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On Intuitionistic Fuzzy Implicative Hyper GR-ideals

Amila P. Macodi[1,*], Archie G. Dorig[2]

[1] *Mathematics Department, Faculty, Mindanao State University, Marawi City, Lanao del Sur, Philippines*
[2] *Department of Fisheries, Marine Biology and Environmental Sciences, Faculty, North Eastern Mindanao State University - Lianga Campus, Lianga, Surigao del Sur, Philippines*

Abstract. The influx of research on hyperstructure theory has encouraged many researchers to introduce new algebras. Notable among these is the work of Indangan and Petalcorin [5] who introduced hyper GR-algebras and that of Macodi and Petalcorin [12] who studied a fuzzification of hyper GR-algebras. Macodi extended the fuzzification of hyper GR-algebras into intuitionistic fuzzification and introduced the concept of an intuitionistic fuzzy hyper GR-ideals [13]. Following the works of Macodi and Petalcorin, this paper established an intuitionistic fuzzy implicative hyper GR-ideals and obtained their characterization using level subsets. Moreover, some properties of intuitionistic fuzzy implicative hyper GR-ideals are presented.

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1. Introduction

Hyperstructure theory, also called multi-algebras, was introduced in 1934 by Marty [14] at the 8^{th} Congress of Scandinavian Mathematicians. Hyperstructures have many applications in several sectors of both pure and applied sciences. It is for this reason that many researchers work on this subject. Jun, et al. [20] applied the hyperstructures to BCK-algebras and introduced the concept of generalizing a BCK-algebra into a hyper BCK-algebra. They also investigated some properties of hyper BCK-algebra. After the introduction of the concept of hyper BCK-algebras, several studies were conducted. Amongst these studies, the most notable are those made by Jun and Long [7], Jun and Shim [8], Borzooei and Bakhsi [2], Borzooei and Jun [3], and Jun and Song [9].

Based on this hyperstructure, Indangan and Petalcorin [5, 6] introduced a new hyperstructure which is called a hyper GR-algebra. They established some results on

*Corresponding author.
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Email addresses: amila.macodi@msumain.edu.ph (A. Macodi), archie.dorig@msumain.edu.ph (A. Dorig)

hyper GR-ideals of a hyper GR-algebra and some hyper homomorphic properties of hyper GR-algebras. In 2019, Macodi and Petalcorin [12] published some results on fuzzy implicative hyper GR-ideals of hyper GR-algebras. They obtained more results on fuzzy structures in Hyper GR-algebras [11] and also established intuitionistic fuzzy hyper GR-ideals in hyper GR-algebras [13].

The concept of intuitionistic fuzzy sets was developed by Atanassov [1] as an extension of Zadeh's fuzzy set [21]. A prominent characteristic of intuitionistic fuzzy sets is that they assign to each element a membership degree and a non-membership degree. Intutionistic fuzzy set theory basically defies the claim that an element x belongs to a given degree say $\mu(x)$ to a fuzzy set A, it naturally follows that x should not belong to A to the extent $1 - \mu(x)$. Consequently, many authors have paid attention to the intuitionistic fuzzy set theory since it has been successfully applied in different areas, such as logic programming [10], decision making problems [4] and medical diagnosis [15], to name a few, and related studies are made in [17, 18]. The concept is also applied in various applications, such as intuitionistic fuzzy neural networks, intuitionistic fuzzy decision making, intuitionistic fuzzy machine learning, and intuitionistic fuzzy semantic representations [19]. Recently, many researchers have applied intuitionistic fuzzy sets to hyper algebras such as hyper BCK-algebra [3], BCI-algebra [16] and hyper GR-algebra [13].

This paper is particularly interested in the concept of intuitionistic fuzzy sets applied to implicative hyper GR-ideal of hyper GR-algebra. The definition of intuitionistic fuzzy implicative hyper GR-ideal is somewhat parallel with the definition of intuitionistic fuzzy hyper GR-ideal which was introduced by Macodi [13].

2. Prelimenaries

This section presents some preliminary concepts and known properties that are needed in this study.

Definition 1. [5] Let H be a nonempty set and $\circledast$ be a hyperoperation on H. Then $(H; \circledast, 0)$ is called a **hyper GR-algebra** if $0 \in H$ and if for all $x, y, z \in H$, the following conditions are satisfied:

(i) $(x \circledast z) \circledast (y \circledast z) \ll x \circledast y$; [HGR1]
(ii) $(x \circledast y) \circledast z = (x \circledast z) \circledast y$; [HGR2]
(iii) $x \ll x$; [HGR3]
(iv) $0 \circledast (0 \circledast x) \ll x$, $x \neq 0$; and, [HGR4]
(v) $(x \circledast y) \circledast z \ll y \circledast z$. [HGR5]

Definition 2. [12] A nonempty subset I of a hyper GR-algebra H is called an **implicative hyper GR-ideal of** H if for any $x, y, z \in H$,

(i) $0 \in I$; and, [IH1]

(ii) $(x \circledast z) \circledast (y \circledast x) \subseteq I$ and $z \in I$ imply that $x \in I$. [IH2]

Definition 3. [21] Let M be a nonempty set. A **fuzzy set** μ in M is the function $\mu : M \to [0,1]$. The **complement of a fuzzy set** μ, denoted by $\bar{\mu}$, is the fuzzy set in M given by $\bar{\mu}(x) = 1 - \mu(x)$ for all $x \in M$.

Definition 4. [12] A fuzzy set μ in a hyper GR-algebra H is called **fuzzy implicative hyper GR-ideal of Type 1** if for $x, y, z \in H$,

$$\mu(0) \geq \mu(x) \geq \min \left\{ \inf_{u \in (x \circledast z) \circledast (y \circledast x)} \mu(u), \mu(z) \right\} \qquad \text{[FIM1]}$$

Definition 5. [1] An **intuitionistic fuzzy set** A in a nonempty set H is an object having the form $A = \{(x, \mu_A(x), \gamma_A(x) : x \in H\}$, where the functions $\mu_A : H \to [0,1]$ and $\gamma_A : H \to [0,1]$ denote the degree of membership and degree of nonmembership, respectively, and for all $x \in H$,

$$0 \leq \mu_A(x) + \gamma_A(x) \leq 1.$$

Furthermore, $\pi_A(x) = 1 - \mu_A(x) - \gamma_A(x)$ is called the intuitionistic fuzzy set index or hesitation margin of x in A. $\pi_A(x)$ is the degree of indeterminancy of $x \in H$ to intuitionistic fuzzy set A and $\pi_A(x) \in [0,1]$. $\pi_A(x)$ expresses the lack of knowledge of whether x belongs to intuitionistic fuzzy set A or not.

For the sake of simplicity, we shall use the symbol $A = (\mu_A(x), \gamma_A(x))$ to denote the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in H\}$.

Definition 6. [16] For an intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in H and $s, t \in [0,1]$, the set $A_{\langle t,s \rangle} = \{x \in H : \mu_A(x) \geq t, \gamma_A(x) \leq s\}$ is called a **level subset of** A.

Definition 7. [16] An **intuitionistic fuzzy relation** on a nonempty set H is an intuitionistic fuzzy set $B = (\mu_B, \gamma_B)$ where

$$\mu_B : H \times H \to [0,1] \text{ and } \gamma_B : H \times H \to [0,1].$$

Definition 8. [16] If B is an intuitionistic fuzzy relation on a nonempty set H and A is an intuitionistic fuzzy set in H, then B is an **intuitionistic fuzzy relation** on a set A if

$$\mu_B(x,y) \leq \min\{\mu_A(x), \mu_A(y)\} \text{ and } \gamma_B(x,y) \geq \max\{\gamma_A(x), \gamma_A(y)\}$$

for all $x, y \in H$.

Definition 9. [16] If A is an intuitionistic fuzzy set in a nonempty set H, the **strongest intuitionistic fuzzy relation** on H is an intuitionistic fuzzy relation on H, denoted by $B_A = ((\mu_B)_{\mu_A}, (\gamma_B)_{\gamma_A})$, given by

$$(\mu_B)_{\mu_A}(x,y) = \min\{\mu_A(x), \mu_A(y)\} \text{ and } (\gamma_B)_{\gamma_A}(x,y) = \max\{\gamma_A(x), \gamma_A(y)\}$$

for all $x, y \in H$.

Definition 10. [16] Let A and B be an intuitionistic fuzzy sets in a nonempty set H. The **Cartesian product of** A **and** B is defined for all $x, y \in H$ as follows

$$(\mu_A \times \mu_B)(x, y) = \min\{\mu_A(x), \mu_B(y)\} \text{ and } (\gamma_A \times \gamma_B)(x, y) = \max\{\gamma_A(x), \gamma_B(y)\}$$

Definition 11. [13] An intuitionistic fuzzy set $A = (\mu_A(x), \gamma_A(x))$ in a hyper GR-algebra H is an **intuitionistic fuzzy hyper GR-ideal** of H if for all $x, y \in H$ the following hold:

(i) $\mu_A(0) \geq \mu_A(x)$ and $\gamma_A(0) \leq \gamma_A(x)$; [IFGR1]

(ii) $\mu_A(x) \geq \min\left\{\inf\limits_{u \in x \circledast y} \mu_A(u), \mu_A(y)\right\}$; and, [IFGR2]

(iii) $\gamma_A(x) \leq \max\left\{\sup\limits_{v \in x \circledast y} \gamma_A(v), \gamma_A(y)\right\}$. [IFGR3]

Theorem 1. [12] *A fuzzy set μ in a hyper GR-algebra H is a fuzzy implicative hyper GR-ideal of Type 1 if and only if μ_t is an implicative hyper GR-ideal of H whenever $\mu_t \neq \varnothing$ and $t \in [0, 1]$.*

Lemma 1. [13] *Let $\mu : H \to [0, 1]$ be a fuzzy set and $S \subseteq H$. Then*

(a) $1 - \sup\limits_{x \in S} \mu(x) = \inf\limits_{x \in S} (1 - \mu(x))$ *and*

(b) $1 - \inf\limits_{x \in S} \mu(x) = \sup\limits_{x \in S} (1 - \mu(x))$.

Corollary 1. [13] *Let $\mu : H \to [0, 1]$ be a fuzzy set and $S \subseteq H$. Then*

(a) $1 - \max\limits_{x \in S} \mu(x) = \min\limits_{x \in S} (1 - \mu(x))$ *and*

(b) $1 - \min\limits_{x \in S} \mu(x) = \max\limits_{x \in S} (1 - \mu(x))$.

3. Intuitionistic Fuzzy Implicative Hyper GR-ideals

Definition 12. An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in a hyper GR-algebra H is an **intuitionistic fuzzy implicative hyper GR-ideal** of H if for all $x, y, z \in H$ the following hold:

(i) $\mu_A(x) \leq \mu_A(0)$ and $\gamma_A(x) \geq \gamma_A(0)$; [IFIGR1]

(ii) $\mu_A(x) \geq \min\left\{\inf\limits_{u \in (x \circledast z) \circledast (y \circledast x)} \mu_A(u), \mu_A(z)\right\}$; and, [IFIGR2]

(iii) $\gamma_A(x) \leq \max\left\{\sup\limits_{v \in (x \circledast z) \circledast (y \circledast x)} \gamma_A(v), \gamma_A(z)\right\}$. [IFIGR3]

The following examples illustrate the dissimilarity of intuitionistic fuzzy implicative hyper GR-ideal from intuitionistic fuzzy hyper GR-ideal.

Example 1. Consider the hyper GR-algebra $H = \{0, 1, 2\}$ and the Cayley table in Table 1.

⊛	0	1	2
0	{0,1}	{0,1}	{0,1}
1	{1}	{0,1}	{0,1}
2	{0,2}	{0,2}	{0,1,2}

Table 1: Hyper GR-algebra

Define the fuzzy sets μ_A and γ_A, respectively by,

$$\mu_A(x) = \begin{cases} 0.5, & \text{if } x = 0 \\ 0.3, & \text{if } x = 1 \\ 0.2 & \text{if } x = 2 \end{cases}$$

and

$$\gamma_A(x) = \begin{cases} 0.4, & \text{if } x = 0 \\ 0.6, & \text{if } x = 1 \\ 0.7 & \text{if } x = 2 \end{cases}.$$

(i) Clearly, $\mu_A(0) \geq \mu_A(x)$ and $\gamma_A(0) \leq \gamma_A(x)$ for all $x \in H$.

(ii) Table 2 and Table 3 show that $\mu_A(x) \geq \min\left\{\inf_{u \in (x \circledast z) \circledast (y \circledast x)} \mu_A(u), \mu_A(z)\right\}$.

x	z	$x \circledast z$	$y \circledast x$			$A = (x \circledast z) \circledast (y \circledast x)$		
			$y = 0$	$y = 1$	$y = 2$	$y = 0$	$y = 1$	$y = 2$
0	0	{0,1}	{0,1}	{1}	{0,2}	{0,1}	{0,1}	{0,1}
0	1	{0,1}	{0,1}	{1}	{0,2}	{0,1}	{0,1}	{0,1}
0	2	{0,1}	{0,1}	{1}	{0,2}	{0,1}	{0,1}	{0,1}
1	0	{1}	{0,1}	{0,1}	{0,2}	{0,1}	{0,1}	{0,1}
1	1	{0,1}	{0,1}	{0,1}	{0,2}	{0,1}	{0,1}	{0,1}
1	2	{0,1}	{0,1}	{0,1}	{0,2}	{0,1}	{0,1}	{0,1}
2	0	{0,2}	{0,1}	{0,1}	H	H	H	H
2	1	{0,2}	{0,1}	{0,1}	H	H	H	H
2	2	H	{0,1}	{0,1}	H	H	H	H

Table 2: IFGR2

x	z	$S = \inf_{u \in (x \circledast z) \circledast (y \circledast x)} \mu_A(u)$			$\min\{S, \mu_A(z)\}$			$\mu_A(x) \geq \min\{S, \mu_A(z)\}$		
		$y=0$	$y=1$	$y=2$	$y=0$	$y=1$	$y=2$	$y=0$	$y=1$	$y=2$
0	0	0.3	0.3	0.3	0.3	0.3	0.3	✓	✓	✓
0	1	0.3	0.3	0.3	0.3	0.3	0.3	✓	✓	✓
0	2	0.3	0.3	0.3	0.2	0.2	0.2	✓	✓	✓
1	0	0.3	0.3	0.3	0.3	0.3	0.3	✓	✓	✓
1	1	0.3	0.3	0.3	0.3	0.3	0.3	✓	✓	✓
1	2	0.3	0.3	0.3	0.2	0.2	0.2	✓	✓	✓
2	0	0.2	0.2	0.2	0.2	0.2	0.2	✓	✓	✓
2	1	0.2	0.2	0.2	0.2	0.2	0.2	✓	✓	✓
2	2	0.2	0.2	0.2	0.2	0.2	0.2	✓	✓	✓

Table 3: IFIGR2(continuation)

(iii) Table 2 and Table 4 show that $\gamma_A(x) \leq \max\left\{ \sup_{v \in (x \circledast z) \circledast (y \circledast x)} \gamma_A(v), \gamma_A(z) \right\}$.

x	z	$T = \sup_{v \in (x \circledast z) \circledast (y \circledast x)} \gamma_A(v)$			$\max\{T, \gamma_A(z)\}$			$\gamma_A(x) \leq \max\{T, \gamma_A(z)\}$		
		$y=0$	$y=1$	$y=2$	$y=0$	$y=1$	$y=2$	$y=0$	$y=1$	$y=2$
0	0	0.6	0.6	0.6	0.6	0.6	0.6	✓	✓	✓
0	1	0.6	0.6	0.6	0.6	0.6	0.6	✓	✓	✓
0	2	0.6	0.6	0.6	0.7	0.7	0.7	✓	✓	✓
1	0	0.6	0.6	0.6	0.6	0.6	0.6	✓	✓	✓
1	1	0.6	0.6	0.6	0.6	0.6	0.6	✓	✓	✓
1	2	0.6	0.6	0.6	0.7	0.7	0.7	✓	✓	✓
2	0	0.7	0.7	0.7	0.7	0.7	0.7	✓	✓	✓
2	1	0.7	0.7	0.7	0.7	0.7	0.7	✓	✓	✓
2	2	0.7	0.7	0.7	0.7	0.7	0.7	✓	✓	✓

Table 4: IFIGR3

Thus, $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy implicative hyper GR-ideal of hyper GR-algebra H.

Example 2. Consider the hyper GR-algebra $H = \{0, 1, 2\}$ and the Cayley table in Table 5.

$\circledast$	0	1	2
0	{0,1}	{0,1}	{0,1}
1	{0}	{0,1}	{0,1}
2	{0,2}	{0,2}	{0,1,2}

Table 5: Hyper GR-algebra

Define the fuzzy sets μ_A and γ_A, respectively by,

$$\mu_A(x) = \begin{cases} 0.5, & \text{if } x = 0 \\ 0.3, & \text{if } x = 1 \\ 0.2 & \text{if } x = 2 \end{cases}$$

and

$$\gamma_A(x) = \begin{cases} 0.4, & \text{if } x = 0 \\ 0.6, & \text{if } x = 1 \\ 0.7 & \text{if } x = 2 \end{cases} .$$

(i) Clearly, $\mu_A(0) \geq \mu_A(x)$ and $\gamma_A(0) \leq \gamma_A(x)$ for all $x \in H$.

(ii) Table 6 and Table 7 show that $\mu_A(x) \geq \min\left\{\inf_{u\in(x\circledast z)\circledast(y\circledast x)} \mu_A(u), \mu_A(z)\right\}$.

x	z	$x \circledast z$	$y \circledast x$			$A = (x \circledast z) \circledast (y \circledast x)$		
			$y = 0$	$y = 1$	$y = 2$	$y = 0$	$y = 1$	$y = 2$
0	0	{0,1}	{0,1}	{0}	{1,2}	{0,1}	{0,1}	{0,1}
0	1	{0,1}	{0,1}	{0}	{1,2}	{0,1}	{0,1}	{0,1}
0	2	{0,1}	{0,1}	{0}	{1,2}	{0,1}	{0,1}	{0,1}
1	0	{0}	{0,1}	{0,1}	{0,2}	{0,1}	{0,1}	{0,1}
1	1	{0,1}	{0,1}	{0,1}	{0,2}	{0,1}	{0,1}	{0,1}
1	2	{0,1}	{0,1}	{0,1}	{0,2}	{0,1}	{0,1}	{0,1}
2	0	{1,2}	{0,1}	{0,1}	H	H	H	H
2	1	{0,2}	{0,1}	{0,1}	H	H	H	H
2	2	H	{0,1}	{0,1}	H	H	H	H

Table 6: IFGR2

x	z	$S = \inf_{u\in(x\circledast z)\circledast(y\circledast x)} \mu_A(u)$			$\min\{S, \mu_A(z)\}$			$\mu_A(x) \geq \min\{S, \mu_A(z)\}$		
		$y = 0$	$y = 1$	$y = 2$	$y = 0$	$y = 1$	$y = 2$	$y = 0$	$y = 1$	$y = 2$
0	0	0.3	0.3	0.3	0.3	0.3	0.3	✓	✓	✓
0	1	0.3	0.3	0.3	0.3	0.3	0.3	✓	✓	✓
0	2	0.3	0.3	0.3	0.2	0.2	0.2	✓	✓	✓
1	0	0.3	0.3	0.3	0.3	0.3	0.3	✓	✓	✓
1	1	0.3	0.3	0.3	0.3	0.3	0.3	✓	✓	✓
1	2	0.3	0.3	0.3	0.2	0.2	0.2	✓	✓	✓
2	0	0.2	0.2	0.2	0.2	0.2	0.2	✓	✓	✓
2	1	0.2	0.2	0.2	0.2	0.2	0.2	✓	✓	✓
2	2	0.2	0.2	0.2	0.2	0.2	0.2	✓	✓	✓

Table 7: IFIGR2(continuation)

(iii) Table 6 and Table 8 show that $\gamma_A(x) \leq \max\left\{\sup_{v\in(x\circledast z)\circledast(y\circledast x)} \gamma_A(v), \gamma_A(z)\right\}$.

x	z	$T = \sup_{v\in(x\circledast z)\circledast(y\circledast x)} \gamma_A(v)$			$\max\{T, \gamma_A(z)\}$			$\gamma_A(x) \leq \max\{T, \gamma_A(z)\}$		
		$y = 0$	$y = 1$	$y = 2$	$y = 0$	$y = 1$	$y = 2$	$y = 0$	$y = 1$	$y = 2$
0	0	0.6	0.6	0.6	0.6	0.6	0.6	✓	✓	✓
0	1	0.6	0.6	0.6	0.6	0.6	0.6	✓	✓	✓
0	2	0.6	0.6	0.6	0.7	0.7	0.7	✓	✓	✓
1	0	0.6	0.6	0.6	0.6	0.6	0.6	✓	✓	✓
1	1	0.6	0.6	0.6	0.6	0.6	0.6	✓	✓	✓
1	2	0.6	0.6	0.6	0.7	0.7	0.7	✓	✓	✓
2	0	0.7	0.7	0.7	0.7	0.7	0.7	✓	✓	✓
2	1	0.7	0.7	0.7	0.7	0.7	0.7	✓	✓	✓
2	2	0.7	0.7	0.7	0.7	0.7	0.7	✓	✓	✓

Table 8: IFIGR3

Thus, $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy implicative hyper GR-ideal of H but is not an intuitionistic fuzzy hyper GR-ideal in H since

$$\mu_A(1) = 0.3 < 0.5 = \min\left\{\inf_{u\in 1\circledast 0} \mu_A(u), \mu_A(0)\right\}$$

and

$$\gamma_A(1) = 0.6 > 0.4 = \max\left\{\sup_{v\in 1\circledast 0} \gamma_A(v), \gamma_A(0)\right\}.$$

Remark 1. *Example 2 shows that not all intuitionistic fuzzy implicative hyper GR-ideal are intuitionistic fuzzy hyper GR-ideal of H.*

The following theorems exhibit a characterization of intuitionistic fuzzy implicative hyper GR-ideal of hyper GR-algebra.

Theorem 2. *An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy implicative hyper GR-ideal of a Hyper GR-algebra H if and only if $A_{\langle t,s\rangle}$ is an implicative hyper GR-ideal of H whenever $A_{\langle t,s\rangle} \neq \varnothing$ and $t, s \in [0, 1]$.*

Proof: Suppose $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy implicative hyper GR-ideal in H. By IFIGR1, $\mu_A(x) \leq \mu_A(0)$ and $\gamma_A(x) \geq \gamma_A(0)$ for all $x \in H$. Let $t, s \in [0, 1]$. By assumption, $A_{\langle t,s\rangle} \neq \varnothing$, that is there exists $x \in A_{\langle t,s\rangle}$ such that $t \leq \mu_A(x) \leq \mu_A(0)$ and $s \geq \gamma_A(x) \geq \gamma_A(0)$. Thus $0 \in A_{\langle t,s\rangle}$. Let $x, y, z \in H$ such that $(x \circledast z) \circledast (y \circledast x) \subseteq A_{\langle t,s\rangle}$ and $z \in A_{\langle t,s\rangle}$. Then $\mu_A(z) \geq t$ and $\gamma_A(z) \leq s$. Also, $\mu_A(u) \geq t$ and $\gamma_A(v) \leq s$ for all $u, v, \in (x \circledast z) \circledast (y \circledast x)$. It implies that t is a lowerbound for $\{\mu_A(u) | u \in (x \circledast z) \circledast (y \circledast x)\}$ and s is an upperbound for $\{\gamma_A(v) | v \in (x \circledast z) \circledast (y \circledast x)\}$. Then,

$$\inf_{u\in(x\circledast z)\circledast(y\circledast x)} \mu_A(u) \geq t \quad \text{and} \quad \sup_{v\in(x\circledast z)\circledast(y\circledast x)} \gamma_A(v) \leq s.$$

By IFIGR2 and IFIGR3,

$$\mu_A(x) \geq \min\left\{\inf_{u\in(x\circledast z)\circledast(y\circledast x)} \mu_A(u), \mu_A(z)\right\} = \min\{t,t\} = t \text{ and}$$
$$\gamma_A(x) \leq \max\left\{\sup_{v\in(x\circledast z)\circledast(y\circledast x)} \gamma_A(v), \gamma_A(z)\right\} = \max\{s,s\} = s.$$

Hence, $x \in A_{\langle t,s\rangle}$. Thus, $A_{\langle t,s\rangle}$ is a hyper GR-ideal of H.
Conversely, suppose $A_{\langle t,s\rangle}$ is a hyper GR-ideal of H where $t,s \in [0,1]$. Let $x \in H$ and $k,l \in [0,1]$ such that $k = \mu_A(x)$ and $l = \gamma_A(x)$. Note that $A_{\langle k,l\rangle}$ is a hyper GR-ideal of H. Thus $0 \in A_{\langle k,l\rangle}$. Then $\mu_A(0) \geq \mu_A(x) = k$ and $\gamma_A(0) \leq \gamma_A(x) = l$.
Let $x,y,z \in H$ and $p,q \in [0,1]$ such that

$$p = \min\left\{\inf_{u\in(x\circledast z)\circledast(y\circledast x)} \mu_A(u), \mu_A(z)\right\} \text{ and } q = \max\left\{\sup_{v\in(x\circledast z)\circledast(y\circledast x)} \gamma_A(v), \gamma_A(z)\right\}.$$

Let $w \in (x \circledast z) \circledast (y \circledast x)$. Then

$$\mu_A(w) \geq \inf_{u\in(x\circledast z)\circledast(y\circledast x)} \mu_A(u) \geq \min\left\{\inf_{u\in(x\circledast z)\circledast(y\circledast x)} \mu_A(u), \mu_A(z)\right\} = p \text{ and}$$
$$\gamma_A(w) \leq \sup_{v\in(x\circledast z)\circledast(y\circledast x)} \gamma_A(v) \leq \max\left\{\sup_{v\in(x\circledast z)\circledast(y\circledast x)} \gamma_A(v), \gamma_A(z)\right\} = q.$$

This implies that $w \in A_{\langle p,q\rangle}$. It follows that $(x \circledast z) \circledast (y \circledast x) \subseteq A_{\langle p,q\rangle}$. Clearly,

$$\mu_A(z) \geq \min\left\{\inf_{u\in(x\circledast z)\circledast(y\circledast x)} \mu_A(u), \mu_A(z)\right\} = p \text{ and}$$
$$\gamma_A(z) \leq \max\left\{\sup_{v\in(x\circledast z)\circledast(y\circledast x)} \gamma_A(v), \gamma_A(z)\right\} = q.$$

Hence, $z \in A_{\langle p,q\rangle}$. Since $A_{\langle p,q\rangle}$ is a implicative hyper GR-ideal of H, $x \in A_{\langle p,q\rangle}$.
It implies that

$$\mu_A(x) \geq \min\left\{\inf_{u\in(x\circledast z)\circledast(y\circledast x)} \mu_A(u), \mu_A(z)\right\} \text{ and}$$
$$\gamma_A(x) \leq \max\left\{\sup_{v\in(x\circledast z)\circledast(y\circledast x)} \gamma_A(v), \gamma_A(z)\right\}.$$

Thus, $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy implicative hyper GR-ideal of H. □

Theorem 3. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy implicative hyper GR-ideal of a hyper GR-algebra H, then the set $J = \{x \in H : \mu_A(x) = \mu_A(0)$ and $\gamma_A(x) = \gamma_A(0)\}$ is an implicative hyper GR-ideal of H.*

Proof: Suppose $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy implicative hyper GR-ideal of a hyper GR-algebra H. Let $x,y,z \in H$ such that $(x \circledast z) \circledast (y \circledast x) \subseteq J$ and $z \in J$. It follows that $0 \in J$. Then, $\mu_A(z) = \mu_A(0), \gamma_A(z) = \gamma_A(0), \mu_A(u) = \mu_A(0)$ and $\gamma_A(v) = \gamma_A(0)$ for any $u,v \in (x \circledast z) \circledast (y \circledast x)$. By IFIGR1, $\mu_A(0) \geq \mu_A(x)$ and $\gamma_A(0) \leq \gamma_A(x)$. By IFIGR2 and IFIGR3,

$$\mu_A(0) \geq \mu_A(x) \geq \min\left\{\inf_{u\in(x\circledast z)\circledast(y\circledast x)} \mu_A(u), \mu_A(z)\right\} = \mu_A(0) \text{ and}$$

$$\gamma_A(0) \leq \gamma_A(x) \leq \max\left\{\sup_{v\in(x\circledast z)\circledast(y\circledast x)} \gamma_A(v), \gamma_A(z)\right\} = \gamma(0).$$

This implies that $\mu_A(x) = \mu_A(0)$ and $\gamma_A(x) = \gamma_A(0)$ and so $x \in J$. Thus, J is an implicative hyper GR-ideal of H. □

Theorem 4. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy implicative hyper GR-ideal of a hyper GR-algebra H, then $\mu_A(x) \geq \inf_{u\in(x\circledast 0)\circledast(y\circledast x)} \mu_A(u)$ and $\gamma_A(x) \leq \sup_{v\in(x\circledast 0)\circledast(y\circledast x)} \gamma_A(v)$ for all $x, y \in H$.*

Proof: Suppose $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy implicative hyper GR-ideal of a hyper GR-algebra H. Let $x, y \in H$. By IFIGR2 and IFIGR1,

$$\begin{aligned}\mu_A(x) &\geq \min\left\{\inf_{u\in(x\circledast 0)\circledast(y\circledast x)} \mu_A(u), \mu_A(0)\right\}\\ &= \inf_{u\in(x\circledast 0)\circledast(y\circledast x)} \mu_A(u).\end{aligned}$$

Also by IFIGR3 and IFIGR1,

$$\begin{aligned}\gamma_A(x) &\leq \max\left\{\sup_{v\in(x\circledast 0)\circledast(y\circledast x)} \gamma_A(v), \gamma_A(0)\right\}\\ &= \sup_{v\in(x\circledast 0)\circledast(y\circledast x)} \gamma_A(v).\end{aligned}$$

□

Lemma 3.1. *An intuitionistic fuzzy sets $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy implicative hyper GR-ideal of a hyper GR-algebra H if and only if the fuzzy sets μ_A and $\bar{\gamma}_A$ are fuzzy implicative hyper GR-ideals of type 1 in H.*

Proof: Suppose $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy implicative hyper GR-ideal of H. By IFIGR1 and IFIGR2, for any $x, y, z \in H$,

$$\mu_A(x) \leq \mu_A(0) \text{ and } \mu_A(x) \geq \min\left\{\inf_{u\in(x\circledast z)\circledast(y\circledast x)} \mu_A(u), \mu_A(z)\right\}$$

Hence, μ_A is a fuzzy implicative hyper GR-ideal of type 1 in H.
Let $x, y, z \in H$. By IFIGR1 and IFIGR3,

$$\gamma_A(x) \geq \gamma_A(0) \text{ and } \gamma_A(x) \leq \max\left\{\sup_{v\in(x\circledast z)\circledast(y\circledast x)} \gamma_A(v), \gamma_A(z)\right\}.$$

Then

$$\bar{\gamma}_A(x) = 1 - \gamma_A(x) \leq 1 - \gamma_A(0) = \bar{\gamma}_A(0),$$

that is,

$$\bar{\gamma}_A(x) \leq \bar{\gamma}_A(0).$$

By Corollary 1(a) and Lemma 1(a),

$$\begin{aligned}\bar{\gamma}_A(x) = 1 - \gamma_A(x) &\geq 1 - \max\left\{\sup_{v\in(x\circledast z)\circledast(y\circledast x)} \gamma_A(v), \gamma_A(z)\right\} \\ &= \min\left\{1 - \sup_{v\in(x\circledast z)\circledast(y\circledast x)} \gamma_A(v), 1 - \gamma_A(z)\right\} \\ &= \min\left\{\inf_{v\in(x\circledast z)\circledast(y\circledast x)} (1 - \gamma_A(v)), 1 - \gamma_A(z)\right\} \\ &= \min\left\{\inf_{v\in(x\circledast z)\circledast(y\circledast x)} \bar{\gamma}_A(v), \bar{\gamma}_A(z)\right\}.\end{aligned}$$

Implies that,

$$\bar{\gamma}_A(x) \geq \min\left\{\inf_{v\in(x\circledast z)\circledast(y\circledast x)} \bar{\gamma}_A(v), \bar{\gamma}_A(z)\right\}.$$

Hence $\bar{\gamma}_A(x)$ is a fuzzy implicative hyper GR-ideal of type 1 in H.
Conversely, suppose that μ_A and $\bar{\gamma}_A$ are fuzzy implicative hyper GR-ideals of type 1 in H.
Let $x, y, z \in H$. By FIM1,

$$\bar{\gamma}_A(x) \leq \bar{\gamma}_A(0)$$

and

$$\gamma_A(x) = 1 - \bar{\gamma}_A(x) \geq 1 - \bar{\gamma}_A(0) = \gamma_A(0).$$

By Corollary 1(b) and Lemma 1(b),

$$\begin{aligned}\gamma_A(x) = 1 - \bar{\gamma}_A(x) &\leq 1 - \min\left\{\inf_{v\in(x\circledast z)\circledast(y\circledast x)} \bar{\gamma}_A(v), \bar{\gamma}_A(z)\right\} \\ &= \max\left\{1 - \inf_{v\in(x\circledast z)\circledast(y\circledast x)} \bar{\gamma}_A(v), 1 - \bar{\gamma}_A(z)\right\} \\ &= \max\left\{\sup_{v\in(x\circledast z)\circledast(y\circledast x)} (1 - \bar{\gamma}_A(v)), \gamma_A(z)\right\} \\ &= \max\left\{\sup_{v\in(x\circledast z)\circledast(y\circledast x)} \gamma_A(v), \gamma_A(z)\right\}.\end{aligned}$$

Thus,

$$\gamma_A(x) \leq \max\left\{\sup_{v\in(x\circledast z)\circledast(y\circledast x)} \gamma_A(v), \gamma_A(z)\right\}.$$

Hence, the fuzzy set $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy implicative hyper GR-ideal of a hyper GR-algebra H. □

The following results establish some characterization and properties of intuitionistic fuzzy implicative hyper GR-ideal of hyper GR-algebra.

Theorem 5. *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy set in a hyper GR-algebra H. Then, A is an intuitionistic fuzzy implicative hyper GR-ideal in H if and only if $\hat{A} = (\mu_A, \bar{\mu}_A)$ and $\breve{A} = (\bar{\gamma}_A, \gamma_A)$ are intuitionistic fuzzy implicative hyper GR-ideals of H.*

Proof: Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy implicative hyper GR-ideal of H. By Lemma 3.1, μ_A and $\bar{\gamma}_A$ are fuzzy implicative hyper GR-ideals of type 1 in H. We are left to show that $\bar{\mu}_A$ will satisfy IFIGR1 and IFIGR3. Let $x, y, z \in H$. Then by FIM1,

$$\mu_A(x) \leq \mu_A(0)$$

and

$$\bar{\mu}_A(x) = 1 - \mu_A(x) \geq 1 - \mu_A(0) = \bar{\mu}_A(0).$$

Thus, IFIGR1 is satisfied.
By Lemma 1(b) and Corollary 1(b),

$$\begin{aligned}
\bar{\mu}_A(x) = 1 - \mu_A(x) &\leq 1 - \min\left\{\inf_{v \in (x \circledast z) \circledast (y \circledast x)} \mu_A(v), \mu_A(z)\right\} \\
&= \max\left\{1 - \inf_{v \in (x \circledast z) \circledast (y \circledast x)} \mu_A(v), 1 - \mu_A(z)\right\} \\
&= \max\left\{\sup_{v \in (x \circledast z) \circledast (y \circledast x)} (1 - \mu_A(v)), 1 - \mu_A(z)\right\} \\
&= \max\left\{\sup_{v \in (x \circledast z) \circledast (y \circledast x)} \bar{\mu}_A(v), \bar{\mu}_A(z)\right\}.
\end{aligned}$$

Implies that

$$\bar{\mu}_A(x) \leq \max\left\{\sup_{v \in (x \circledast z) \circledast (y \circledast x)} \bar{\mu}_A(v), \bar{\mu}_A(z)\right\}.$$

Thus, IFIGR3 is satisfied. Hence, $\hat{A} = (\mu_A, \bar{\mu}_A)$ and $\breve{A} = (\bar{\gamma}_A, \gamma_A)$ are intuitionistic fuzzy implicative hyper GR-ideals of H.
Conversely, suppose $\hat{A} = (\mu_A, \bar{\mu}_A)$ and $\breve{A} = (\bar{\gamma}_A, \gamma_A)$ are intuitionistic fuzzy implicative hyper GR-ideals in H. Then by Lemma 3.1, μ_A and $\bar{\gamma}_A$ are fuzzy implicative hyper GR-ideals of type 1 in H and $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy implicative hyper GR-ideal of a hyper GR-algebra H. □

Corollary 3.2. *For any subset I of a hyper GR-algebra H, let $A(I) = (\mu_{A(I)}, \gamma_{A(I)})$ be an intuitionistic fuzzy set in H where*

$$\mu_{A(I)}(x) = \begin{cases} m_1, & \text{if } x \in I \\ m_2, & \text{otherwise} \end{cases}$$

and

$$\gamma_{A(I)}(x) = \begin{cases} n_1, & \textit{if } x \in I \\ n_2, & \textit{otherwise} \end{cases}$$

for all $x \in H$, where $m_1, m_2, n_1, n_2 \in [0,1]$ with $m_1 > m_2$ and $n_1 < n_2$, $m_i + n_i \leq 1$ for $i = 1, 2$. Then I is an implicative hyper GR-ideal of H if and only if $A(I) = (\mu_{A(I)}, \gamma_{A(I)})$ is an intuitionistic fuzzy implicative hyper GR-ideal H.

Proof: Note that the level subsets of $\mu_{A(I)}$ is

$$(\mu_{A(I)})_{t_1} = \begin{cases} \varnothing, & \text{if } m_1 < t_1 \leq 1 \\ I, & \text{if } m_2 < t_1 \leq m_1 \\ H, & \text{if } 0 \leq t_1 \leq m_2 \end{cases}.$$

Since

$$\bar{\gamma}_{A(I)}(x) = \begin{cases} 1 - n_1, & \text{if } x \in I \\ 1 - n_2, & \textit{otherwise.} \end{cases}$$

and $1 - n_1 > 1 - n_2$,

$$(\bar{\gamma}_{A(I)})_{t_2} = \begin{cases} \varnothing, & \text{if } 1 - n_1 < t_2 \leq 1 \\ I, & \text{if } 1 - n_2 < t_2 \leq 1 - n_1 \\ H, & \text{if } 0 \leq t_2 \leq 1 - n_2 \end{cases}.$$

Let I be an implicative hyper GR-ideal of H. Then the nonempty level subsets $(\mu_{A(I)})_{t_1}$ and $(\bar{\gamma}_{A(I)})_{t_2}$ are impllicative hyper GR-ideals of H. By Theorem 1, $\mu_{A(I)}$ and $\bar{\gamma}_{A(I)}$ are fuzzy implicative ideals of type 1 in H. By Lemma 3.1, $A(I) = (\mu_{A(I)}, \gamma_{A(I)})$ is an intuitionistic fuzzy implicative hyper GR-ideal of H. Conversely, suppose $A(I) = (\mu_{A(I)}, \gamma_{A(I)})$ is an intuitionistic fuzzy implicative hyper GR-ideal of H. By Lemma 3.1, $\mu_{A(I)}$ and $\bar{\gamma}_{A(I)}$ are fuzzy implicative ideals of H. Consequently by Theorem 1, $I = (\mu_{A(I)})_{t_1}$ is an implicative hyper GR-ideal of H.

□

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