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Random Wave Equation for the Stochastic Quantum Zakharov-Kuznetsov Equation and Their Exact Solutions

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Abstract. In this study, we consider the stochastic Quantum Zakharov-Kuznetsov equation (SQZKE) perturbed in the Itô sense by multiplicative Brownian motion. We employ a suitable transformation for changing the SQZKE to another QZKE with random variable coefficients (QZKE-RVCs). Utilizing the modified extended tanh function method and the Jacobi elliptic function approach, we get novel hyperbolic, elliptic, trigonometric, and rational solutions for QZKE-RVCs. The SQZKE solutions can then be obtained. Additionally, we present many graphic representations to demonstrate how multiplicative Brownian motion affects the exact solutions of the SQZKE.

2020 Mathematics Subject Classifications: 35A20, 83C15, 60H10, 60H15, 35Q51

Key Words and Phrases: Brownian motion, Zakharov-Kuznetsov equation, modified extended tanh function method

1. Introduction

The quantum Zakharov-Kuznetsov equation (QZKE) is a fundamental equation in the field of quantum plasmas, which explains the behavior of small-scale electromagnetic waves in a plasma. It is an extension of the classical Zakharov-Kuznetsov equation, which itself describes the propagation of nonlinear ion acoustic solitary waves in a plasma [7].

The QZKE takes into account the quantum nature of the plasma particles, introducing the concept of quantum effects such as quantum pressure and quantum Bohm potential. These quantum effects play a important role in the dynamics of the plasma, as they affect the behavior of the solitary waves. The equation provides insights into the quantum effects and their impact on the structure and stability of solitary waves in the plasma.

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The study of the QZKE is important in various fields, such as astrophysics, laboratory plasma experiments, and quantum device applications. Understanding the behavior of quantum plasmas is crucial in astrophysical phenomena like the dynamics of astrophysical jets and the behavior of compact objects such as neutron stars. In addition, laboratory experiments that aim to simulate quantum plasma behavior can benefit from the insights provided by the equation.

On the other side, random fluctuations in the QZKE arise due to the inherent quantum nature of the plasma. In classical physics, fluctuations are often considered as random noise that does not contribute significantly to the dynamics of the system. However, in the quantum regime, these fluctuations become significant as they are associated with the intrinsic uncertainty of the quantum state. These random fluctuations can affect the wave propagation and dynamics of the plasma, leading to fluctuations in the plasma density, velocity, and electromagnetic fields.

Understanding and characterizing random fluctuations in the QZKE is essential for various applications. For instance, in plasma physics, studying the behavior of random fluctuations can give insights into the stability and turbulence of the plasma. These fluctuations can also affect the transport of energy and momentum in the plasma, influencing various physical phenomena such as wave-particle interactions, particle acceleration, and plasma heating.

In this study, we consider the following stochastic quantum Zakharov-Kuznetsov equation (SQZKE) forced by multiplicative noise as follows:

$$\mathcal{G}_t + a\mathcal{G}\mathcal{G}_x + b\mathcal{G}_{zzz} + c\mathcal{G}_{zxx} + c\mathcal{G}_{zyy} = \delta\mathcal{G}\mathcal{B}_t, \tag{1}$$

where $\mathcal{G}(x, y, z, t)$ represents the electrostatic potential, $\mathcal{B}(t)$ is the Brownian motion, $\mathcal{B}_t = \frac{\partial\mathcal{B}}{\partial t}$ and δ is the noise intensity.

Various methods for getting the exact solutions for the QZKE (1) with $\delta = 0$ for instance extended F-expansion method [5], generalized unified method [10], exp-function and modified F-expansion methods [6], Hirota bilinear and auxiliary equation [15], generalized (G'/G)-expansion and Jacobi elliptic equation [14, 16]. While, the exact solutions of Eq. (1) with stochastic term has obtained in [9] by utilizing modified F-expansion and Jacobi elliptic function methods.

Moreover, there are many different analytical and numerical methods for solving various partial differential equations for example meshless method [1, 12], Jacobi elliptic function method [2], perturbation method [8], mapping method [4], (G'/G)-expansion [3], and so on.

The main aim of this research is to discover the exact stochastic solutions to the SQZKE (1). To do this, we apply a suitable transformation to change the SQZKE into another QZKE with random variable coefficients (QZKE-RVCs). Following that, we get accurate solutions for QZKE-RVCs using the modified extended tanh function method (METF-method) and the Jacobi elliptic function method (JEF-method). In the end, by using the transformation we utilized, we may derive stochastic solutions to the SQZKE. As far as we know, this is the first time we have supposed that the wave equation's solution is stochastic; in all other research such as [9], the solution was considered to be deterministic.

These obtained solutions are essential for comprehending a number of challenging physical phenomena because the SQZKE (1) is so important in a dense quantum magnetoplasma, astrophysics, laboratory plasma experiments, and quantum device applications. We utilize MATLAB tools for creating some figures that illustrate the impact of the stochastic term.

This is the format for the rest of the paper: We derive QZKE-RVCs from SQZKE (1) in Section 2, and we use the JEF-method and METF-method to get exact solutions of QZKE-RVCs. We obtain the solutions to SQZKE (1) in Section 3. A discussion of our results is provided in Section 4. Lastly, we provide the conclusions of the article.

2. QZKE-RVCs and Its Solutions

In this section, we derive the QZKE-RVCs. To do this, we use the following transformation

$$\mathcal{G}(x,y,z,t) = \mathcal{U}(x,y,z,t)e^{\delta\mathcal{B}(t)}, \tag{2}$$

to get QZKE-RVCs as follows

$$\mathcal{U}_t + b\mathcal{U}_{zzz} + c\mathcal{U}_{zxx} + c\mathcal{U}_{zyy} + A(t)\mathcal{U}\mathcal{U}_x + \frac{1}{2}\delta^2\mathcal{U} = 0, \tag{3}$$

where we used the the Itô derivatives rule, $\mathcal{U}$ is a stochastic real function and $A(t) = ae^{\delta\mathcal{B}(t)}$.

2.1. METF-method

To acquire the solutions of the QZKE-RVCs, we use the modified extended tanh function method (METF-method) that is stated in [13]. First, let us assume the solutions of Eq. (3) has the form

$$\mathcal{U}(x,y,z,t) = \sum_{k=0}^{M} \alpha_k(t)\mathcal{Q}^k(\xi), \quad \xi = kx + my + nz + \int_0^t \lambda(s)ds, \tag{4}$$

where

$$\mathcal{Q}' = \mathcal{Q}^2 + p. \tag{5}$$

First let us calculate the parameter M by balancing $\mathcal{U}_{zzz}$ with $\mathcal{U}\mathcal{U}_x$ as follows

$$M = 2.$$

Rewriting Eq. (4) as

$$\mathcal{U}(x,y,z,t) = \alpha_0(t) + \alpha_1(t)\mathcal{Q}(\xi) + \alpha_2(t)\mathcal{Q}^2(\xi). \tag{6}$$

We have by differentiating Eq. (6) with regards to t, x, y and z:

$$\begin{aligned}
\mathcal{U}_t &= (\dot{\alpha}_0 + p\alpha_1\lambda) + (\dot{\alpha}_1 + 2p\lambda\alpha_2)\mathcal{Q} + (\lambda\alpha_1 + \dot{\alpha}_2)\mathcal{Q}^2 + 2\lambda\alpha_2\mathcal{Q}^3, \\
\mathcal{U}_x &= k[2\alpha_2\mathcal{Q}^3 + \alpha_1\mathcal{Q}^2 + (2p\alpha_2)\mathcal{Q} + p\alpha_1],
\end{aligned}$$

$$\begin{aligned}
\mathcal{U}_{zzz} &= n^3[24\alpha_2\mathcal{Q}^5 + 6\alpha_1\mathcal{Q}^4 + 48p\alpha_2\mathcal{Q}^3 + 8p\alpha_1\mathcal{Q}^2 + 2p^2\alpha_1],\\
\mathcal{U}_{zxx} &= k^2n[24\alpha_2\mathcal{Q}^5 + 6\alpha_1\mathcal{Q}^4 + 48p\alpha_2\mathcal{Q}^3 + 8p\alpha_1\mathcal{Q}^2 + 2p^2\alpha_1],\\
\mathcal{U}_{zyy} &= m^2n[24\alpha_2\mathcal{Q}^5 + 6\alpha_1\mathcal{Q}^4 + 48p\alpha_2\mathcal{Q}^3 + 8p\alpha_1\mathcal{Q}^2 + 2p^2\alpha_1],\\
\mathcal{U}\mathcal{U}_x &= k[2\alpha_2^2\mathcal{Q}^5 + \alpha_1\alpha_2\mathcal{Q}^4 + (2\alpha_0\alpha_2 + \alpha_1^2 + 2p\alpha_2^2)\mathcal{Q}^3\\
&\quad +(\alpha_0\alpha_1 + 3p\alpha_2\alpha_1)\mathcal{Q}^2 + (2p\alpha_0\alpha_2 + p\alpha_1^2)\mathcal{Q} + p\alpha_0\alpha_1].
\end{aligned} \tag{7}$$

Plugging Eqs. (6) and (7) into Eq. (3), we get a polynomial of degree 5 in $\mathcal{Q}$ as follows

$$\begin{aligned}
&[24\hbar\alpha_2 + 2Ak\alpha_2^2]\mathcal{Q}^5 + [6\hbar\alpha_1 + kA\alpha_1\alpha_2]\mathcal{Q}^4\\
&+[2\lambda\alpha_2 + 48p\hbar\alpha_2 + 2kA\alpha_0\alpha_2 + kA\alpha_1^2 + 2kAp\alpha_2^2]\mathcal{Q}^3\\
&+[\lambda\alpha_1 + \dot{\alpha}_2 + 8p\hbar\alpha_1 + kA\alpha_0\alpha_1 + 3kAp\alpha_2\alpha_1 + \frac{1}{2}\delta^2\alpha_2]\mathcal{Q}^2
\end{aligned}$$

$$\begin{aligned}
&[\dot{\alpha}_1 + 2p\lambda\alpha_2 + 2pkA\alpha_0\alpha_2 + pkA\alpha_0^2 + \frac{1}{2}\delta^2\alpha_1]\mathcal{Q}\\
&+[\dot{\alpha}_0 + p\alpha_1\lambda + 2\hbar p^2\alpha_1 + pkA\alpha_0\alpha_1 + \frac{1}{2}\delta^2\alpha_0] = 0,
\end{aligned}$$

where $\hbar = bn^3 + cnm^2 + cnk^2$. Setting each coefficient of $\mathcal{Q}^i$ to zero, we attain

$$24\hbar\alpha_2 + 2Ak\alpha_2^2 = 0,$$

$$6\hbar\alpha_1 + kA\alpha_1\alpha_2 = 0,$$

$$2\lambda\alpha_2 + 48p\hbar\alpha_2 + 2kA\alpha_0\alpha_2 + kA\alpha_1^2 + 2kAp\alpha_2^2 = 0,$$

$$\lambda\alpha_1 + \dot{\alpha}_2 + 8p\hbar\alpha_1 + kA\alpha_0\alpha_1 + 3kAp\alpha_2\alpha_1 + \frac{1}{2}\delta^2\alpha_2 = 0,$$

$$\dot{\alpha}_1 + 2p\lambda\alpha_2 + 2pkA\alpha_0\alpha_2 + pkA\alpha_1^2 + \frac{1}{2}\delta^2\alpha_1 = 0,$$

and

$$\dot{\alpha}_0 + p\alpha_1\lambda + 2\hbar p^2\alpha_1 + pkA\alpha_0\alpha_1 + \frac{1}{2}\delta^2\alpha_0 = 0.$$

These equations are solved to obtain

$$\alpha_0(t) = \ell_0 e^{-\frac{1}{2}\delta^2 t}, \quad \alpha_1 = 0, \ \alpha_2 = \ell_2 e^{-\frac{1}{2}\delta^2 t}, \qquad \lambda(t) = -ak\ell_0 e^{\delta\mathcal{B}(t) - \frac{1}{2}\delta^2 t},$$

and

$$bn^3 + cnm^2 + cnk^2 = \frac{-ak\ell_2}{12} e^{\delta\mathcal{B}(t) - \frac{1}{2}\delta^2 t},$$

where ℓ_0 and ℓ_2 are constants. Thus the solutions of QZKE-RVCs (3), by using Eq. (6), are

$$\mathcal{U}(x,y,z,t) = (\ell_0 + \ell_2\mathcal{Q}^2(\xi))e^{-\frac{1}{2}\delta^2 t}, \ \xi = kx + my + nz - ak\ell_0 \int_0^t e^{\delta\mathcal{B}(\tau) - \frac{1}{2}\delta^2\tau} d\tau. \tag{8}$$

To obtain $\mathcal{Q}$, there are a different sets for the solution of Eq. (5) relying on p as follows:

Set 1: When $p > 0$, thus the solutions of Eq. (5) are:

$$\mathcal{Q}_1(\xi) = \sqrt{p}\tan\left(\sqrt{p}\xi\right),$$

$$\mathcal{Q}_2(\xi) = -\sqrt{p}\cot\left(\sqrt{p}\xi\right),$$

$$\mathcal{Q}_3(\xi) = \sqrt{p}\Big(\tan(\sqrt{4p}\xi) \pm \sec(\sqrt{4p}\xi)\Big),$$

$$\mathcal{Q}_4(\xi) = -\sqrt{p}\Big(\cot(\sqrt{4p}\xi) \pm \csc(\sqrt{4p}\xi)\Big),$$

$$\mathcal{Q}_5(\xi) = \frac{1}{2}\sqrt{p}\Big(\tan(\frac{1}{2}\sqrt{p}\xi) - \cot(\frac{1}{2}\sqrt{p}\xi)\Big),$$

Then, QZKE-RVCs (3) has the trigonometric function solutions:

$$\mathcal{U}_1(x,y,z,t) = \Big(\ell_0 + \ell_2 p\tan^2(\sqrt{p}\xi)\Big)e^{-\frac{1}{2}\delta^2 t}, \tag{9}$$

$$\mathcal{U}_2(x,y,z,t) = \Big(\ell_0 - \ell_2 p\cot^2(\sqrt{p}\xi)\Big)e^{-\frac{1}{2}\delta^2 t}, \tag{10}$$

$$\mathcal{U}_3(x,y,z,t) = \Big(\ell_0 + \ell_2 p\Big(\tan(\sqrt{4p}\xi) \pm \sec(\sqrt{4p}\xi)\Big)^2\Big)e^{-\frac{1}{2}\delta^2 t}, \tag{11}$$

$$\mathcal{U}_4(x,y,z,t) = \Big(\ell_0 - \ell_2 p\Big(\cot(\sqrt{4p}\xi) \pm \csc(\sqrt{4p}\xi)\Big)^2\Big)e^{-\frac{1}{2}\delta^2 t}, \tag{12}$$

$$\mathcal{U}_5(x,y,z,t) = \Big(\ell_0 + \frac{\ell_2 p}{2}\Big(\tan(\frac{1}{2}\sqrt{p}\xi) - \cot(\frac{1}{2}\sqrt{p}\xi)\Big)^2\Big)e^{-\frac{1}{2}\delta^2 t}, \tag{13}$$

where $\xi = kx + my + nz - ak\ell_0\int_0^t e^{\delta\mathcal{B}(\tau)-\frac{1}{2}\delta^2\tau}d\tau$.

Set 2: When $p < 0$, the solutions of Eq. (5) are:

$$\mathcal{Q}_6(\xi) = -\sqrt{-p}\tanh\left(\sqrt{-p}\xi\right),$$

$$\mathcal{Q}_7(\xi) = -\sqrt{-p}\coth\left(\sqrt{-p}\xi\right),$$

$$\mathcal{Q}_8(\xi) = -\sqrt{-p}\Big(\coth(\sqrt{-4p}\xi) \pm \operatorname{csch}(\sqrt{-4p}\xi)\Big),$$

$$\mathcal{Q}_9(\xi) = \frac{-1}{2}\sqrt{-p}\Big(\tanh(\frac{1}{2}\sqrt{-p}\xi) + \coth(\frac{1}{2}\sqrt{-p}\xi)\Big).$$

Then, the hyperbolic function solutions of QZKE-RVCs (3) are:

$$\mathcal{U}_6(x,y,z,t) = \Big(\ell_0 + \ell_2 p\tanh^2(\sqrt{-p}\xi)\Big)e^{-\frac{1}{2}\delta^2 t}, \tag{14}$$

$$\mathcal{U}_7(x,y,z,t) = \Big(\ell_0 + \ell_2 p\coth^2(\sqrt{-p}\xi)\Big)e^{-\frac{1}{2}\delta^2 t}, \tag{15}$$

$$\mathcal{U}_8(x,y,z,t)=\Big(\ell_0+\ell_2 p\Big(\coth(\sqrt{-4p}\xi)\pm\operatorname{csch}(\sqrt{-4p}\xi)\Big)^2\Big)e^{-\frac{1}{2}\delta^2 t}, \tag{16}$$

$$\mathcal{U}_9(x,y,z,t)=\Big(\ell_0+\frac{\ell_2 p}{2}\Big(\tanh(\frac{1}{2}\sqrt{-p}\xi)+\coth(\frac{1}{2}\sqrt{-p}\xi)\Big)^2\Big)e^{-\frac{1}{2}\delta^2 t}, \tag{17}$$

where $\xi=kx+my+nz-ak\ell_0\int_0^t e^{\delta\mathcal{B}(\tau)-\frac{1}{2}\delta^2\tau}d\tau$.

Set 3: When $p=0$, then the solution of Eq. (5) is

$$\mathcal{Q}_{10}(\xi)=\frac{-1}{\xi}.$$

Then, we get the rational function solution of the QZKE-RVCs (3) as

$$\mathcal{U}_{10}(x,y,z,t)=\Big(\ell_0+\frac{\ell_2}{\xi^2}\Big)e^{-\frac{1}{2}\delta^2 t}, \tag{18}$$

where $\xi=kx+my+nz-ak\ell_0\int_0^t e^{\delta\mathcal{B}(\tau)-\frac{1}{2}\delta^2\tau}d\tau$.

2.2. JEF-method

We utilize the JEF-method, as mentioned in [11]. Let the solutions of QZKE-RVCs (3), with $M=2$, have the form

$$\mathcal{U}(x,y,z,t)=a_0(t)+a_1(t)\mathcal{J}(\xi)+a_2(t)\mathcal{J}^2(\xi),\ \xi=kx+my+nz-ak\ell_0\int_0^t e^{\delta\mathcal{B}(\tau)-\frac{1}{2}\delta^2\tau}d\tau. \tag{19}$$

where $\mathcal{J}(\xi)$ indicates one of these elliptic functions: $cn(\varpi\zeta,\tilde{n})$, $sn(\varpi\zeta,\tilde{n})$ or $dn(\varpi\zeta,\tilde{n})$. We have by differentiating Eq. (19) with respects to t, x, y and z:

$$\begin{aligned}
\mathcal{U}_t &= \dot{a}_0+\dot{a}_1\mathcal{J}+\varpi\lambda a_1\mathcal{J}'+\dot{a}_2\mathcal{J}^2+2\varpi\lambda a_2\mathcal{J}'\mathcal{J},\\
\mathcal{U}_x &= (\varpi k a_1+2\varpi k a_2\mathcal{J})\mathcal{J}',\\
\mathcal{U}_{zz} &= n^2(a_1+2a_2)(B_1\mathcal{J}+B_2\mathcal{J}^3)+2\varpi^2k^2a_2\mathcal{J}'^2,\\
\mathcal{U}_{zzz} &= \varpi n^3a_1(B_1+3B_2\mathcal{J}^2)\mathcal{J}'+4\varpi n^3a_2(2B_1\mathcal{J}+3B_2\mathcal{J}^3)\mathcal{J}',\\
\mathcal{U}_{zxx} &= \varpi nk^2a_1(B_1+3B_2\mathcal{J}^2)\mathcal{J}'+4\varpi nk^2a_2(2B_1\mathcal{J}+3B_2\mathcal{J}^3)\mathcal{J}',\\
\mathcal{U}_{zyy} &= \varpi nm^2a_1(B_1+3B_2\mathcal{J}^2)\mathcal{J}'+4\varpi nm^2a_2(2B_1\mathcal{J}+3B_2\mathcal{J}^3)\mathcal{J}',\\
\mathcal{U}_x\mathcal{U} &= \varpi k(a_0a_1+2a_0a_2\mathcal{J}+a_1^2\mathcal{J}+3a_1a_2\mathcal{J}^2+2a_2^2\mathcal{J}^3)\mathcal{J}',
\end{aligned} \tag{20}$$

where B_1 and B_2 are constants relaying on ϖ, $\tilde{n}$. They will be described later. Substituting Eqs. (19) and (20) into KdVE-RVCs (3). Once all of the coefficients in $\mathcal{J}'\mathcal{J}^n$ are set to zero, we obtain

$$\begin{aligned}
\mathcal{J}^0 &:\quad \dot{a}_0+\frac{1}{2}\delta^2a_0=0,\\
\mathcal{J} &:\quad \dot{a}_1+\frac{1}{2}\delta^2a_1=0,
\end{aligned}$$

$$\begin{aligned}\mathcal{J}^2 &: \quad \dot{a}_2+\frac{1}{2}\delta^2 a_2=0,\\ \mathcal{J}^0\mathcal{J}' &: \quad \varpi a_1[\lambda+4\hbar B_1+ka_0A(t)]=0,\\ \mathcal{J}\mathcal{J}' &: \quad 2\varpi\lambda a_2+8\varpi\hbar a_2B_1+A\varpi k a_1^2+2\varpi kA(t)a_0a_2=0,\\ \mathcal{J}^2\mathcal{J}' &: \quad 12\varpi\hbar a_1B_2+3\varpi k a_1a_2A(t)=0,\end{aligned}$$

and

$$\mathcal{J}^3\mathcal{J}': \quad 12\varpi\hbar B_2a_2+2\varpi kA(t)a_2^2=0,$$

where $\hbar=bn^3+cnk^2+cnm^2$. Solving these equations yields

$$a_0(t)=\ell_0e^{-\frac{1}{2}\delta^2t},\quad a_1=0,\ a_2(t)=\ell_2e^{-\frac{1}{2}\delta^2t},$$

$$\hbar=\frac{-\ell_2kA(t)}{6B_2}e^{-\frac{1}{2}\delta^2t},\text{ and }\ \lambda(t)=k(\frac{2\ell_2B_1}{3B_2}-\ell_0)A(t)e^{-\frac{1}{2}\delta^2t},$$

where ℓ_0 and ℓ_2 are constants. Therefore, the solution of the QZKE-RVCs (3) is

$$\mathcal{U}(x,y,z,t)=[\ell_0+\ell_2\mathcal{J}^2(\zeta)]e^{-\frac{1}{2}\delta^2t},\ \zeta=\varpi kx+\varpi my+\varpi nz+k(\frac{2\ell_2B_1}{3B_2}-\ell_0)\int_0^t e^{\delta\mathcal{B}(\tau)-\frac{1}{2}\delta^2\tau}d\tau. \tag{21}$$

In the following, we define $\mathcal{J}(\zeta)$ as:

Set 1: When $\mathcal{J}(\zeta)=sn(\varpi\zeta,\tilde{n})$, hence Eq. (21) becomes

$$\begin{aligned}\mathcal{U}(x,y,z,t) &= \Big(\ell_0+\ell_2\Big(sn(\varpi kx+\varpi my+\varpi nz\\ &\quad +k\varpi(\frac{2\ell_2B_1}{3B_2}-\ell_0)\int_0^t e^{\delta\mathcal{B}(\tau)-\frac{1}{2}\delta^2\tau}d\tau,\tilde{n})\Big)^2\Big)e^{-\frac{1}{2}\delta^2t},\end{aligned} \tag{22}$$

where

$$B_1=-\varpi^2(1+\tilde{n}^2)\text{ and }B_2=2\varpi^2\tilde{n}^2.$$

Set 2: When $\mathcal{J}(\zeta)=cn(\varpi\zeta,\tilde{n})$, hence Eq. (21) becomes

$$\begin{aligned}\mathcal{U}(x,y,z,t) &= \Big(\ell_0+\ell_2\Big(cn(\varpi kx+\varpi my+\varpi nz\\ &\quad -k\varpi(\frac{2\ell_2B_1}{3B_2}+\ell_0)\int_0^t e^{\delta\mathcal{B}(\tau)-\frac{1}{2}\delta^2\tau}d\tau,\tilde{n})\Big)^2\Big)e^{-\frac{1}{2}\delta^2t},\end{aligned} \tag{23}$$

where

$$B_1=\varpi^2(1-2\tilde{n}^2)\text{ and }B_2=-2\varpi^2\tilde{n}^2.$$

Set 3: When $\mathcal{J}(\zeta)=dn(\varpi\zeta,\tilde{n})$, hence Eq. (21) becomes

$$\begin{aligned}\mathcal{U}(x,y,z,t) &= \Big(\ell_0+\ell_2\Big(dn(\varpi kx+\varpi my+\varpi nz\\ &\quad +k\varpi(\frac{2\ell_2B_1}{3B_2}-\ell_0)\int_0^t e^{\delta\mathcal{B}(\tau)-\frac{1}{2}\delta^2\tau}d\tau,\tilde{n})\Big)^2\Big)e^{-\frac{1}{2}\delta^2t},\end{aligned} \tag{24}$$

where

$$B_1=\varpi^2(2-\tilde{n}^2)\text{ and }B_2=2\varpi^2.$$

3. Exact Solutions of SQZKE

We use previously results and the transformation (2) to acquire the solutions of SQZKE (1) as follows:

3.1. METF-method

Putting Eq. (8) into Eq. (2), we have the solution of SQZKE (1) as

$$\mathcal{G}(x,y,z,t) = \mathcal{U}(\xi)e^{\delta\mathcal{B}(t)},\ \xi = kx + my + nz - ak\ell_0\int_0^t e^{\delta\mathcal{B}(\tau)-\frac{1}{2}\delta^2\tau}d\tau. \tag{25}$$

When $p > 0$, hence the trigonometric function solutions of the SQZKE (1), utilizing (9)-(13), are:

$$\mathcal{G}_1(x,y,z,t) = \Big(\ell_0 + \ell_2 p\tan^2(\sqrt{p}\xi)\Big)e^{[\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t]}, \tag{26}$$

$$\mathcal{G}_2(x,y,z,t) = \Big(\ell_0 - \ell_2 p\cot^2(\sqrt{p}\xi)\Big)e^{[\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t]}, \tag{27}$$

$$\mathcal{G}_3(x,y,z,t) = \Big(\ell_0 + \ell_2 p\Big(\tan(\sqrt{4p}\xi) \pm \sec(\sqrt{4p}\xi)\Big)^2\Big)e^{[\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t]}, \tag{28}$$

$$\mathcal{G}_4(x,y,z,t) = \Big(\ell_0 - \ell_2 p\Big(\cot(\sqrt{4p}\xi) \pm \csc(\sqrt{4p}\xi)\Big)^2\Big)e^{[\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t]}, \tag{29}$$

$$\mathcal{G}_5(x,y,z,t) = \Big(\ell_0 + \frac{\ell_2 p}{2}\Big(\tan(\frac{1}{2}\sqrt{p}\xi) - \cot(\frac{1}{2}\sqrt{p}\xi)\Big)^2\Big)e^{[\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t]}, \tag{30}$$

While, if $p < 0$, then the hyperbolic function solutions of the SQZKE (1), utilizing (14)-(17), are:

$$\mathcal{G}_6(x,y,z,t) = \Big(\ell_0 + \ell_2 p\tanh^2(\sqrt{-p}\xi)\Big)e^{[\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t]}, \tag{31}$$

$$\mathcal{G}_7(x,y,z,t) = \Big(\ell_0 + \ell_2 p\coth^2(\sqrt{-p}\xi)\Big)e^{[\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t]}, \tag{32}$$

$$\mathcal{G}_8(x,y,z,t) = \Big(\ell_0 + \ell_2 p\Big(\coth(\sqrt{-4p}\xi) \pm \operatorname{csch}(\sqrt{-4p}\xi)\Big)^2\Big)e^{[\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t]}, \tag{33}$$

$$\mathcal{G}_9(x,y,z,t) = \Big(\ell_0 + \frac{\ell_2 p}{2}\Big(\tanh(\frac{1}{2}\sqrt{-p}\xi) + \coth(\frac{1}{2}\sqrt{-p}\xi)\Big)^2\Big)e^{[\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t]}. \tag{34}$$

When $p = 0$, then the rational function solution SQZKE (1), utilizing (18), is:

$$\mathcal{G}_{10}(x,y,z,t) = \Big(\frac{-1}{\xi}\Big)e^{[\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t]}, \tag{35}$$

where $\xi = kx + my + nz - ak\ell_0\int_0^t e^{\delta\mathcal{B}(\tau)-\frac{1}{2}\delta^2\tau}d\tau$.

Remark 1. *Setting $\delta = 0$ in Eqs. (26)-(35) and choosing suitable values for ℓ_0 and ℓ_2,we obtain the same solutions that in [6].*

3.2. JEF-method

Plugging Eqs (22)-(24) into Eq. (2), we get the elliptic function solutions for SQZKE (1):

$$\begin{aligned}\mathcal{G}(x,y,z,t) &= \Big(\ell_0+\ell_2\Big(sn(\varpi kx+\varpi my+\varpi nz\\ &\quad -k\varpi(\frac{\ell_2(1+\tilde{n}^2)}{3\tilde{n}^2}+\ell_0)\int_0^t e^{\delta\mathcal{B}(\tau)-\frac{1}{2}\delta^2\tau}d\tau,\tilde{n})\Big)^2\Big)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t},\end{aligned} \tag{36}$$

$$\begin{aligned}\mathcal{G}(x,y,z,t) &= \Big(\ell_0+\ell_2\Big(cn(\varpi kx+\varpi my+\varpi n\\ &\quad +k\varpi(\frac{\ell_2(2\tilde{n}^2-1)}{3\tilde{n}^2}-\ell_0)\int_0^t e^{\delta\mathcal{B}(\tau)-\frac{1}{2}\delta^2\tau}d\tau,\tilde{n})\Big)^2\Big)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t},\end{aligned} \tag{37}$$

and

$$\begin{aligned}\mathcal{G}(x,y,z,t) &= \Big(\ell_0+\ell_2\Big(dn(\varpi kx+\varpi my+\varpi nz\\ &\quad +k\varpi(\frac{\ell_2(2-\tilde{n}^2)}{3B_2}-\ell_0)\int_0^t e^{\delta\mathcal{B}(\tau)-\frac{1}{2}\delta^2\tau}d\tau,\tilde{n})\Big)^2\Big)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t},\end{aligned} \tag{38}$$

If $\tilde{n}\to 1$, then the Eqs (36)-(38) turn into the following hyperbolic function solutions

$$\begin{aligned}\mathcal{G}(x,y,z,t) &= \Big(\ell_0+\ell_2\Big(\tanh(\varpi kx+\varpi my+\varpi n\\ &\quad +k\varpi(\frac{2\ell_2}{3}-\ell_0)\int_0^t e^{\delta\mathcal{B}(\tau)-\frac{1}{2}\delta^2\tau}d\tau)\Big)^2\Big)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t},\end{aligned} \tag{39}$$

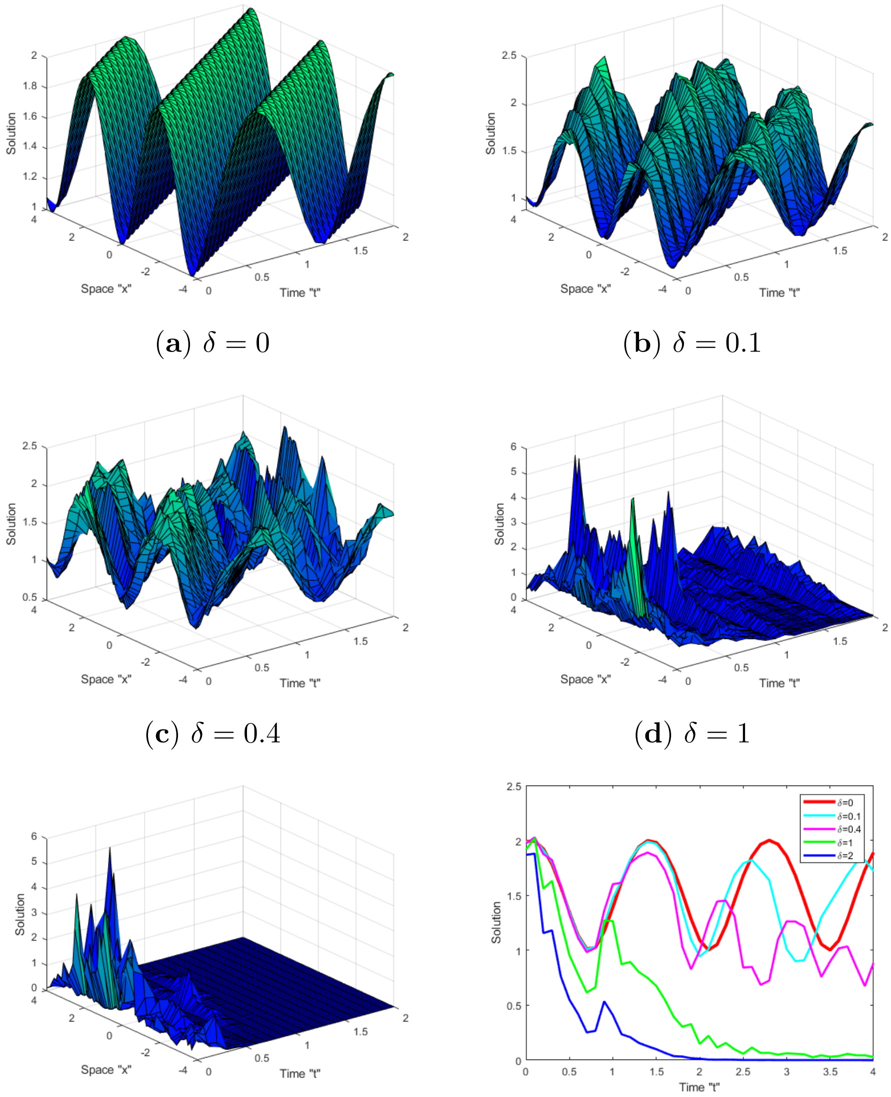
and

$$\begin{aligned}\mathcal{G}(x,y,z,t) &= \Big(\ell_0+\ell_2\Big(\text{sech}(\varpi kx+\varpi my+\varpi n\\ &\quad -k\varpi(\frac{\ell_2}{3}+\ell_0)\int_0^t e^{\delta\mathcal{B}(\tau)-\frac{1}{2}\delta^2\tau}d\tau)\Big)^2\Big)e^{\delta\mathcal{B}(t)-\frac{1}{2}\delta^2 t}.\end{aligned} \tag{40}$$

4. Discussion and impacts of noise

Discussion: We attained in this study the solutions of the SQZKE (1). We employed the JEF and METF methods, which yielded a variety of solutions, including solitary trigonometric solutions (26)-(31), solitary hyperbolic solutions (31)-(33), solitary rational solution (34) and solitary elliptic solutions (35)-(38). Solitary solutions of Quantum Zakharov-Kuznets equations are a crucial area of research with far-reaching implications for our understanding of plasma waves in quantum systems. By studying these solutions, researchers can gain valuable insights into the behavior of quantum plasma waves, develop new technologies based on their unique properties, and uncover new fundamental physics principles governing quantum systems. As technology continues to advance, the study of solitary solutions of QZK will likely play a key role in shaping our understanding of plasma dynamics in quantum regimes.

Impacts of noise: Here, we investigate how multiplicative noise affects the exact solution of SQZKE (1). Several graphic representations of possible solutions with varying noise intensities are displayed. Figures 1, 2 and 3 show the solutions $\mathcal{G}(x, y, z, t)$ stated in Eqs (35), (39) and (40), respectively, for various intensity of noise δ as follows:

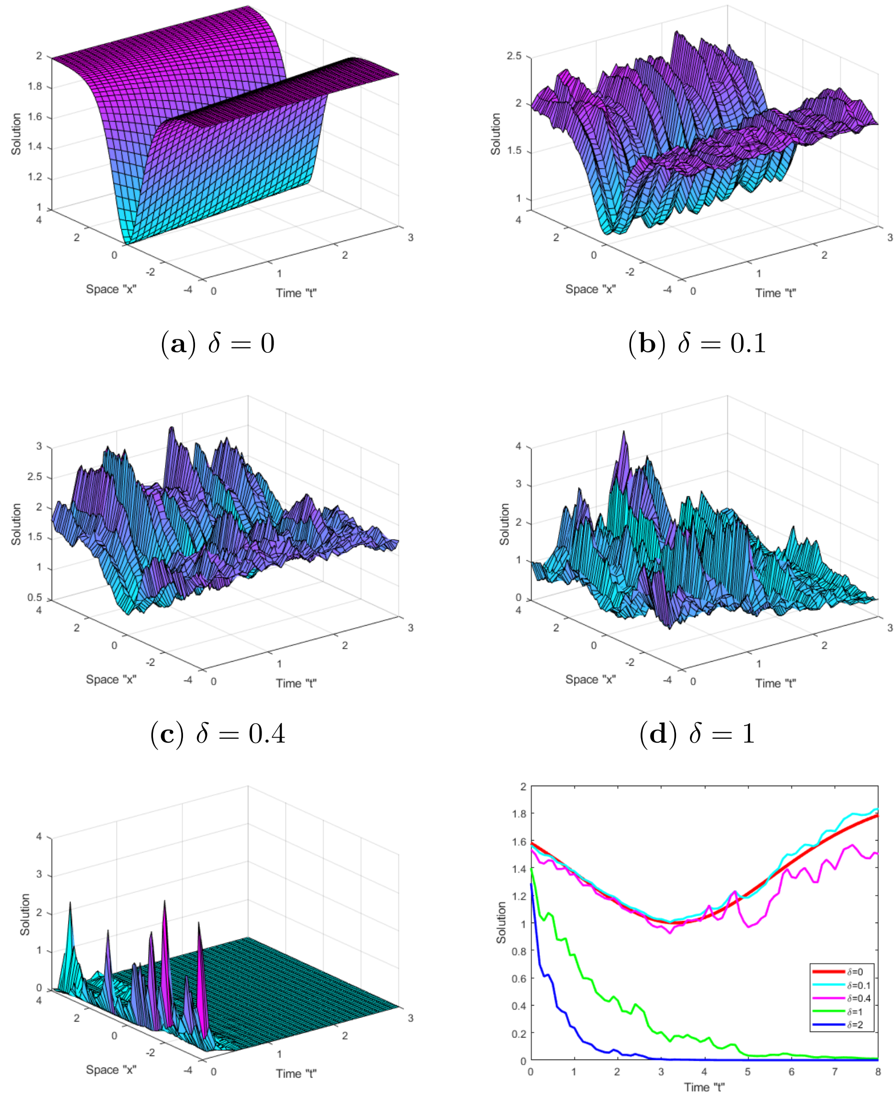


(**a**) $\delta = 0$ (**b**) $\delta = 0.1$

(**c**) $\delta = 0.4$ (**d**) $\delta = 1$

(**e**) $\delta = 2$ (**f**) $\delta = 0, 0.1, 0.4,\ 1,\ 2$

Figure 1: (a-e) describe 3-dimension shape of $\mathcal{G}(x, y, z, t)$ reported in Eq. (35) with $\ell_0 = \ell_2 = 1,\ \check{n} = 0.5,$ $\varpi = k = n = m = 1,\ y = z = 0,\ a = b = c = 1, t \in [0, 2]$ and $x \in [-4, 4]$ (f) presents the 2-dimension shape of the Eq. (35) with various δ

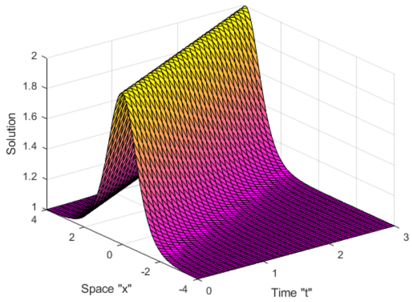


(**a**) $\delta = 0$ (**b**) $\delta = 0.1$

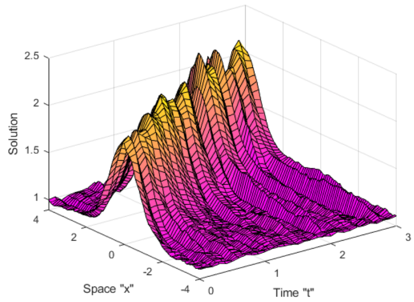
(**c**) $\delta = 0.4$ (**d**) $\delta = 1$

(**e**) $\delta = 2$ (**f**) $\delta = 0, 0.1, 0.4,\ 1,\ 2$

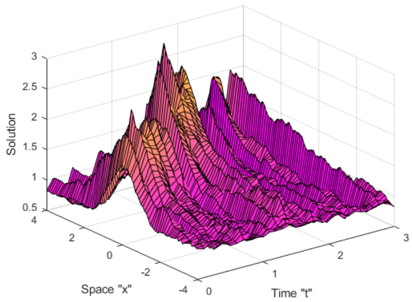
Figure 2: (a-e) present 3-dimension shape of $\mathcal{G}(x, y, z, t)$ described in Eq. (39) with $\ell = \delta = 1$, $\varpi = k = n = m = 1,\ y = z = 0,\ \gamma_1 = 1,\ t \in [0, 3]$, and $x \in [-4, 4]$ (f) presents 2-dimension shape of Eq. (39) with various δ



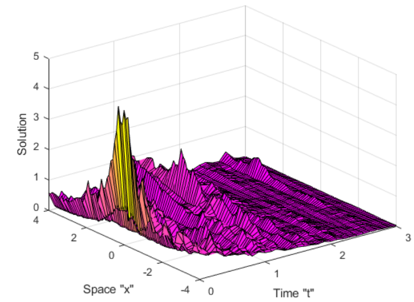
(**a**) $\delta = 0$



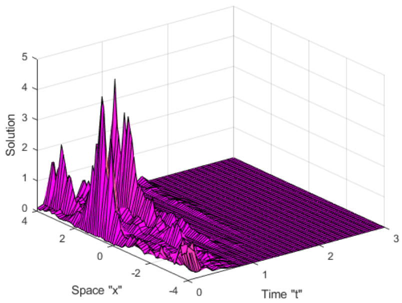
(**b**) $\delta = 0.1$



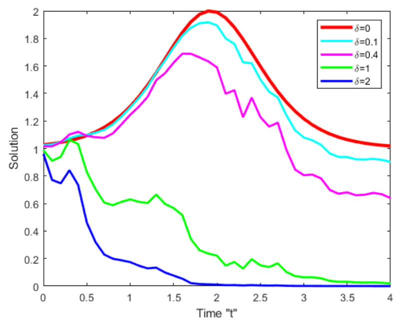
(**c**) $\delta = 0.4$



(**d**) $\delta = 1$



(**e**) $\delta = 2$



(**f**) $\delta = 0, 0.1, 0.4,\ 1,\ 2$

Figure 3: (a-e) shows 3-dimension shape of $\mathcal{G}(x, y, z, t)$ reported in Eq. (40) with $\ell = \delta = 1,\ \varpi = k = n = m = 1,\ y = z = 0,\ \gamma_1 = 1, t \in [0, 3]$, and $x \in [-4, 4]$ (f) presents 2-dimension shape of Eq. (40) with various δ

Figures 1–3 show that many kinds of solutions, such as solitary periodic solution, solitary bright solution, solitary dark solution, appear when noise is disregarded (i.e., $\delta = 0$). After a few transit patterns, the surface flattens when noise is added at $\delta = 0.1,\ 0.4,\ 1,\ 2$. This results demonstrates the stabilization of the solutions of SQZKE (1) around zero due to multiplicative Brownian motion.

5. Conclusions

In this study, we considered the stochastic QZKE (SQZKE) (1) forced in the Itô sense by multiplicative Brownian motion. We transformed the SQZKE into a different QZKE-RVCs (3) by applying the proper transformation. We employed the METF and JEF approaches to develop new exact stochastic solutions for QZKE-RVCs in the form of rational, hyperbolic, trigonometric, elliptic functions. Following that, we got the solutions of SQZKE (1). Although all previous studies assumed that the solutions of wave equation were deterministic, here we have considered that it is stochastic. In addition, we generated some earlier solutions, including the solutions presented in [6]. Due to the significance of QZKE in a dense quantum magnetoplasma, astrophysics, laboratory plasma experiments, and quantum device applications, the achieved solutions are essential in recognizing various challenging physical phenomena. Lastly, various graphs were included to show how the Brownian motion affected the exact solutions of SQZKE.

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