Upper and Lower $s-(\tau_1, \tau_2)p$-Continuous Multifunctions

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Abstract. Our main purpose is to introduce the concepts of upper and lower $s-(\tau_1, \tau_2)p$-continuous multifunctions. Furthermore, several characterizations of upper and lower $s-(\tau_1, \tau_2)p$-continuous multifunctions are investigated.

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1. Introduction

In 1965, Lee [27] studied the notion of semiconnected functions. Kohli [24] introduced the notion of $s$-continuous functions and investigated several characterizations of semilocally connected spaces in terms of $s$-continuous functions. The class of $s$-continuity is a generalization of continuity and semiconnectedness. Furthermore, Kohli [25] introduced the concepts of $s$-regular spaces and completely $s$-regular spaces and proved that $s$-regularity and complete $s$-regularity are preserved under certain $s$-continuous functions. Duangphui et al. [21] introduced and investigated the notion of almost $(\mu, \mu')(m,n)$-continuous functions. Thongmoon and Boonpok [35] introduced and studied the notion of strongly $\theta(\Lambda,p)$-continuous functions. Moreover, several characterizations of almost $(\Lambda, p)$-continuous functions, almost strongly $\theta(\Lambda,p)$-continuous functions, $\theta(\Lambda,p)$-continuous functions, weakly $(\Lambda,b)$-continuous functions, $\theta(\ast)$-precontinuous functions, $s$-continuous functions, $\theta$-$\mathcal{I}$-continuous functions, almost $(g,m)$-continuous functions, $(\Lambda, sp)$-continuous functions, $\delta p(\Lambda, s)$-continuous functions, $(\Lambda, p(\ast))$-continuous functions, pairwise almost $M$-continuous functions, $(\tau_1, \tau_2)$-continuous functions, almost $(\tau_1, \tau_2)$-continuous functions and weakly $(\tau_1,\tau_2)$-continuous functions were presented in [33], [11], [31], [16], [10], [9], [5], [2], [37], [34], [8], [3], [17], [15] and [12], respectively.

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In 1989, Lipski [28] extended the concept of $s$-continuous functions to the setting of multifunctions. Popa [29] introduced the concept of precontinuous multifunctions and showed that $H$-almost continuity and precontinuity are equivalent for multifunctions. Ewert and Lipski [22] introduced and investigated the concept of $s$-quasi-continuous multifunctions. Popa and Noiri [30] introduced and studied the notion of $s$-precontinuous multifunctions as a generalization of $s$-continuous multifunctions and precontinuous multifunctions. Laprom et al. [26] introduced and investigated the concept of $\beta(\tau_1, \tau_2)$-continuous multifunctions. In particular, some characterizations of $(\tau_1, \tau_2)$-continuous multifunctions, almost weakly $\star$-continuous multifunctions, weakly $\star\star$-continuous multifunctions, $\tau^*$-continuous multifunctions, almost $\beta(\star)$-continuous multifunctions, almost weakly $(\tau_1, \tau_2)$-continuous multifunctions, weakly $\alpha$$\star$-continuous multifunctions, $r^*$-continuous multifunctions, and $(\tau_1, \tau_2)$$\alpha$-continuous multifunctions were established in [6], [18], [4], [14], [13], [7], [19], [23] and [36], respectively. Pue-on et al. [32] introduce and studied the concepts of upper and lower $s-(\tau_1, \tau_2)p$-continuous multifunctions. We also investigate several characterizations of upper and lower $s-(\tau_1, \tau_2)p$-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ (or simply $X$ and $Y$) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let $A$ be a subset of a bitopological space $(X, \tau_1, \tau_2)$. The closure of $A$ and the interior of $A$ with respect to $\tau_i$ are denoted by $\tau_i$-$\text{Cl}(A)$ and $\tau_i$-$\text{Int}(A)$, respectively, for $i = 1, 2$. A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_1\tau_2$-$\text{closed}$ [20] if $A = \tau_1$-$\text{Cl}(\tau_2$-$\text{Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-$\text{open}$. The intersection of all $\tau_1\tau_2$-closed sets of $X$ containing $A$ is called the $\tau_1\tau_2$-$\text{closure}$ [20] of $A$ and is denoted by $\tau_1\tau_2$-$\text{Cl}(A)$. The union of all $\tau_1\tau_2$-open sets of $X$ contained in $A$ is called the $\tau_1\tau_2$-$\text{interior}$ [20] of $A$ and is denoted by $\tau_1\tau_2$-$\text{Int}(A)$.

Lemma 1. [20] Let $A$ and $B$ be subsets of a bitopological space $(X, \tau_1, \tau_2)$. For the $\tau_1\tau_2$-closure, the following properties hold:

1. $A \subseteq \tau_1\tau_2$-$\text{Cl}(A)$ and $\tau_1\tau_2$-$\text{Cl}(\tau_1\tau_2$-$\text{Cl}(A)) = \tau_1\tau_2$-$\text{Cl}(A)$.

2. If $A \subseteq B$, then $\tau_1\tau_2$-$\text{Cl}(A) \subseteq \tau_1\tau_2$-$\text{Cl}(B)$.

3. $\tau_1\tau_2$-$\text{Cl}(A)$ is $\tau_1\tau_2$-$\text{closed}$.

4. $A$ is $\tau_1\tau_2$-$\text{closed}$ if and only if $A = \tau_1\tau_2$-$\text{Cl}(A)$.

5. $\tau_1\tau_2$-$\text{Cl}(X - A) = X - \tau_1\tau_2$-$\text{Int}(A)$.

A bitopological space $(X, \tau_1, \tau_2)$ is said to be $\tau_1\tau_2$-$\text{connected}$ [20] if $X$ cannot be written as the union of two nonempty disjoint $\tau_1\tau_2$-open sets. A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $(\tau_1, \tau_2)r$-$\text{open}$ [36] (resp. $(\tau_1, \tau_2)s$-$\text{open}$ [6], $(\tau_1, \tau_2)p$-$\text{open}$ [6], $(\tau_1, \tau_2)\beta$-$\text{open}$ [6], $\alpha(\tau_1, \tau_2)$-$\text{open}$ [38]) if $A = \tau_1\tau_2$-$\text{Int}(\tau_1\tau_2$-$\text{Cl}(A))$ (resp. $A \subseteq$
In particular, $Y$, and we always assume that $F$ following [1] we shall denote the upper and lower inverse of a set $F$ of multifunctions. Moreover, some characterizations of upper and lower $s$-continuous multifunctions are discussed.

**Definition 1.** A multifunction $F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be upper $s-(\tau_1, \tau_2)p$-continuous if for $x \in X$ and each $\sigma_1\sigma_2$-open set $V$ of $Y$ containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement, there exists a $(\tau_1, \tau_2)p$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq V$.

**Theorem 1.** For a multifunction $F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) $F$ is upper $s-(\tau_1, \tau_2)p$-continuous;

(2) $F^+(V)$ is $(\tau_1, \tau_2)p$-open in $X$ for every $\sigma_1\sigma_2$-open set $V$ of $Y$ having $\sigma_1\sigma_2$-connected complement.

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**Lemma 2.** For a subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$, the following properties hold:

(1) $A$ is $(\tau_1, \tau_2)p$-closed if and only if $(\tau_1, \tau_2)-pCl(A) = A$;

(2) $(\tau_1, \tau_2)-pCl(A) = \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)) \cup A$;

(3) $(\tau_1, \tau_2)-pCl((\tau_1, \tau_2)pCl(A)) = (\tau_1, \tau_2)pCl(A)$.

By a multifunction $F : X \rightarrow Y$, we mean a point-to-set correspondence from $X$ into $Y$, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \rightarrow Y$, following [1] we shall denote the upper and lower inverse of a set $B$ of $Y$ by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{ x \in X \mid F(x) \subseteq B \}$ and $F^-(B) = \{ x \in X \mid F(x) \cap B \neq \emptyset \}$.

In particular, $F^-(y) = \{ x \in X \mid y \in F(x) \}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \bigcup_{x \in A} F(x)$.

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3. Upper and lower $s-(\tau_1, \tau_2)p$-continuous multifunctions

In this section, we introduce the notions of upper and lower $s-(\tau_1, \tau_2)p$-continuous multifunctions. Moreover, some characterizations of upper and lower $s-(\tau_1, \tau_2)p$-continuous multifunctions are discussed.

**Definition 1.** A multifunction $F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be upper $s-(\tau_1, \tau_2)p$-continuous if for $x \in X$ and each $\sigma_1\sigma_2$-open set $V$ of $Y$ containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement, there exists a $(\tau_1, \tau_2)p$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq V$.

**Theorem 1.** For a multifunction $F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) $F$ is upper $s-(\tau_1, \tau_2)p$-continuous;

(2) $F^+(V)$ is $(\tau_1, \tau_2)p$-open in $X$ for every $\sigma_1\sigma_2$-open set $V$ of $Y$ having $\sigma_1\sigma_2$-connected complement;
(3) $F^-(K)$ is $(\tau_1, \tau_2)p$-closed in $X$ for every $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closed set $K$ of $Y$;

(4) $\tau_1\tau_2-Cl(\tau_1\tau_2-Int(F^-(B))) \subseteq F^-(\sigma_1\sigma_2-Cl(B))$ for every subset $B$ of $Y$ having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure;

(5) $(\tau_1, \tau_2)-pCl(F^-(B)) \subseteq F^-(\sigma_1\sigma_2-Cl(B))$ for every subset $B$ of $Y$ having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure;

(6) $F^+(\sigma_1\sigma_2-Int(B)) \subseteq (\tau_1, \tau_2)-pInt(F^+(B))$ for every subset $B$ of $Y$ such that $Y - \sigma_1\sigma_2-Int(B)$ is $\sigma_1\sigma_2$-connected.

Proof. (1) $\Rightarrow$ (2): Let $V$ be any $\sigma_1\sigma_2$-open set of $Y$ having $\sigma_1\sigma_2$-connected complement and $x \in F^+(V)$. Then, there exists a $(\tau_1, \tau_2)p$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq V$. Therefore, we have $x \in U \subseteq \tau_1\tau_2-Int(\tau_1\tau_2-Cl(F^+(V)))$. Thus,

$$F^+(V) \subseteq \tau_1\tau_2-Int(\tau_1\tau_2-Cl(F^+(V)))$$

and hence $F^+(V)$ is $(\tau_1, \tau_2)p$-open in $X$.

(2) $\Rightarrow$ (3): The proof follows immediately from the fact that $F^+(Y - B) = X - F^-(B)$ for every subset $B$ of $Y$.

(3) $\Rightarrow$ (4): Let $B$ be any subset of $Y$ having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure. Then, $F^-(\sigma_1\sigma_2-Cl(B))$ is a $(\tau_1, \tau_2)p$-closed set of $X$. By Lemma 2, we have

$$\tau_1\tau_2-Cl(\tau_1\tau_2-Int(F^-(B))) \subseteq \tau_1\tau_2-Cl(\tau_1\tau_2-Int(F^-(\sigma_1\sigma_2-Cl(B))))$$

$$\subseteq (\tau_1, \tau_2)-pCl(F^-(\sigma_1\sigma_2-Cl(B)))$$

$$= F^-(\sigma_1\sigma_2-Cl(B)).$$

(4) $\Rightarrow$ (5): Let $B$ be any subset of $Y$ having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure. It follows from Lemma 2 that

$$(\tau_1, \tau_2)-pCl(F^-(B)) = F^-(B) \cup \tau_1\tau_2-Cl(\tau_1\tau_2-Int(F^-(B)))$$

$$\subseteq F^-(\sigma_1\sigma_2-Cl(B)).$$

(5) $\Rightarrow$ (6): Let $B$ be any subset of $Y$ such that $Y - \sigma_1\sigma_2-Int(B)$ is $\sigma_1\sigma_2$-connected. By (5),

$$X - (\tau_1, \tau_2)-Int(F^+(B)) = (\tau_1, \tau_2)-pCl(X - F^+(B))$$

$$= (\tau_1, \tau_2)-pCl(F^-(Y - B))$$

$$\subseteq F^-(\sigma_1\sigma_2-Cl(Y - B))$$

$$= F^-(Y - \sigma_1\sigma_2-Int(B))$$

$$= X - F^+(\sigma_1\sigma_2-Int(B)).$$
Thus, $F^+(\sigma_1\sigma_2{-}\text{Int}(B)) \subseteq (\tau_1, \tau_2){-}\text{pInt}(F^+(B))$.

(6) $\Rightarrow$ (1): Let $x \in X$ and $V$ be any $\sigma_1\sigma_2$-open set of $Y$ containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement. By (6), we have

$$F^+(V) = F^+(\sigma_1\sigma_2{-}\text{Int}(V)) \subseteq (\tau_1, \tau_2){-}\text{pInt}(F^+(V)).$$

Put $U = (\tau_1, \tau_2){-}\text{pInt}(F^+(V))$. Then, $U$ is a $(\tau_1, \tau_2)$-open set of $X$ containing $x$ such that $F(U) \subseteq V$. This shows that $F$ is upper $s(\tau_1, \tau_2)$-continuous.

**Definition 2.** A multifunction $F:\ (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower $s(\tau_1, \tau_2)$-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set $V$ of $Y$ having $\sigma_1\sigma_2$-connected complement such that $F(x) \cap V \neq \emptyset$, there exists a $(\tau_1, \tau_2)$-open set $U$ of $X$ containing $x$ such that $F(z) \cap V \neq \emptyset$ for each $z \in U$.

**Theorem 2.** For a multifunction $F:\ (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

1. $F$ is lower $s(\tau_1, \tau_2)$-continuous;
2. $F^{-}(V)$ is $(\tau_1, \tau_2)$-open in $X$ for every $\sigma_1\sigma_2$-open set $V$ of $Y$ having $\sigma_1\sigma_2$-connected complement;
3. $F^{+}(K)$ is $(\tau_1, \tau_2)$-closed in $X$ for every $\sigma_1\sigma_2$-closed $\sigma_1\sigma_2$-connected set $K$ of $Y$;
4. $\tau_1\tau_2{-}\text{Cl}(\tau_1\tau_2{-}\text{Int}(F^+(B))) \subseteq F^+(\sigma_1\sigma_2{-}\text{Cl}(B))$ for every subset $B$ of $Y$ having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure;
5. $(\tau_1, \tau_2){-}\text{pCl}(F^+(B)) \subseteq F^+(\sigma_1\sigma_2{-}\text{Cl}(B))$ for every subset $B$ of $Y$ having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure;
6. $F^{-}(\sigma_1\sigma_2{-}\text{Int}(B)) \subseteq (\tau_1, \tau_2){-}\text{pInt}(F^-(B))$ for every subset $B$ of $Y$ such that $Y - \sigma_1\sigma_2{-}\text{Int}(B)$ is $\sigma_1\sigma_2$-connected.

**Proof.** The proof is similar to that of Theorem 1.

**Definition 3.** A function $f:\ (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower $s(\tau_1, \tau_2)$-continuous if for each point $x \in X$ and each $\sigma_1\sigma_2$-open set $V$ of $Y$ containing $f(x)$ and having $\sigma_1\sigma_2$-connected complement, there exists a $(\tau_1, \tau_2)$-open set $U$ of $X$ containing $x$ such that $f(U) \subseteq V$.

**Corollary 1.** For a function $f:\ (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

1. $f$ is lower $s(\tau_1, \tau_2)$-continuous.
(2) $f^{-1}(V)$ is $(\tau_1, \tau_2)p$-open in $X$ for every $\sigma_1\sigma_2$-open set $V$ of $Y$ having $\sigma_1\sigma_2$-connected complement;

(3) $f^{-1}(K)$ is $(\tau_1, \tau_2)p$-closed in $X$ for every $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closed set $K$ of $Y$;

(4) $\tau_1\tau_2$-$\text{Cl}(\tau_1\tau_2$-$\text{Int}(f^{-1}(B))) \subseteq f^{-1}(\sigma_1\sigma_2$-$\text{Cl}(B))$ for every subset $B$ of $Y$ having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure;

(5) $(\tau_1, \tau_2)$-$p\text{Cl}(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2$-$\text{Cl}(B))$ for every subset $B$ of $Y$ having the $\sigma_1\sigma_2$-connected $\sigma_1\sigma_2$-closure;

(6) $f^{-1}(\sigma_1\sigma_2$-$\text{Int}(B)) \subseteq (\tau_1, \tau_2)$-$p\text{Int}(f^{-1}(B))$ for every subset $B$ of $Y$ such that $Y - \sigma_1\sigma_2$-$\text{Int}(B)$ is $\sigma_1\sigma_2$-connected.

**Corollary 2.** A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper $s$-$(\tau_1, \tau_2)p$-continuous if $F^{-1}(V)$ is $(\tau_1, \tau_2)p$-closed in $X$ for every $\sigma_1\sigma_2$-connected set $V$ of $Y$.

**Proof.** Let $V$ be any $\sigma_1\sigma_2$-open set of $Y$ having $\sigma_1\sigma_2$-connected complement. Then, $Y - V$ is $\sigma_1\sigma_2$-connected and $F^{-}(Y - V)$ is $(\tau_1, \tau_2)p$-closed in $X$. Thus, $F^{-}(V)$ is $(\tau_1, \tau_2)p$-open in $X$ and by Theorem 1, $F$ is upper $s$-$(\tau_1, \tau_2)p$-continuous.

**Corollary 3.** A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower $s$-$(\tau_1, \tau_2)p$-continuous if $F^{+}(V)$ is $(\tau_1, \tau_2)p$-closed in $X$ for every $\sigma_1\sigma_2$-connected set $V$ of $Y$.

**Proof.** The proof is similar to that of Corollary 2.

For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, by $\text{Cl}_{\delta}F_{\delta} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ [20] we denote a multifunction defined as follows: $\text{Cl}_{\delta}F_{\delta}(x) = \sigma_1\sigma_2$-$\text{Cl}(F(x))$ for each $x \in X$.

**Definition 4.** [20] A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be:

(1) $\tau_1\tau_2$-paracompact if every cover of $A$ by $\tau_1\tau_2$-open sets of $X$ is refined by a cover of $A$ which consists of $\tau_1\tau_2$-open sets of $X$ and is $\tau_1\tau_2$-locally finite in $X$;

(2) $\tau_1\tau_2$-regular if for each $x \in A$ and each $\tau_1\tau_2$-open set $U$ of $X$ containing $x$, there exists a $\tau_1\tau_2$-open set $V$ of $X$ such that $x \in V \subseteq \tau_1\tau_2$-$\text{Cl}(V) \subseteq U$.

**Lemma 3.** [20] If $A$ is a $\tau_1\tau_2$-regular $\tau_1\tau_2$-paracompact set of a bitopological space $(X, \tau_1, \tau_2)$ and $U$ is a $\tau_1\tau_2$-open neighbourhood of $A$, then there exists a $\tau_1\tau_2$-open set $V$ of $X$ such that $A \subseteq V \subseteq \tau_1\tau_2$-$\text{Cl}(V) \subseteq U$.

**Lemma 4.** [20] If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a multifunction such that $F(x)$ is $\tau_1\tau_2$-regular and $\tau_1\tau_2$-paracompact for each $x \in X$, then $\text{Cl}_{\delta}F^{+}_{\delta}(V) = F^{+}(V)$ for each $\sigma_1\sigma_2$-open set $V$ of $Y$. 
Theorem 3. Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a multifunction such that $F(x)$ is $\sigma_1\sigma_2$-paracompact and $\sigma_1\sigma_2$-regular for each $x \in X$. Then, the following properties are equivalent:

(1) $F$ is upper $s-(\tau_1, \tau_2)p$-continuous;

(2) $Cl F_0$ is upper $s-(\tau_1, \tau_2)p$-continuous.

Proof. We put $G = Cl F_0$. Suppose that $F$ is upper $s-(\tau_1, \tau_2)p$-continuous. Let $x \in X$ and $V$ be any $\sigma_1\sigma_2$-open set of $Y$ containing $G(x)$ and having $\sigma_1\sigma_2$-connected complement. By Lemma 4, we have $x \in G^+(V) = F^+(V)$ and hence there exists a $\tau_1\tau_2$-open set $U$ of $X$ containing $x$ such that $F(U) \subseteq V$. Since $F(z)$ is $\sigma_1\sigma_2$-paracompact and $\sigma_1\sigma_2$-regular for each $z \in U$, by Lemma 3 there exists a $\tau_1\tau_2$-open set $W$ of $X$ such that $F(z) \subseteq W \subseteq \sigma_1\sigma_2-Cl(W) \subseteq V$; hence $G(z) \subseteq \sigma_1\sigma_2-Cl(W) \subseteq V$ for each $z \in U$. Thus, $G(U) \subseteq V$ and hence $G$ is upper $s-(\tau_1, \tau_2)p$-continuous.

Conversely, suppose that $G$ is upper $s-(\tau_1, \tau_2)p$-continuous. Let $x \in X$ and $V$ be any $\sigma_1\sigma_2$-open set of $Y$ containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement. By Lemma 4, we have $x \in F^+(V) = G^+(V)$ and hence $G(x) \subseteq V$. There exists a $\tau_1\tau_2$-open set $U$ of $X$ containing $x$ such that $G(U) \subseteq V$. Thus, $U \subseteq G^+(V) = F^+(V)$ and so $F(U) \subseteq V$. This shows that $F$ is upper $s-(\tau_1, \tau_2)p$-continuous.

Lemma 5. [20] For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, $Cl F_0^-(V) = F^-(V)$ for each $\sigma_1\sigma_2$-open set $V$ of $Y$.

Theorem 4. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

(1) $F$ is lower $s-(\tau_1, \tau_2)p$-continuous;

(2) $Cl F_0$ is lower $s-(\tau_1, \tau_2)p$-continuous.

Proof. By using Lemma 5 this can be shown similarly to that of Theorem 3.

The $(\tau_1, \tau_2)p$-frontier of a subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$, denoted by $(\tau_1, \tau_2)-pfr(A)$, is defined by

$$(\tau_1, \tau_2)-pfr(A) = (\tau_1, \tau_2)-pCl(A) \cap (\tau_1, \tau_2)-pCl(X - A)$$

$$= (\tau_1, \tau_2)-pCl(A) - (\tau_1, \tau_2)-pInt(A).$$

Theorem 5. The set of all points $x$ of $X$ at which a multifunction

$$F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$$

is not upper $s-(\tau_1, \tau_2)p$-continuous is identical with the union of the $(\tau_1, \tau_2)p$-frontier of the upper inverse images of the $\sigma_1\sigma_2$-closures of $\sigma_1\sigma_2$-open sets containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement.
Proof. Suppose that $F$ is not upper $s-(\tau_1, \tau_2)p$-continuous at $x \in X$. Then, there exists a $\sigma_1\sigma_2$-open set $V$ of $Y$ containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement such that $U \cap (X - F^+(V)) \neq \emptyset$ for every $(\tau_1, \tau_2)p$-open set $U$ of $X$ containing $x$. Therefore, we have $x \in (\tau_1, \tau_2)pCl(X - F^+(V))$. On the other hand, we have

$$x \in F^+(V) \subseteq (\tau_1, \tau_2)pCl(F^+(V))$$

and hence $x \in (\tau_1, \tau_2)pfr(F^+(V))$.

Conversely, suppose that $V$ is a $\sigma_1\sigma_2$-open set of $Y$ containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement such that $x \in (\tau_1, \tau_2)pfr(F^+(V))$. If $F$ is upper $s-(\tau_1, \tau_2)p$-continuous at $x \in X$, there exists a $(\tau_1, \tau_2)p$-open set $U$ of $X$ containing $x$ such that $U \subseteq F^+(V)$; hence $x \in (\tau_1, \tau_2)pInt(F^+(V))$. This is a contradiction and so $F$ is not upper $s-(\tau_1, \tau_2)p$-continuous at $x$.

**Theorem 6.** The set of all points $x$ of $X$ at which a multifunction

$$F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$$

is not lower $s-(\tau_1, \tau_2)p$-continuous is identical with the union of the $(\tau_1, \tau_2)p$-frontier of the lower inverse images of the $\sigma_1\sigma_2$-closures of $\sigma_1\sigma_2$-open sets meeting $F(x)$ and having $\sigma_1\sigma_2$-connected complement.

Proof. The proof is similar to that of Theorem 5.

**4. Conclusion**

This paper deals with the notions of upper and lower $s-(\tau_1, \tau_2)p$-continuous multifunctions. Furthermore, some characterizations and several properties concerning upper and lower $s-(\tau_1, \tau_2)p$-continuous multifunctions are established. In the upcoming work, we plan to apply the concepts initiated in this paper to study a new generalization of upper (lower) $s-(\tau_1, \tau_2)p$-continuous multifunctions, namely upper (lower) almost $s-(\tau_1, \tau_2)p$-continuous multifunctions. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called upper (lower) almost $s-(\tau_1, \tau_2)p$-continuous multifunctions if for each $x \in X$ and each $\sigma_1\sigma_2$-open set $V$ of $Y$ having $\sigma_1\sigma_2$-connected complement such that $x \in F^+(V)$ ($x \in F^-(V)$), there exists a $(\tau_1, \tau_2)p$-open set $U$ of $X$ containing $x$ such that $U \subseteq F^+(\sigma_1\sigma_2-Int(\sigma_1\sigma_2-Cl(V)))$ ($U \subseteq F^-(\sigma_1\sigma_2-Int(\sigma_1\sigma_2-Cl(V)))$). The class of upper (lower) $s-(\tau_1, \tau_2)p$-continuous multifunctions included in the class of upper (lower) almost $s-(\tau_1, \tau_2)p$-continuous multifunctions.

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References


