

(4) $x \in \tau_1\tau_2$-Int$(F^-(B))$ for each subset B of Y such that $Y - \sigma_1\sigma_2$-Int(B) is $\sigma_1\sigma_2$-compact and $x \in F^-(\sigma_1\sigma_2$-Int$(B))$.

Proof. The proof is similar to that of Theorem 1.

Definition 3. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be c-(τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$ and having $\sigma_1\sigma_2$-compact complement, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be c-(τ_1, τ_2)-continuous if f has this property at every point of X.*

Corollary 1. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is c-(τ_1, τ_2)-continuous at $x \in X$;

(2) $x \in \tau_1\tau_2$-Int$(f^{-1}(V))$ for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$ and having $\sigma_1\sigma_2$-compact complement;

(3) $x \in f^{-1}(\sigma_1\sigma_2$-Cl$(B))$ for each subset B of Y having the $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closure such that $x \in \tau_1\tau_2$-Cl$(f^{-1}(B))$;

(4) $x \in \tau_1\tau_2$-Int$(f^{-1}(B))$ for each subset B of Y such that $Y - \sigma_1\sigma_2$-Int(B) is $\sigma_1\sigma_2$-compact and $x \in f^{-1}(\sigma_1\sigma_2$-Int$(B))$.

Theorem 3. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper c-(τ_1, τ_2)-continuous;

(2) $F^+(V)$ is $\tau_1\tau_2$-open in X for each $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-compact complement;

(3) $F^-(K)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2$-Cl$(F^-(B)) \subseteq F^-(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y having the $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closure;

(5) $F^+(\sigma_1\sigma_2$-Int$(B)) \subseteq \tau_1\tau_2$-Int$(F^+(B))$ for every subset B of Y such that $Y - \sigma_1\sigma_2$-Int(B) is $\sigma_1\sigma_2$-compact.

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$ and having $\sigma_1\sigma_2$-compact complement and $x \in F^+(V)$. Then, we have $F(x) \subseteq V$. By Theorem 1, $x \in \tau_1\tau_2\text{-Int}(F^+(V))$. Thus, $F^+(V) \subseteq \tau_1\tau_2\text{-Int}(F^+(V))$ and hence $F^+(V)$ is $\tau_1\tau_2$-open in X.

(2) ⇒ (3): The proof follows immediately from the fact that $F^+(Y - B) = Y - F^-(B)$ for every subset B of Y.

(3) ⇒ (4): Let B be any subset of Y having the $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closure. Then, $\sigma_1\sigma_2\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed and by (3), $F^-(\sigma_1\sigma_2\text{-Cl}(B))$ is $\tau_1\tau_2$-closed in X. Thus,

$$\begin{aligned} F^-(B) &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B)) \\ &= \tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(B))) \end{aligned}$$

and hence $\tau_1\tau_2\text{-Cl}(F^-(B)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B))$.

(4) ⇒ (5): Let B be any subset of Y such that $Y - \sigma_1\sigma_2\text{-Int}(B)$ is $\sigma_1\sigma_2$-compact. By (4), we have

$$\begin{aligned} X - \tau_1\tau_2\text{-Int}(F^+(B)) &= \tau_1\tau_2\text{-Cl}(X - F^+(B)) \\ &= \tau_1\tau_2\text{-Cl}(F^-(Y - B)) \\ &\subseteq \tau_1\tau_2\text{-Cl}(F^-(Y - \sigma_1\sigma_2\text{-Int}(B))) \\ &\subseteq F^-(Y - \sigma_1\sigma_2\text{-Int}(B)) \\ &= X - F^+(\sigma_1\sigma_2\text{-Int}(B)). \end{aligned}$$

Thus, $F^+(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \tau_1\tau_2\text{-Int}(F^+(B))$.

(5) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$ and having $\sigma_1\sigma_2$-compact complement. Then, $x \in F^+(V) = F^+(\sigma_1\sigma_2\text{-Int}(V)) \subseteq \tau_1\tau_2\text{-Int}(F^+(V))$. By Theorem 1, F is upper c-(τ_1, τ_2)-continuous at x. This shows that F is upper c-(τ_1, τ_2)-continuous.

Theorem 4. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower c-(τ_1, τ_2)-continuous;

(2) $F^-(V)$ is $\tau_1\tau_2$-open in X for each $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-compact complement;

(3) $F^+(K)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2\text{-}Cl(F^+(B)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y having the $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closure;

(5) $F^-(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \tau_1\tau_2\text{-}Int(F^-(B))$ for every subset B of Y such that $Y - \sigma_1\sigma_2\text{-}Int(B)$ is $\sigma_1\sigma_2$-compact.

Proof. The proof is similar to that of Theorem 3.

Corollary 2. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is c-(τ_1, τ_2)-continuous;

(2) $f^{-1}(V)$ is $\tau_1\tau_2$-open in X for each $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-compact complement;

(3) $f^{-1}(K)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2$-Cl$(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y having the $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closure;

(5) $f^{-1}(\sigma_1\sigma_2$-Int$(B)) \subseteq \tau_1\tau_2$-Int$(f^{-1}(B))$ for every subset B of Y such that $Y - \sigma_1\sigma_2$-Int(B) is $\sigma_1\sigma_2$-compact.

4. Some characterizations

The $\tau_1\tau_2$-*frontier* [21] of a subset A of a bitopological space (X, τ_1, τ_2), denoted by $\tau_1\tau_2$-fr(A), is defined by

$$\tau_1\tau_2\text{-fr}(A) = \tau_1\tau_2\text{-Cl}(A) \cap \tau_1\tau_2\text{-Cl}(X - A) = \tau_1\tau_2\text{-Cl}(A) - \tau_1\tau_2\text{-Int}(A).$$

Theorem 5. *The set of all points $x \in X$ at which a multifunction*

$$F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$$

is not upper c-(τ_1, τ_2)-continuous is identical with the union of the $\tau_1\tau_2$-frontier of the upper inverse images of the $\sigma_1\sigma_2$-closures of $\sigma_1\sigma_2$-open sets containing $F(x)$ and having $\sigma_1\sigma_2$-compact complement.

Proof. Suppose that F is not upper c-(τ_1, τ_2)-continuous at $x \in X$. Then, there exists a $\sigma_1\sigma_2$-open set V of Y containing $F(x)$ and having $\sigma_1\sigma_2$-compact complement such that $U \cap (X - F^+(V)) \neq \emptyset$ for every $\tau_1\tau_2$-open set U of X containing x. Then, we have $x \in \tau_1\tau_2$-Cl$(X - F^+(V))$. On the other hand, we have $x \in F^+(V) \subseteq \tau_1\tau_2$-Cl$(F^+(V))$ and hence $x \in \tau_1\tau_2$-fr$(F^+(V))$.

Conversely, suppose that V is a $\sigma_1\sigma_2$-open set of Y containing $F(x)$ and having $\sigma_1\sigma_2$-compact complement such that $x \in \tau_1\tau_2$-fr$(F^+(V))$. If F is upper c-(τ_1, τ_2)-continuous at $x \in X$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $U \subseteq F^+(V)$ and hence $x \in \tau_1\tau_2$-Int$(F^+(V))$. This is a contradiction and so F is not upper c-(τ_1, τ_2)-continuous at x.

Theorem 6. *The set of all points $x \in X$ at which a multifunction*

$$F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$$

is not lower c-(τ_1, τ_2)-continuous is identical with the union of the $\tau_1\tau_2$-frontier of the lower inverse images of the $\sigma_1\sigma_2$-closures of $\sigma_1\sigma_2$-open sets meeting $F(x)$ and having $\sigma_1\sigma_2$-compact complement.

Proof. The proof is similar to that of Theorem 5.

For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, by $\mathrm{Cl}F_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ [21] we denote a multifunction defined as follows: $\mathrm{Cl}F_{\circledast}(x) = \sigma_1\sigma_2\text{-Cl}(F(x))$ for each $x \in X$.

Definition 4. [21] *A subset A of a bitopological space (X, τ_1, τ_2) is said to be:*

(1) $\tau_1\tau_2$-paracompact if every cover of A by $\tau_1\tau_2$-open sets of X is refined by a cover of A which consists of $\tau_1\tau_2$-open sets of X and is $\tau_1\tau_2$-locally finite in X;

(2) $\tau_1\tau_2$-regular if for each $x \in A$ and each $\tau_1\tau_2$-open set U of X containing x, there exists a $\tau_1\tau_2$-open set V of X such that $x \in V \subseteq \tau_1\tau_2\text{-Cl}(V) \subseteq U$.

Lemma 2. [21] *If A is a $\tau_1\tau_2$-regular $\tau_1\tau_2$-paracompact set of a bitopological space (X, τ_1, τ_2) and U is a $\tau_1\tau_2$-open neighbourhood of A, then there exists a $\tau_1\tau_2$-open set V of X such that $A \subseteq V \subseteq \tau_1\tau_2\text{-Cl}(V) \subseteq U$.*

Lemma 3. [21] *If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a multifunction such that $F(x)$ is $\tau_1\tau_2$-regular and $\tau_1\tau_2$-paracompact for each $x \in X$, then $\mathrm{Cl}F^{+}_{\circledast}(V) = F^{+}(V)$ for each $\sigma_1\sigma_2$-open set V of Y.*

Theorem 7. *Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a multifunction such that $F(x)$ is $\sigma_1\sigma_2$-paracompact and $\sigma_1\sigma_2$-regular for each $x \in X$. Then, the following properties are equivalent:*

(1) F is upper c-(τ_1, τ_2)-continuous;

(2) $\mathrm{Cl}F_{\circledast}$ is upper c-(τ_1, τ_2)-continuous.

Proof. We put $G = \mathrm{Cl}F_{\circledast}$. Suppose that F is upper c-(τ_1, τ_2)-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $G(x)$ and having $\sigma_1\sigma_2$-compact complement. By Lemma 3, we have $x \in G^{+}(V) = F^{+}(V)$ and hence there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq V$. Since $F(z)$ is $\sigma_1\sigma_2$-paracompact and $\sigma_1\sigma_2$-regular for each $z \in U$, by Lemma 2 there exists a $\tau_1\tau_2$-open set W of X such that $F(z) \subseteq W \subseteq \sigma_1\sigma_2\text{-Cl}(W) \subseteq V$; hence $G(z) \subseteq \sigma_1\sigma_2\text{-Cl}(W) \subseteq V$ for each $z \in U$. Thus, $G(U) \subseteq V$ and hence G is upper c-(τ_1, τ_2)-continuous.

Conversely, suppose that G is upper c-(τ_1, τ_2)-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$ and having $\sigma_1\sigma_2$-compact complement. By Lemma 3, we have $x \in F^{+}(V) = G^{+}(V)$ and hence $G(x) \subseteq V$. There exists a $\tau_1\tau_2$-open set U of X containing x such that $G(U) \subseteq V$. Thus, $U \subseteq G^{+}(V) = F^{+}(V)$ and so $F(U) \subseteq V$. This shows that F is upper c-(τ_1, τ_2)-continuous.

Lemma 4. [21] *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, $\mathrm{Cl}F^{-}_{\circledast}(V) = F^{-}(V)$ for each $\sigma_1\sigma_2$-open set V of Y.*

Theorem 8. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower c-(τ_1, τ_2)-continuous;

(2) Cl$F_{\circledast}$ is lower c-(τ_1, τ_2)-continuous.

Proof. By using Lemma 4 this can be shown similarly to that of Theorem 7.

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