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More on Classes of Strongly Indexable Graphs

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Abstract. Given any positive integer k, a (p,q)-graph $G=(V,E)$ is *strongly k-indexable* if there exists a bijection $f: V \to \{0,1,2,\ldots,p-1\}$ such that $f^+(E(G)) = \{k, k+1, k+2, \ldots, k+q-1,\}$ where $f^+(uv) = f(u)+f(v)$ for any edge $uv \in E$; in particular, G is said to be *strongly indexable* when $k=1$. For any strongly k-indexable (p,q)-graph G, $q \le 2p-3$ and if, in particular, $q = 2p-3$ then G is called a *maximal strongly indexable graph*. In this paper, our main focus is to construct more classes of k-strongly indexable graphs.

2000 Mathematics Subject Classifications: 05C78

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1. Introduction

Unless mentioned otherwise, by a *graph* we shall mean in this paper a finite, undirected, connected graph without loops or multiple edges. Terms not defined here are used in the sense of Harary [11].

Acharya *et.al* [2] introduced the concept of an 'indexer' of a graph as a special case of *arithmetic labelings*. A *labeling* of a graph $G=(V,E)$ is an assignment f of distinct nonnegative integers to the vertices of G; it is an *indexer* of G if the induced 'edge function' $f^+: E(G) \to \mathbb{N}$, from $E(G)$ into the set $\mathbb{N}$ of natural numbers, defined by the rule: $f^+(uv) = f(u) + f(v)$, $\forall\ uv \in E(G)$, is also injective. It is known that every finite graph has an indexer; hence, an indexer f is said to be *optimal* if $f[G] := max_{v\in V(G)}\{f(v)\}$ has the least possible value $\upsilon(G)$ amongst all the indexers of G. Clearly, $\upsilon(G) \ge |V(G)|$ for any graph G with a countable number of vertices. For any given positive integer k, an indexer f of G is called a *k-indexer* if $f^+(E(G)) := \{f^+(uv) : uv \in E(G)\} = \{k, k+1, k+2, \ldots,\}$. Not every graph is k-indexable as indicated by the following theorem for finite graphs.

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Theorem 1. *[2]: Let $G = (V, E)$ be any (p, q)-graph and f be any k-indexer of G, where k is odd. Then, there exists an 'equitable partition' of V into two subsets V_o and V_e such that there are exactly $\lceil \frac{q+k-1}{2} \rceil$ edges each of which joins a vertex of V_o with one of V_e, where $\lceil . \rceil$ denotes the least integer function.*

Theorem 2. *[2]: For any indexable (p, q)-graph G, $q \leq 2p - 3$, calling G a* maximally indexable graph *if $q = 2p - 3$.*

Acharya and Germina [3] characterized the classes of maximal strongly indexable graphs, satisfying $q = 2p - 3$, particularly, such outerplanar graphs.

We shall need the following known results.

Theorem 3. *[1]: For any graph $G = (V, E)$ and for any additive vertex function $f : V(G) \to N$, $\Sigma_{e \in E} f^+(e) = \Sigma_{u \in V} f(u)d(u)$*

Theorem 4. *[2]: Every strongly indexable finite graph has at most one nontrivial component which is either a star or has a triangle.*

Lemma 1. *[4]: Let $G = (V, E)$ be a maximal outerplanar graph with $p > 7$. Let $H = (u_1, u_2, u_3, \ldots, u_p)$ be a Hamiltonian cycle in G. Let $V_1 = \{u_1, u_2, u_3, \ldots, u_{\lfloor \frac{p}{2} \rfloor}\}$ and $V_2 = \{u_{\lfloor \frac{p}{2} \rfloor + 1}, u_{\lfloor \frac{p}{2} \rfloor + 2}, u_{\lfloor \frac{p}{2} \rfloor + 3}, \ldots, u_p\}$ constitute an equitable partition of the vertex set of G. Then, no chord of G has both vertices in V_1 or V_2 if and only if $\Delta(G) = \lfloor \frac{p}{2} \rfloor + 2$ and there exist exactly two vertices of degree 2.*

Theorem 5. *[4]: Let $G = (V, E)$ be a maximal outerplanar graph with $p > 7$. Then, G is strongly indexable if and only if $\Delta(G) = \lfloor \frac{p}{2} \rfloor + 2$ and there exist exactly two vertices of degree 2.*

2. Construction of Strongly Indexable Graphs

Definition 1. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then the* join *$G = (V, E)$ of G_1 and G_2 is defines as the $V = V_1 + V_2$ and the edge set E of G is the edges in $G_1 \cup G_2$ and all edges joining G_1 and G_2.*

In this section we study the properties of some important families of graphs such as fans, ladders, and generalized prisms that are strongly indexable (for some value of k).

Theorem 6. *The fan $P_n + K_1$ is strongly indexable if and only if $n \in \{1, 2, 3, 4, 5, 6\}$.*

Proof. The strongly indexable labellings of $P_n + K_1$ for $n \in \{1, 2, 3, 4, 5, 6\}$ is depicted in Figure 1

Conversely, note that $P_n + K_1$, for $n \geq 2$ is a maximal outer planar graph with $q = 2p - 3$ and there exists a vertex of full degree. Hence invoking Lemma 1 (see, [4]), G is strongly indexable if and only if $p \leq 7$. Hence the proof follows.

Theorem 7. *$P_n + K_2$ is strongly indexable if and only if $n \leq 2$.*

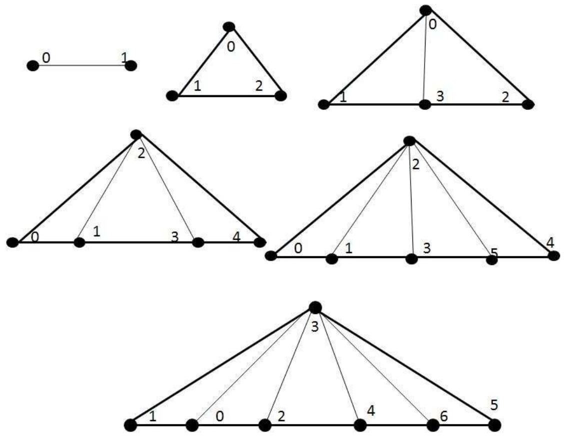

Figure 1

Proof. The strongly indexable labellings of $P_n + K_2$ for $n = 1, 2$ is depicted in Figure 2

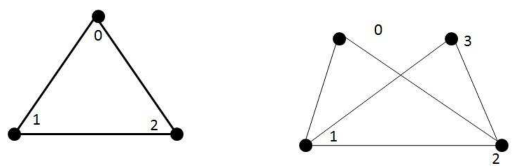

Figure 2

Converse follows from the fact that for any indexable (p,q)-graph G, $q \leq 2p - 3$ (See [1, 2]), since $|E(P_n + K_2)| > 2|V(P_n + K_2)| - 3$.

In general, we have the following Theorem

Theorem 8. *$P_n + K_i$ is strongly indexable if and only if $n \leq 2$, when $i \leq 2$, and $n \leq 6$, when $i = 1$*

Lemma 2. *For every positive integer n, the graph $K_2 + nK_1$ is strongly indexable.*

Proof. Let $V(K_2) = \{v_1, v_2\}$ and $V(nK_1) = \{u_1, u_2, \ldots, u_n\}$. Let $f : V(K_2 + nK_1) \rightarrow \{0, 1, 2 \ldots, n+1\}$ defined by
$f(v_1) = 0$; $f(v_2) = n + 1$; $f(u_i) = i,\ 1 \leq i \leq p - 2$

Remark 1. *Lemma 2 establishes the sharpness of the Theorem 2 and hence we obtain a sequence of strongly indexable graphs as follows: Take the labellling f defined for $K_2 + nK_1$. Remove the edge with maximum labelling (here $2p - 3$) and continue this process of removing the edge with maximum labelling until we arrive at $K_{1,n}$. Hence we are able to characterize all the strongly indexable complete m-bipartite graphs as follows.*

Theorem 9. *The only strongly indexable complete m-partite graphs are $K_{1,n}$ and $K_{1,1,n}$, for all integers $n \geq 1$*

Proof. It is easy to see that $K_{1,n}$ is strongly indexable by assigning 0 to the central vertex and the integers $1, 2, \ldots n-1$ to the non-central vertices in a one-one manner. Furthermore, the complete tripartite graph $K_{1,1,n} \cong K_2 + nK_1$ is strongly indexable by Lemma 2.

For the uniqueness of $K_{1,1,n}$ let $G \cong K_{n_1,n_2,n_3}$ be a complete tripartite graph with $n_1, n_2, n_3 \geq 1$. Now, assume the contrary that $n_2 \geq 2$ and G is strongly indexable. The order of G is $n_1 + n_2 + n_3$ and the size of G is $n_1n_2 + n_1n_3 + n_2n_3$. Since K_{n_1,n_2,n_3} is strongly indexable by assumption, $n_1n_2 + n_1n_3 + n_2n_3 \leq 2(n_1 + n_2 + n_3) - 3$, which in turn implies $n_1n_3 < 2n_2 - 3$, since $n_2 \geq 2$. Hence $n_2n_3 \leq 2n_2$, which implies $2 - n_3 > 0$, from which we conclude that $n_3 = 1$. By similar argument we get $n_1 = 1$.

Now to show that there are no strongly indexable complete m-partite graphs for $m \geq 4$, observe that $K_{1,1,1,n}$ is such that $|E(K_{1,1,1,n})| > 2|V(K_{1,1,1,n})| - 3$. ($|V(K_{1,1,1,n}| = 3 + n$ and $|E(K_{1,1,1,n}| = 3n + 3$).

Definition 2. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then the Cartesian product $G = (V, E)$ of G_1 and G_2 is defined as : Consider any two nodes $u = u_1u_2$ and v_1v_2 in $V = V_1 \times V_2$. Then u and v are adjacent in $G_1 \times G_2$, whenever $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$.*

The Ladder $L_n \cong P_n \times P_2$ is not strongly indexable for all $n \geq 2$, since L_2 contains no triangle. However there exists an integer k such that L_2 is k-strongly indexable.

Theorem 10. *The ladder $L_n \cong P_n \times P_2$ is $\lceil \frac{n}{2} \rceil$-strongly indexable, if n is odd.*

Proof. Let $V(P_2) = \{v_1, v_2\}$ and $V(P_n) = \{u_i : 1 \leq i \leq n\}$. Define $f : V(P_n \times P_2) \rightarrow \{0, 1, 2, \ldots, 2n\}$ defined by

$$f(u_i, v_1) = \begin{cases} \frac{i-1}{2} & 1 \leq i \leq n,\ i \text{ odd} \\ \frac{n+i-1}{2} & 1 \leq i \leq n,\ i \text{ even} \end{cases}$$

and

$$f(u_i, v_2) = \begin{cases} f(u_{n-1}v_1) + \frac{i}{2} & 1 \leq i \leq n,\ i \text{ even} \\ f(u_{n-1}v_1) + \frac{n+i}{2} & 1 \leq i \leq n,\ i \text{ odd} \end{cases}$$

Remark 2. *The converse of Theorem 10 is not true. $L_2 \cong C_4$ is not k-strongly indexable. However $P_4 \times P_2$ and $P_6 \times P_2$ are 3-strongly indexable and 4-strongly indexable respectively. (See Figure 3)*

Theorem 11. *$K_3 \times P_n$ is strongly indexable*

Proof. Let $V(K_3) = \{u_i : 1 \leq i \leq 3\}$ and $V(P_n) = \{x_i : 1 \leq i \leq n\}$.
Define $f : V(K_3 \times P_n) \rightarrow \{0, 1, , 2, \ldots, 3n\}$ defined by
$f(u_1x_i) = \{0, 4, 6, 10, 12, 16, 18, 22, 24, 30, 32, 36, \ldots\}$
$f(u_2x_i) = \{2, 3, 8, 9, 14, 15, 20, 21, 26, 27, 32, \ldots\}$ and
$f(u_3x_i) = \{1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, \ldots\}$

Figure 4 illustrates the strongly indexable labelling of $K_3 \times P_4$

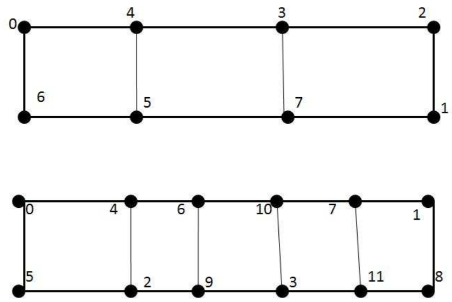

Figure 3

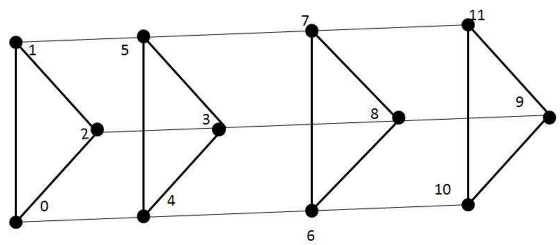

Figure 4

Theorem 12. *In general $K_n \times P_k$ is strongly indexable if and only $n = 3$.*

Proof. Necessary part follows from Theorem 11.
Converse follows from the fact that $|E(K_n \times P_k)| > 2|V(K_n \times P_k)| - 3$.

Theorem 13. *$C_m \times P_n$ is 2-strongly indexable if m is odd and $n \geq 2$*

Proof. Assume $V(C_m) = \{v_i, 1 \leq i \leq m\}$ and $V(P_n) = \{u_j, 1 \leq j \leq n.\}$ Now, the 2-strongly indexable labeling f is defined as follows

$$f(v_i, u_j) = \begin{cases} \frac{i+m-1}{2} & 1 \leq i \leq m,\ i \text{ even } j = 1 \\ \frac{i-1+m(2j-1)}{2} & 2 \leq j \leq n,\ j \text{ odd, } 1 \leq i \leq m \\ \frac{i-2+m(2j-1)}{2} & 2 \leq j \leq n,\ j \text{ even, } 1 \leq i \leq m,\ i \text{ odd} \\ \frac{i-2+m(2j-2)}{2} & 2 \leq j \leq n,\ j \text{ even } 1 \leq i \leq m,\ i \text{ even} \end{cases}$$

Remark 3. *Even though $C_m \times P_n$ is not strongly indexable we can generate infinitely many classes of strongly indexable graph G by adjoining two vertices say u and v with the vertex assignments 0 and 1 respectively. Hence $C_m \times P_n \cup \{uv\}$, where u and v having the vertex assignments 0 and 1 are classes of strongly indexable graphs. In fact, this constriction of adjoining an edge uv where, $f(u) = 0$and $f(v) = 1$ of a 2-strongly indexable graphs results in to a strongly indexable graph.*

Theorem 14. *Given any k-strongly indexable graph $G = (p, q)$, there exists a strongly indexable graph $H = (p, q + k - 1)$ graph, with G a spanning subgraph of H.*

Proof. Let $G = (p,q)$, be k-strongly indexable and let f be the k-strong indexer of G. Hence, $f(V(G) = \{0,1,2,\ldots,p-1\}$ and $f^+(E(G) = \{k,k+1,\ldots,k+q-1\}$. Let $u \in V(G)$ be such that $f(u) = 0$ and let $u_i, 1 \le i \le k-1$ be the vertices of G with $f(u_i) = i,\ 1 \le i \le k-1$. Now construct the edges by joining uu_i so that $f(uu_i) = \{1,2,\ldots,k-1\}$. The new graph H constructed is a $(p,q+k-1)$ graph with $f(V(H) = \{0,1,2,p-1\}$ and $f^+(E(H) = \{1,2,\ldots,k+q-1\}$ and hence f is a strong indexer of H.

Figure 5 and Figure 6 gives the strongly indexable labellings of $C_5 \times P_n$ and $C_5 \times P_n \cup \{uv\}$

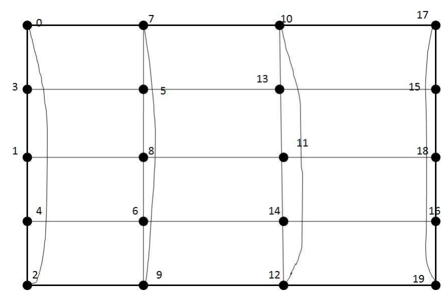

Figure 5

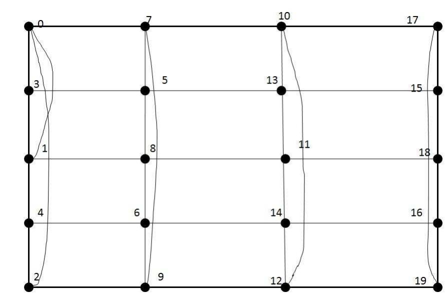

Figure 6

Definition 3. *For three or more disjoint graph $G_1, G_2, \ldots, G_k$ sequential join $G_1 + G_2 + \ldots G_k$ is the graph $(G_1 + G_2) \cup (G_2 + G_3) \cup \cdots \cup (G_{k-1} + G_n)$*

Lemma 3. *Let $G_i \cong K_1$, $1 \le i \le n$. Then the sequential join $(G_1+G_2) \cup (G_2+G_3) \cup \cdots \cup (G_{n-1}+G_n)$ is strongly indexable if and only if $n \le 3$.*

Proof. The proof follows from the fact that P_n is strongly indexable if and only if $n \le 3$.

Lemma 4. *Let $G_i \cong K_1$, $1 \le i \le n$. Then the sequential join $(G_1+G_2) \cup (G_2+G_3) \cup \cdots \cup (G_{n-1}+G_n)$ is $\lceil \frac{n}{2} \rceil$- strongly indexable for all n.*

Proof. The proof follows from the fact that P_n is $\lceil \frac{n}{2} \rceil$-strongly indexable for all n, where the $\lceil \frac{n}{2} \rceil$-strongly indexable labelling of P_n is as follows

Define $f : V(P_m) \to \{0,1,2,\ldots,m-1\}$ defined by

$$f(u_i) = \begin{cases} \frac{i-1}{2} & 1 \leq i \leq m,\ i \text{ odd},\ m \text{ odd} \\ \frac{i-1}{2} & 1 \leq i \leq m-1,\ i \text{ odd},\ m \text{ even} \\ \frac{n-i-1}{2} & 2 \leq i \leq m-1,\ i \text{ even},\ m \text{ odd} \\ \frac{n-i-2}{2} & 2 \leq j \leq m-1,\ i \text{ even},\ m \text{ even} \end{cases}$$

Lemma 5. *Let $G_1 \cong K_1$, and $G_i \cong K_2$, $2 \leq i \leq n$. The sequential join $(G_1 + G_2) \cup (G_2 + G_3) \cup \cdots \cup (G_{n-1} + G_n)$ is strongly indexable if and only if $n = 2$. However, P_n is $\lceil \frac{n}{2} \rceil$-strongly indexable.*

Proof. The proof follows from the fact that P_n is strongly indexable if and only if $n \leq 3$ and that P_n is $\lceil \frac{n}{2} \rceil$-strongly indexable

Theorem 15. *$K_{1,n} + K_i$ is not strongly indexable for $n \geq 2$, $i \geq 1$.*

Proof. The proof follows from the fact that $|E(K_{1,n} + K_i)| > 2V(K_{1,n} + K_i)| - 3$.

Theorem 16. *Let $G_i \cong K_{1,n}$, $1 \leq i \leq n$. The sequential join $G \cong (G_1 + G_2) \cup (G_2 + G_3) \cup \cdots \cup (G_{n-1} + G_n)$ is strongly indexable if and only if, either $i = n = 1$ or $i = 2$ and $n = 1$ or $i = 1, n = 3$.*

Proof. Let $i = n = 1$, then $G \cong P_2$, which is strongly indexable and when $i = 2$ and $n = 1$ or $i = 1, n = 3$, $G \cong P_3$ which is again strongly indexable.
Converse follows from the fact that, whenever $i = n > 1$ or $i > 2$ and $n > 1$ or $i > 1, n > 3$, $|E(G)| > |V(G) - 3$.

Definition 4. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then the* union $G = (V, E)$ *of G_1 and G_2 is defines as the $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$.*

Theorem 17. *For any integer $n \geq 3$, the linear forest $P_1 \cup P_n$ is strongly indexable if and only if $n \leq 3$.*

Proof. Clearly, for $n \in \{0,1,2\}$, $P_1 \cup P_n$ is strongly indexable.
Conversely, since P_n, $n \geq 4$ is not strongly indexable since we can not have a strongly indexable labelling of P_n.

Theorem 18. *For any integer $n \geq 3$, the linear forest $P_1 \cup P_n$ is $\lceil \frac{n}{2} \rceil$-strongly indexable.*

Proof. The proof is immediate as P_n, $n \geq 4$ is $\lceil \frac{n}{2} \rceil$-strongly indexable.

Theorem 19. *The linear forest $P_2 \cup P_n$ is not strongly indexable. However $P_2 \cup P_n$ is $\lceil \frac{n+3}{2} \rceil$-strongly indexable.*

Proof. Let $P_2 \cup P_n$ be a linear forest. Let $V(P_2 \cup P_n) = \{u_1, u_2\} \cup \{v_i,\ 1 \le i \le n\}$ so that $E(P_2 \cup P_n) = \{u_1u_2\} \cup \{v_iv_{i+1},\ 1 \le i \le n-1\}$. Now, $V(P_2 \cup P_n) = n+2$ and $E(P_2 \cup P_n) = n$.

Define $f : V(P_2 \cup P_n) \to \{0, 1, 2, \ldots, n+1\}$ defined in the following cases.

Case 1 $n \equiv 0(mod4)$
$f(u_1) = 0$; $f(u_2) = \frac{n}{2} + 2$

$$f(v_j) = \begin{cases} \frac{n}{2}+1 & \text{if } j = 1 \\ \frac{n}{2}+3 & \text{if } j = 3 \\ 2i-1 & \text{if } j = 4i \text{ and } 1 \le i \le \frac{n}{4} \\ \frac{n}{2}+2i+3 & \text{if } j = 4i+2 \text{ and } 1 \le i \le \frac{n-4}{4} \\ 2i+2 & \text{if } j = 4i \text{ and } 1 \le i \le \frac{n-4}{4} \\ \frac{n}{2}+2i+2 & \text{if } j = 4i+3 \text{ and } 1 \le i \le \frac{n-4}{4} \end{cases}$$

Case 2 $n \equiv 1(mod4)$
$f(u_1) = 0$; $f(u_2) = n+1$

$$f(v_j) = \begin{cases} \frac{n+2j+1}{4} & \text{if } j \text{ is odd and } 1 \le j \le n \\ \frac{3n+2j+1}{4}+3 & \text{if } j \text{ is even and } 2 \le j \le \frac{n-1}{2} \\ \frac{2j-n+1}{4} & \text{if } j \text{ is even and } \frac{n+3}{2} \le j \le \frac{n-1}{2} \end{cases}$$

Case 3 $n \equiv 2(mod4)$
$f(u_1) = 0$; $f(u_2) = \frac{n}{2} + 1$

$$f(v_j) = \begin{cases} n+1 & \text{if } j = 1 \\ n-1 & \text{if } j = 3 \\ n & \text{if } j = n \\ \frac{n}{2}-2i+1 & \text{if } j = 4i \text{ and } 1 \le i \le \frac{n-2}{4} \\ n-2i-1 & \text{if } j = 4i+1 \text{ and } 1 \le i \le \frac{n-2}{4} \\ \frac{n}{2}-2i-2 & \text{if } j = 4i+2 \text{ and } 0 \le i \le \frac{n-6}{4} \\ n-2i & \text{if } j = 4i+3 \text{ and } 1 \le i \le \frac{n-6}{4} \end{cases}$$

Case 4 $n \equiv 3(mod4)$
$f(u_1) = 0$; $f(u_2) = n+1$

$$f(v_j) = \begin{cases} \frac{j+1}{2} & \text{if } j \text{ is odd and } 1 \le j \le \frac{n-1}{2} \\ \frac{j+n}{2} + 3 & \text{if } j \text{ is odd and } \frac{n+3}{2} \le j \le n \\ \frac{j+n+1}{2} & \text{if } j \text{ is even and } 2 \le j \le \frac{n-3}{2} \\ \frac{j+2}{2} + 3 & \text{if } j \text{ is even and } \frac{n+1}{2} \le j \le n-1 \end{cases}$$

In all the cases f extends a $\lceil \frac{n+3}{2} \rceil$-strongly indexable labelling, as the edge values are consecutive integers from $\lceil \frac{n+3}{2} \rceil$ to $\lceil \frac{3n+1}{2} \rceil$.

Theorem 20. *For an integer $m \ge 2$, $P_3 \cup mP_2$ is k-strongly indexable, where $k = \frac{3m+3}{2}$ if m is odd and $k = \frac{3m+2}{2}$ if m is even.*

Proof. **Case 1** m is odd
Define $f : V(P_3 \cup mP_2) \to \{0, 1, 2, \ldots, 2m+2\}$ defined by

$$f(x) = \begin{cases} \frac{3m-5}{2} & \text{if } x = u \\ m+1 & \text{if } x = v \\ \frac{m+3}{2} & \text{if } x = w \\ i-1 & \text{if } x = u_i \text{ and } \frac{m-1}{2} \le i \le \frac{m+3}{2} \\ i & \text{if } x = u_i \text{ and } \frac{m+5}{2} \le i \le m \\ i + \frac{3m-2}{2} + 2 & \text{if } x = v_i \text{ and } 1 \le i \le \frac{m+3}{2} \\ i + \frac{m-1}{2} & \text{if } x = v_i \text{ and } \frac{m+5}{2} \le i \le m \end{cases}$$

Clearly f is infective and the edge values are consecutive numbers from $\frac{3m+3}{2}$ to $\frac{5m-1}{2}$.

Case 2 m is even
Define $f : V(P_3 \cup mP_2) \to \{0, 1, 2, \ldots, 2m+2\}$ defined by

$$f(x) = \begin{cases} m+1 & \text{if } x = u \\ m & \text{if } x = v \\ m+2 & \text{if } x = w \\ i-1 & \text{if } x = u_i \text{ and } 1 \le i \le \frac{m+2}{2} \\ i & \text{if } x = u_i \text{ and } \frac{m}{2} \le i \le m \\ i + \frac{3m}{2} + 2 & \text{if } x = v_i \text{ and } 1 \le i \le \frac{m}{2} \\ i + \frac{m}{2} + 2 & \text{if } x = v_i \text{ and } \frac{m+2}{2} \le i \le m \end{cases}$$

Clearly f is infective and the edge values are consecutive numbers from $\frac{3m+2}{2}$ to $\frac{5m}{2} + 2$.

Theorem 21. $K_{1,n} \cup K_{1,n+1}$, $n \ge 1$ *is strongly* 3*-indexable*

Proof. Let $V(K_{1,n} \cup K_{1,n+1}) = \{u, u_i,\ 1 \le i \le n\} \cup \{v, v_i,\ 1 \le i \le n+1\}$ and $E(K_{1,n} \cup K_{1,n+1}) = \{uu_i : 1 \le i \le n\} \cup \{vv_i : 1 \le i \le n+1\}$.

Define $f : V(K_{1,n} \cup K_{1,n+1}) \rightarrow \{0, 1, 2, \ldots, 2n+2\}$ defined by
$f(u) = 0$; $f(u_i) = 2(i+1),\ 1 \leq i \leq n$
$f(v) = 2$; $f(v_i) = 2i - 1,\ 1 \leq i \leq n+1$.

Now, clearly $f(V(K_{1,n} \cup K_{1,n+1})) = \{0, 1, 2, \ldots 2n+2\}$. Also, the minimum and maximum edge value induced at the edges are $f^+(vv_1) = 2 + f(v_1) = 3$ and $f^+(vv_{n+1}) = 2 + f(v_{n+1}) = 2n+3$; Also note that $f(u) + f(u_i)$ is always even and $f(v) + f(v_i)$ is always odd. Again, $f(u) + f(u_i) \neq f(u) + f(u_j)$ for all $i \neq j$, and $f(v) + f(v_i) \neq f(v) + f(v_j)$ for all $i \neq j$. Hence, the induced edge values are consecutive integers from 3 to $2n+2$, which implies f is a 3-strong indexer of $K_{1,n} \cup K_{1,n+1}$.

Theorem 21 can be expanded to the following form

Theorem 22. *For the integers m, n, $mK_{1,n}$, is $\frac{3m-1}{2}$-strongly indexable, whenever m is odd.*

Proof. Let $V(mK_{1,n}) = \{u_i : 1 \leq i \leq m\} \cup \{v_{i,j} : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ and
$E(mK_{1,n}) = \{u_i v_{i,j} : 1 \leq i \leq m,\ 1 \leq j \leq n\}$.
Define $f : V(mK_{1,n}) \rightarrow \{0, 1, 2 \ldots, m(n+1) - 1\}$ defined by
$f(u_i) = i - 1,\ \ 1 \leq i \leq m$;

$$f(v_{i,1}) = \begin{cases} i + \frac{3m-3}{2} & \text{if } 1 \leq i \leq \frac{m+1}{2} \\ i + \frac{m-3}{2} & \text{if } \frac{m+1}{2} < i \leq m \end{cases}$$

and $f(v_{i,j}) = f(v_{i,1}) + m(j-1),\ \ 1 \leq i \leq m,\ \ 2 \leq j \leq n$;

Clearly, f is a $\frac{3m-1}{2}$-strong indexer of $mK_{1,n}$.

Figure 7 gives the strong indexer of $5K_4$.

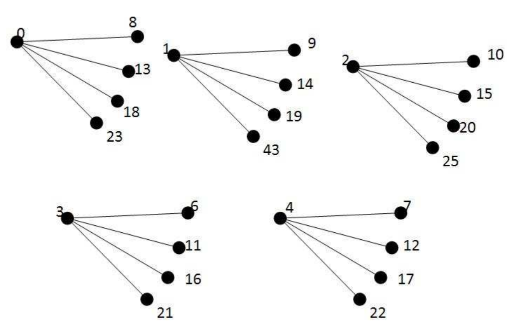

Figure 7

Theorem 23. *The graph mC_n is $m\lfloor \frac{n}{2} \rfloor$-strongly indexable for all $m \geq 1$ and $n \geq 3$.*

Proof. Let both m and n be odd integers. When $m = 1$ the graph is an odd cycle which is k-strongly indexable.
Let $V(C_n) = \{0, 1, 2, \ldots, n-1\}$
Define the strong indexer of C_n as follows:

$$f(v_i) = \begin{cases} \frac{i-2}{2} & \text{if } 1 \equiv 0(mod2) \\ \frac{n+i-2}{2} & \text{if } i \equiv 1(mod2) \end{cases}$$

Then f is a $\lfloor \frac{n}{2} \rfloor$-strong indexer of C_n. Now, let $m \geq 3$.
Let $V(mC_n) = \{u_{i,j} : 1 \leq i \leq m, \ \ 1 \leq j \leq n\}$ and
$E(mC_n) = \{u_{i,j}u_{i,j+1} : 1 \leq i \leq m, \ \ 1 \leq j \leq n-1\} \cup \{u_{i,n}u_{i,1} : 1 \leq i \leq m\}$.

Define $f : V(mC_n) \to \{0, 1, 2, \ldots, mn\}$ defined by

$f(u_{i,1}) = i - 1, \ 1 \leq i \leq m$ and

$$f(v_{i,j}) = \begin{cases} m(\lceil \frac{n}{2} \rceil + \frac{j-2}{2}) + \frac{2i+m-1}{2} & \text{if } 1 \leq i \leq \frac{m-1}{2} \text{ and } j \text{ even} \\ m(\lceil \frac{n}{2} \rceil + \frac{j-2}{2}) + \frac{2i-m-1}{2} & \text{if } \frac{m+1}{2} \leq i \leq m \text{ and } j \text{ even} \\ m(\frac{j-i}{2} + 1) - 2i & \text{if } 1 \leq i \leq \frac{m-1}{2} \text{ and } j \neq 1 \text{ is odd} \\ m(\frac{j-i}{2} + 2) - 2i & \text{if } \frac{m+1}{2} \leq i \leq m \text{ and } j \neq 1 \text{ is odd} \end{cases}$$

It is not difficult to check that f is a strong indexer of mC_n

Figure 8 is strongly indexable labelling of $7C_5$

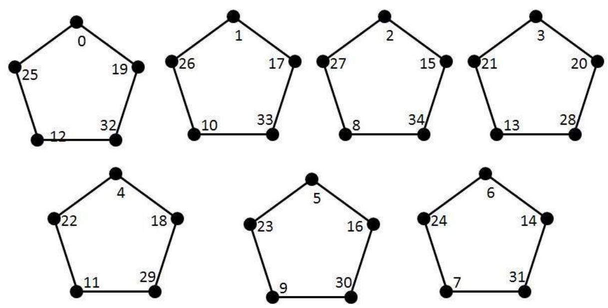

Figure 8

Remark 4. *Invoking Theorem 3 due to Acharya [1] which state that "if G is r-regular k-strongly indexable (p, q)-graph ($r \geq 1$), then q is odd" we see that the converse of Theorem 23 also holds good.*

From Theorem 23 and Remark 4 we have the following Theorems

Theorem 24. *The 2-regular graph mC_n is k-strongly indexable if and only if $m \geq 1$ and $n \geq 3$ are odd*

Theorem 25. *Any 3-regular graph $G = (p, q)$ is k-strongly indexable then $p \equiv 2(mod4)$*

Proof. Assume $G = (p, q)$ be 3-regular k-strongly indexable. Since G is 3-regular, p should necessarily be even so that either $p \equiv 0(mod4)$ or $p \equiv 2(mod4)$.

When $p \equiv 0(mod4)$, $p = 4t$ say, for some positive integer t so that $q = \frac{12t}{2} = 6t$
If f is the strong indexer of G, then by Theorem 3 [1]

$\Sigma_{i=0}^{p-1} id(u_i) = k + k + 1 + \cdots + k + q - 1$
Hence, $\frac{3p(p-1)}{2} = kq + \frac{q(q-1)}{2}$, applying $p = 4t,\ \ q = 6t \Rightarrow 21t(4t-1) = 12tk + 6t(6t-1) \Rightarrow$ $t = \frac{2k+1}{2}$, a contradiction.

Hence, $p \equiv 2(mod4)$.

3. Conclusion and scope

Graph labelings, where the vertices and edges are assigned, real values subject to certain conditions, have often been motivated by their utility to various applied fields and their intrinsic mathematical interest (logico - mathematical). Graph labelings are applied in determination of crystal structure from X-ray diffraction data [6, 10, 14, 15, 16], the design of certain important classes of good non periodic codes for pulse radar and missile guidance [7], and in the problem in radio-astronomy that a few movable antennae are required to be located in several successive array configurations to receive various spatial frequencies relative to some area of the sky [5]. Harper formulated this design Optimization Problem in graph labeling terms and solved some cases using this technic for minimum-confusion code design [12]. k-strongly indexable graphs are used in the construction of polygons of same internal angle and distinct sides: Using strongly k-Indexable labelings of a cycle C_{2n+1}, one can construct a polygon P_{4n+2} with $4n+2$ sides such that all the internal angles are equal and lengths of the sides are distinct [13].

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References

[1] B.D. Acharya. *On the construction of graphs with given constant valence- difference(s) on each of their lines*, Wiss. Z. Th. Ilmenau, **23**, 33-60, 1977.

[2] B.D. Acharya and S.M. Hegde. *Arithmetic graphs*, J. Graph Theory, **14**, 275-299, 1990.

[3] B.D. Acharya and Germina K.A. *Strongly k-indexable unicyclic graphs*, Graph Theory Notes of New York,45-49, LV:2008.

[4] B.D. Acharya and Germina K.A. *Maximal Strongly indexable Graphs*, Ars. Combinatorics, To appear.

[5] J.C. Bermond, A. Kotzig, and J. Turgeon. On a combinatorial problem of antennas in radio-astronomy. Colloquium of the Mathematical Society, Janos Bolyai 18, Combinatorics, Hungary, 135-149, 1976.

[6] G.S. Bloom. Numbered undirected graphs and their uses: A survey of unifying scientific and engineering concepts and its use in developing a theory of non-redundant homometric sets relating to some ambiguities in x-ray diffraction analysis, Ph. D., dissertation, Univ. of Southern California, Los Angeles, 1975

[7] A.R. Eckler. The construction of missile guidance codes resistant to random interference, Bell.Syst. Tech., J, Vol. 39, 973-994, 1960

[8] H. Enomoto. A.S. Llado, T. Nakamigawa, and G. Ringel, *Super edge-magic graphs*, SUT J. Math., 34,105-109, 1998.

[9] R. Figueroa-Centeno, R. Ichishima, and F. Muntaner-Batle. *The place of super-edge-magic labellings among other classes of labellings*, Discrete Math., **231**, 153-168, 2001.

[10] J.N. Franklin. ambiguities in the X-ray analysis of crystal structures, Acta Cryst., Vol. A 30, 698-702, Nov. 1974.

[11] F. Harary. **Graph Theory**, Addison Wesley,Reading, Massachusetts, 1969.

[12] L. H. Harper. DSIF Integrated circuit layout and Iso-perimetric Problems, JPL space program summary 37-66 Vol. 2, pp. 37-42, sept. 1970

[13] S.M. Hegde and Sudhakar Shetty. Strongly indexable graphs and applications, Discrete Mathematics, To appear.

[14] J. Leech. on the representation of $1, 2, \dots, n$ by differences, J. London math. Soc., Vol. 31, 160-169, Apr. 1956

[15] J.C. P. Miller, differences basis, three problems in additive number theory, A.D.L. Atkin and B.J. Birch, Eds, London, Academic Press,299-322, 1971.

[16] G. Ringel. Labeling problems, *Proceedings of the Eighth International Conference on Graph Theory, Combinatorics, Algorithms and Applications*. 1996