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Gradient Descent and Twice Differentiable Simpson-Type Inequalities via K-Riemann-Liouville Fractional Operators in Function Spaces

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Abstract. This paper investigates novel properties of Hilbert spaces through tensor operations and establishes new bounds for Simpson-type inequalities using fractional integral operators. The results contribute to advancing the theoretical understanding of these mathematical structures and their applications in functional analysis and related fields.

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1. Introduction

The concept of convexity is integral to numerous disciplines within mathematics and applied sciences, including optimization, machine learning, and energy systems. Recent advancements have highlighted the utility of convex functions in addressing complex real-world problems. For instance, machine learning techniques leverage convexity for improving market surveillance and customer relationship management systems to estimate

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error terms using inequalities [9]. Convexity principles are also applied to transportation systems, such as in fatigue detection for drivers using adaptive fuzzy classifiers [44]. In industrial optimization, convexity plays a critical role in designing squeeze casting parameters with neural networks [43]. Moreover, it contributes to the efficient design of perovskite solar cells by enabling specific material configurations [42]. In the field of control systems, convex optimization is employed to achieve robust stability designs for inverters [51]. For autonomous systems, convexity-based methods improve docking control for underwater vehicles through adaptive reinforcement learning [64]. Lastly, convex functions have enabled advancements in biotechnology, such as efficient co-production of hydrogen and methane through microbial regulation [63]. These applications demonstrate the extensive and varied impact of convexity across multiple domains [33, 60, 65].

Recent years have witnessed remarkable progress in fractional calculus, influencing diverse domains of mathematics and applied sciences [26, 29, 35, 52]. The development of novel definitions for fractional integrals and derivatives, extending classical approaches, has attracted significant research interest. These new definitions have become a central focus in mathematical analysis, paving the way for innovative methodologies. The adaptability of fractional calculus [23, 24, 61] has enabled researchers to establish convex integral inequalities, which play a crucial role in approximation theory. Inequalities such as Jensen's [49], Simpson's [37], Ostrowski's [8], Hermite–Hadamard's [48], and trapezoidal [1] inequalities are frequently employed to derive error bounds for numerical integration methods. To construct such inequalities, researchers utilize different ways, including maps, operators, relations, and other analytical techniques, demonstrating the profound impact of fractional calculus in modern mathematical analysis.

For instance, in [5], the creators used symmetric curved composed capabilities to foster Hermite-Hadamard imbalances. In [55], partial Riemann-Liouville integrals were utilized to determine Newton-type disparities for summed up arched capabilities. The work in [53] introduced Simpson-type results utilizing different classes of convexities, while [66] presented Bullen-type results utilizing various novel convex mappings. The authors of [19] upgraded Young's disparity by giving interesting limits and applications, while [30] broadened Hölder's type result by addressing defer differential conditions through mean congruity and demonstrating their uniqueness. Moreover, in [16], Ostrowski-type imbalances were created utilizing differentiable s-arched mappings, and [54] presented trapezoidal-type disparities utilizing quantum integrals.

Simpson's disparity is pivotal as it not just lays out a hypothetical reason for surveying the accuracy of mathematical combination procedures yet in addition helps scientists in choosing ideal techniques in view of the particular properties of the capabilities being scrutinized, especially in the domains of quadrature mistake assessment and complex distinct integrals. Beginning from crafted by eighteenth century mathematician Thomas Simpson, Simpson's standard supports this disparity. The standard approximates a capability utilizing a quadratic polynomial, giving a viable means to gauge its vital. In particular, for a capability $\Im$ that is ceaseless more than the span $[\nu_1, \nu_2]$, Simpson's $\frac{3}{8}$ rule approximates the basic as observes:

$$\int_{\nu_1}^{\nu_2} \Im(\mathcal{W})\mathrm{d}\mathcal{W} \approx \frac{\nu_2-\nu_1}{8}\left[\Im(\nu_1)+3\Im\left(\frac{2\nu_1+\nu_2}{3}\right)+3\Im\left(\frac{\nu_1+2\nu_2}{3}\right)+\Im(\nu_2)\right].$$

The most frequently utilized Simpson-type disparity has the accompanying definition.

Theorem 1 (See [36]). *Assume* $\Im : [\nu_1, \nu_2] \to \mathsf{R}$ *be a real-valued convex map, and suppose that* $\left\|\Im^{(4)}\right\|_\infty = \sup_{\mathcal{W}\in(\nu_1,\nu_2)} \left|\Im^{(4)}(\mathcal{W})\right| < \infty$. *Then, we have*
$\left|\frac{1}{6}\left[\Im(\nu_1)+4\Im\left(\frac{\nu_1+\nu_2}{2}\right)+\Im(\nu_2)\right]-\frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2}\Im(\mathcal{W})\mathrm{d}\mathcal{W}\right| \le \frac{1}{2880}\left\|\Im^{(4)}\right\|_\infty(\nu_2-\nu_1)^4.$

Scholars have utilized various approaches to investigate Simpson's inequality. For example, the authors of [12] established several novel inequalities using bi-dimensional convex functions using quantum integral operators. In [21], researchers employed various non-integer integral operators to, uncovering numerous enhanced bounds. The work in [15] focused on refinements and reversals of Simpson's inequality by utilizing preinvex mappings and quantum calculus. Similarly, the authors of [11] explored the concept of tempered fractional integral operators, while [45] employed multiplicative calculus to derive a range of bounds and reversals for such inequalities. For further insights into these related developments, readers are encouraged to consult [6, 10, 18, 20, 38, 39, 47] and the associated references.

Self-adjoint operators, a crucial concept in mathematics and physics, enable the extension of classical numerical inequalities to linear operators on Hilbert spaces. These operators, generalizing Hermitian matrices, are characterized by their symmetry, ensuring real eigenvalues and orthogonal eigenvectors. The study of such inequalities has significant applications in areas such as functional analysis, quantum mechanics, operator theory, and optimization. Recently, researchers have focused on adapting classical inequalities like Hermite–Hadamard, Jensen, and Hölder inequalities to the operator framework, deepening their applicability in quantum physics, matrix theory, and variational methods within Hilbert space settings. For instance, in [46], operators within Hilbert spaces were utilized to derive numerical-type inequalities, highlighting their significance in functional analysis and optimization. Similarly, [31] introduced various means inequalities for bounded operators, further enriching the theoretical framework of operator inequalities in Hilbert spaces. In [27], Hölder form inequalities involving power series were proposed, revealing intriguing applications within Hilbert space settings. Additionally, [59] investigated variational problems linked to inequalities and graph structures in Hilbert spaces, showcasing their utility in diverse mathematical and physical contexts. For additional insights and related results, readers are referred to the references in [4, 7, 14, 17, 34, 40, 62, 67].

Dragomir [28] introduces various innovative modifications and refinements of the following double inequality within the tensorial framework.

Theorem 2 (See [28]). *Let* $\mathcal{D}$ *be a complete inner product space. Suppose the operators (adjoint)* $\mathcal{U}$ *and* $\mathcal{D}$ *satisfy the condition* $0 < \mathcal{S}_1 \le \mathcal{U}, \mathcal{D} \le \mathcal{S}_2$, *where* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *are positive* *constants. Then*

$$\begin{aligned} 0 &\leq \frac{\mathcal{S}_1}{\mathcal{S}_2^2}\mathcal{W}(1-\mathcal{W})\left(\frac{\mathcal{U}^2\otimes 1+1\otimes\mathcal{D}^2}{2}-\mathcal{U}\otimes\mathcal{D}\right)\\ &\leq (1-\mathcal{W})\mathcal{U}\otimes 1+\mathcal{W}1\otimes\mathcal{D}-\mathcal{U}^{1-\mathcal{W}}\otimes\mathcal{D}^{\mathcal{W}}\\ &\leq \frac{\mathcal{S}_2}{\mathcal{S}_1^2}\mathcal{W}(1-\mathcal{W})\left(\frac{\mathcal{U}^2\otimes 1+1\otimes\mathcal{D}^2}{2}-\mathcal{U}\otimes\mathcal{D}\right). \end{aligned}$$

Theorem 3 (See [2])**.** *Assume that $\mathcal{U}$ and $\mathcal{D}$ are operators operators (adjoint) with corresponding spectra $\mathcal{SP}(\mathcal{U}),\mathcal{SP}(\mathcal{D})\subset\Delta$. Let $\Im$ is a continous function on Δ, then we have*

$$\begin{aligned} &2\min\{\kappa,1-\kappa\}\left[\frac{\Im(\mathcal{U})\otimes 1+1\otimes\Im(\mathcal{D})}{2}-\Im\left(\frac{2\mathcal{U}\otimes 1\otimes\mathcal{D}\otimes 1}{\mathcal{U}\otimes 1+1\otimes\mathcal{D}}\right)\right]\\ &\leq \kappa\Im(\mathcal{U})\otimes 1+(1-\kappa)1\otimes\Im(\mathcal{D})-\Im(\kappa\mathcal{U}\otimes 1+(1-\kappa)1\otimes\mathcal{D})\\ &\leq 2\min\{\kappa,1-\kappa\}\left[\frac{\Im(\mathcal{U})\otimes 1+1\otimes\Im(\mathcal{D})}{2}-\Im\left(\frac{2\mathcal{U}\otimes 1\otimes\mathcal{D}\otimes 1}{\mathcal{U}\otimes 1+1\otimes\mathcal{D}}\right)\right]. \end{aligned}$$

The author of [56] used standard operators and differentiable mappings to generate the following type of double inequalities.

Theorem 4 (See [2, 56])**.** *Assume that $\mathcal{U}$ and $\mathcal{D}$ are selfadjoint operators with associated sepctrums*
$\mathcal{SP}(\mathcal{U}),\mathcal{SP}(\mathcal{D})\subset\Delta$. Let $\Im$ is a continous function on Δ, then we have

$$\begin{aligned} &\int_0^1\Im((1-\mathcal{W})\mathcal{U}\otimes 1+\mathcal{W}1\otimes\mathcal{D})\mathtt{d}\mathcal{W}-\Im\left(\frac{\mathcal{U}\otimes 1+1\otimes\mathcal{D}}{2}\right)\\ =&\frac{(1\otimes\mathcal{D}-\mathcal{U}\otimes 1)^2}{16}\left[\int_0^1\mathcal{W}^2\Im''\left((1-\mathcal{W})\mathcal{U}\otimes 1+\mathcal{W}1\otimes\mathcal{D}\right)\mathtt{d}\mathcal{W}\right.\\ &\left.+\int_0^1(\mathcal{W}-1)^2\Im''\left(\left(\frac{1-\mathcal{W}}{2}\right)\mathcal{U}\otimes 1+\left(\frac{1+\mathcal{W}}{2}\right)1\otimes\mathcal{D}\right)\mathtt{d}\mathcal{W}\right]. \end{aligned}$$

The following double inequality was derived by the authors using positive semidefinite operators on a Hilbert space.

Theorem 5 (See [25])**.** *Let $\mathcal{U}$ and $\mathcal{D}$ be positive and semidefinite operators, with associated spectrums $\mathcal{SP}(\mathcal{U}),\mathcal{SP}(\mathcal{D})\subset\Delta$. Then*

$$\begin{aligned} (\mathcal{U}\#\mathcal{D})\otimes(\mathcal{U}\#\mathcal{D}) &\leqslant \frac{1}{2}\left\{(\mathcal{U}\sigma\mathcal{D})\otimes\left(\mathcal{U}\sigma^{\perp}\mathcal{D}\right)+\left(\mathcal{U}\sigma^{\perp}\mathcal{D}\right)\otimes(\mathcal{U}\sigma\mathcal{D})\right\}\\ &\leqslant \frac{1}{2}\{(\mathcal{U}\otimes\mathcal{D})+(\mathcal{D}\otimes\mathcal{U})\}. \end{aligned}$$

Significance of the study

Tensor inequalities extend the concepts of scalar and matrix inequalities to higher-dimensional spaces, offering a framework to handle multi-dimensional data. his generalization allows researchers to study and model problems in more complex settings where scalar or matrix representations are insufficient. They often play a role in estimating operator norms and proving convergence results in functional spaces. In optimization, tensor inequalities provide constraints and bounds that facilitate the solution of multi-dimensional problems, including those involving tensor decompositions or eigenvalue problems. Tensor inequalities provide bounds for quantum information measures, such as entanglement entropy and mutual information. Tensor inequalities are vital in understanding geometric structures like curvature in Riemannian geometry, where tensors such as the Riemann curvature tensor play a central role.

Our motivation to develop an advanced and novel version of various inequalities in tensor Hilbert spaces is largely inspired by the works of [2, 22, 57, 68]. By incorporating innovative techniques and perspectives that have been explored in only a limited number of studies, this work aims to significantly expand and enhance the existing theory of inequalities. The paper is organized into five sections. Section 2 presents a summary of fundamental concepts associated with Hilbert spaces and their basic operations. In Section 3, we derive various new bounds for numerical integral inequalities. Section 4 explores non-trivial examples and general observations. Finally, Section 5 discusses the main findings and potential directions for future research related to these results.

2. Preliminaries

In this section, we revisit fundamental concepts related to Hilbert spaces and extended convex mappings. For further details and additional insights, readers are encouraged to consult [14].

Definition 1 (See [41]). *A pre-Hilbert space on* $\mathtt{R}$ *is defined as*

$$(\cdot,\cdot):\mathtt{R}\times\mathtt{R}\to\mathcal{C},$$

for all $\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3\in\mathtt{R}$ *and* $\lambda\in\mathcal{C}$*, we have*

$$\begin{aligned}
\langle\mathcal{W}_1+\mathcal{W}_2,\mathcal{W}_3\rangle &= \langle\mathcal{W}_1,\mathcal{W}_3\rangle+\langle\mathcal{W}_2+\mathcal{W}_3\rangle\\
\langle\lambda\mathcal{W}_1,\mathcal{W}_2\rangle &= \lambda\langle\mathcal{W}_1,\mathcal{W}_2\rangle\\
\langle\mathcal{W}_1,\mathcal{W}_2\rangle &= \overline{\langle\mathcal{W}_2,\mathcal{W}_1\rangle}\\
\langle\mathcal{W}_1,\mathcal{W}_1\rangle &\geq 0,\quad \langle\mathcal{W}_1,\mathcal{W}_1\rangle=0\Longleftrightarrow\mathcal{W}_1=0.
\end{aligned}$$

Definition 2 (See [41]). *Assume* $\Im:\mathcal{U}\times\mathcal{D}\to\mathtt{R}$ *be a function. The corresponding product of* $\mathcal{U}$ *with* $\mathcal{D}$ *on Hilbert space* $\mathtt{R}$ *is denoted as*

- *the space* $\mathtt{R}$ *be a collections of all vectors* $\Im(\nu_1,\nu_2)(\nu_1\in\mathcal{U},\nu_2\in\mathcal{D})$ *such that* $\mathtt{R}$ *is generated;*
- $(\Im(\nu_1,\nu_2)\mid\Im(\nu_3,\nu_4))=(\nu_1\mid\nu_2)(\nu_3\mid\nu_4)$ *for* $\nu_1,\nu_2\in\mathcal{U},\nu_3,\nu_4\in\mathcal{D}$*. If* $(\mathtt{K},\Im)$ *is taken to be product of* $\mathcal{U}$ *and* $\mathcal{D}$*, it is write* $\nu_1\otimes\nu_2$ *inplace of* $\Im(\nu_1,\nu_2)$*. A product*

$\mathcal{U}\otimes\mathcal{D}$ and a mapping $(\nu_1,\nu_2)\mapsto\nu_1\otimes\nu_2$ of $\mathcal{U}\times\mathcal{D}$ into $\mathcal{U}\otimes\mathcal{D}$, holds

$$\begin{aligned}(\nu_1+\nu_2)\otimes\nu_2 &= \nu_1\otimes\nu_2+\nu_2\otimes\nu_2\\ (\lambda\nu_1)\otimes\nu_2 &= \lambda(\nu_1\otimes\nu_2)\\ \nu_1\otimes(\nu_3+\nu_4) &= \nu_1\otimes\nu_3+\nu_1\otimes\nu_3\\ \nu_1\otimes(\lambda\nu_2) &= \lambda(\nu_1\otimes\nu_2),\end{aligned}$$

where $\lambda\in\mathtt{R}$.

Assume $\Im:\Delta_1\times\ldots\times\Delta_{\mathtt{p}}\to\mathsf{R}$ be mapping defined over intervals. Assume that $\mathcal{P}=\left(\mathcal{P}_1,\ldots,\mathcal{P}_{\mathtt{p}}\right)$ be a collections of operators with $\mathtt{E}_1,\ldots,\mathtt{E}_{\mathtt{p}}$ are associated Hilbert spaces. Then

$$\mathcal{P}_{\mathtt{i}}=\int_{\Delta_{\mathtt{i}}}\mathcal{W}_{\mathtt{i}}\mathtt{dE}_{\mathtt{i}}\left(\mathcal{W}_{\mathtt{i}}\right)$$

is the spectra of possible operators for $\mathtt{i}=1,\ldots,\mathtt{p}$; following [25], we define $\mathcal{P}_{\mathtt{i}}$ as follows:

$$\Im\left(\mathcal{P}_1,\ldots,\mathcal{P}_{\mathtt{p}}\right):=\int_{\Delta_1}\cdots\int_{\Delta_{\mathtt{p}}}\Im\left(\mathcal{W}_1,\ldots,\mathcal{W}_{\mathtt{p}}\right)\mathtt{dE}_1\left(\mathcal{W}_1\right)\otimes\ldots\otimes\mathtt{dE}_{\mathtt{p}}\left(\mathcal{W}_{\mathtt{p}}\right).$$

Integrating processes into finite summations can significantly reduce the complexity of many complex processes when the dimensions of Hilbert spaces are finite. In [68], the author elaborates on the construction [25] and defines it as follows:

$$\Im\left(\mathcal{P}_1,\ldots,\mathcal{P}_{\mathtt{p}}\right)=\Im_1\left(\mathcal{P}_1\right)\otimes\ldots\otimes\Im_{\mathtt{p}}\left(\mathcal{P}_{\mathtt{p}}\right),$$

where $\Im$ is a product of one variable mapping and can be split. $\Im\left(\mathtt{a}_1,\ldots,\mathtt{a}_{\mathtt{p}}\right)=\Im_1\left(\mathtt{a}_1\right)\ldots\Im_{\mathtt{p}}\left(\mathtt{a}_{\mathtt{p}}\right)$.

If $\Im$ is sub(super)-multiplicative across the interval Δ, then

$$\Im(\nu_1\nu_2)\geq(\leq)\Im(\nu_1)\Im(\nu_2)\text{ for all }\nu_1\nu_2\in[0,\infty).$$

If $\Im$ is continuous on the interval $[0,\infty)$, then

$$\Im(\mathcal{U}\otimes\mathcal{D})\geq(\leq)\Im(\mathcal{U})\otimes\Im(\mathcal{D})\text{ for all }\mathcal{U},\mathcal{D}\geq 0.$$

This leads to the conclusion that, if

$$\mathcal{U}=\int_{[0,\infty)}\nu_1\mathtt{dE}(\nu_1)\text{ and }\mathcal{D}=\int_{[0,\infty)}\nu_2\mathtt{dF}(\nu_2)$$

are the assocaited spectrums.

$$\Im(\mathcal{U}\otimes\mathcal{D})=\int_{[0,\infty)}\int_{[0,\infty)}\Im(\nu_1\nu_2)\mathtt{dE}(\nu_1)\otimes\mathtt{dF}(\nu_2).$$

The geometric property of linear bounded operator $\mathcal{U},\mathcal{D}>0$ is defined as follows.

$$\mathcal{U}\#_{\mathtt{p}}\mathcal{D} := \mathcal{U}^{1/2}\left(\mathcal{U}^{-1/2}\mathcal{D}\mathcal{U}^{-1/2}\right)^{\mathtt{p}}\mathcal{U}^{1/2},$$

where $\mathtt{p} \in [0,1]$ and

$$\mathcal{U}\#\mathcal{D} := \mathcal{U}^{1/2}\left(\mathcal{U}^{-1/2}\mathcal{D}\mathcal{U}^{-1/2}\right)^{1/2}\mathcal{U}^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$\mathcal{U}\#\mathcal{D} = \mathcal{D}\#\mathcal{U} \text{ and } (\mathcal{U}\#\mathcal{D}) \otimes (\mathcal{D}\#\mathcal{U}) = (\mathcal{U} \otimes \mathcal{D})\#(\mathcal{D} \otimes \mathcal{U}).$$

Consider the eventually similar to the tensorial product :

$$(\mathcal{U}\beta) \otimes (\mathcal{D}\mathcal{S}) = (\mathcal{U} \otimes \mathcal{D})(\mathcal{U} \otimes \mathcal{S}),$$

that holds $\forall \mathcal{U}, \mathcal{D}, \beta, \mathcal{S} \in \mathtt{B}(\nu_2)$. If we take $\beta = \mathcal{U}$ and $\mathcal{S} = \mathcal{D}$, then we get

$$\mathcal{U}^2 \otimes \mathcal{D}^2 = (\mathcal{U} \otimes \mathcal{D})^2.$$

Through induction, we have

$$\mathcal{U}^{\mathtt{p}} \otimes \mathcal{D}^{\mathtt{p}} = (\mathcal{U} \otimes \mathcal{D})^{\mathtt{p}} \text{ for natural number } \sigma \geq 0.$$

Specifically

$$\mathcal{U}^{\kappa} \otimes 1 = (\mathcal{U} \otimes 1)^{\kappa} \text{ and } 1 \otimes \mathcal{D}^{\kappa} = (1 \otimes \mathcal{D})^{\kappa}$$

for all $\kappa \geq 0$. Additionally, we note that the $1 \otimes \mathcal{D}$ and $\mathcal{U} \otimes 1$ are commutative with each other

$$(\mathcal{U} \otimes 1)(1 \otimes \mathcal{D}) = (1 \otimes \mathcal{D})(\mathcal{U} \otimes 1) = \mathcal{U} \otimes \mathcal{D}.$$

Moreover, for any two natural numbers κ_1, κ_2

$$(\mathcal{U} \otimes 1)^{\kappa_1}(1 \otimes \mathcal{D})^{\kappa_2} = (1 \otimes \mathcal{D})^{\kappa_1}(\mathcal{U} \otimes 1)^{\kappa_2} = \mathcal{U}^{\kappa_2} \otimes \mathcal{D}^{\kappa_1}.$$

Definition 3 (See [3]). *A function $\Im : \Delta \to \mathsf{R}$ is known as convex over Δ, if*

$$\Im(\mathcal{W}\nu_1 + (1-\mathcal{W})\nu_2) \leq (\geq)\mathcal{W}\Im(\nu_1) + (1-\mathcal{W})\Im(\nu_2)$$

valid for all $\nu_1, \nu_2 \in \Delta$ and $\mathcal{W} \in [0,1]$.

Definition 4 (See [3]). *A mapping $\Im : \Delta \to \mathsf{R}$ is known as convex in quasi sense, if*

$$\Im((1-\mathcal{W})\nu_1 + \mathcal{W}\nu_2) \leq \max\{\Im(\nu_2), \Im(\nu_1)\} = \frac{1}{2}(\Im(\nu_2) + \Im(\nu_1) + |\Im(\nu_2) - \mathcal{D}(\nu_1)|)$$

for all $\nu_1, \nu_2 \in \Delta$ and $\mathcal{W} \in [0,1]$.

Some Needed Technical Lemmas

Lemma 1. *Let $\mathcal{U}$ and $\mathcal{D}$ be self-adjoint operators such that the spectra of $\mathcal{U}$ and $\mathcal{D}$ are contained within the set Δ (i.e., $\mathcal{SP}(\mathcal{U}) \subset \Delta$ and $\mathcal{SP}(\mathcal{D}) \subset \Delta$). If $\Im$ is a convex and differentiable function defined on Δ, the following inequality holds:*

$$\begin{aligned}\left(\Im'(\mathcal{U}) \otimes 1\right)(\mathcal{U} \otimes 1 - 1 \otimes \mathcal{D}) &\leq \Im(\mathcal{U}) \otimes 1 - 1 \otimes \Im(\mathcal{D}) \\ &\leq (\mathcal{U} \otimes 1 - 1 \otimes \mathcal{D})\left(1 \otimes \Im'(\mathcal{D})\right). \end{aligned} \tag{2.1}$$

Proof. Taking into account th following double inequality for $\Im$ on Δ, we derive

$$\Im'(\kappa)(\kappa - \gamma) \leq \Im(\kappa) - \Im(\gamma) \leq \Im'(\gamma)(\kappa - \gamma)$$

for all $\kappa, \gamma \in \Delta$. Since

$$\mathcal{U} = \int_{\Delta} \kappa \mathtt{dE}(\kappa) \text{ and } \mathcal{D} = \int_{\Delta} \gamma \mathtt{dF}(\gamma).$$

This imply that

$$\begin{aligned}\int_{\Delta}\int_{\Delta} \Im'(\kappa)(\kappa-\gamma)\mathtt{dE}_{\kappa} \otimes \mathtt{dE}_{\gamma} &\leq \int_{\Delta}\int_{\Delta} (\Im(\kappa) - \Im(\gamma))\mathtt{dE}_{\kappa} \otimes \mathtt{dE}_{\gamma} \\ &\leq \int_{\Delta}\int_{\Delta} \Im'(\gamma)(\kappa-\gamma)\mathtt{dE}_{\kappa} \otimes \mathtt{dE}_{\gamma}. \end{aligned} \tag{2.2}$$

Observe that

$$\begin{aligned}&\int_{\Delta}\int_{\Delta} \Im'(\kappa)(\kappa-\gamma)\mathtt{dE}_{\kappa} \otimes \mathtt{dE}_{\gamma} \\ &= \int_{\Delta}\int_{\Delta} \left(\Im'(\kappa)\kappa - \Im'(\kappa)\gamma\right) \mathtt{dE}_{\kappa} \otimes \mathtt{dE}_{\gamma} \\ &= \int_{\Delta}\int_{\Delta} \Im'(\kappa) t \mathtt{dE}_{\kappa} \otimes \mathtt{dE}_{\gamma} - \int_{\Delta}\int_{\Delta} \Im'(\kappa)\gamma \mathtt{dE}_{\kappa} \otimes \mathtt{dE}_{\gamma} \\ &= \left(\Im'(\mathcal{U})\mathcal{U}\right) \otimes 1 - \Im'(\mathcal{U}) \otimes \mathcal{D} \\ &\int_{\Delta}\int_{\Delta} (\Im(\kappa) - \Im(\gamma))\mathtt{dE}_{\kappa} \otimes \mathtt{dE}_{\gamma} = \Im(\mathcal{U}) \otimes 1 - 1 \otimes \Im(\mathcal{D}) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned}&\int_{\Delta}\int_{\Delta} \Im'(\gamma)(\kappa-\gamma)\mathtt{dE}_{\kappa} \otimes \mathtt{dE}_{\gamma} \\ &= \int_{\Delta}\int_{\Delta} \left(\kappa\Im'(\gamma) - \Im'(\gamma)\gamma\right) \mathtt{dE}_{\kappa} \otimes \mathtt{dE}_{\gamma} \\ &= \int_{\Delta}\int_{\Delta} \kappa\Im'(\gamma) \mathtt{dE}_{\kappa} \otimes \mathtt{dE}_{\gamma} - \int_{\Delta}\int_{\Delta} \Im'(\gamma)\gamma \mathtt{dE}_{\kappa} \otimes \mathtt{dE}_{\gamma} \\ &= \mathcal{U} \otimes \Im'(\mathcal{D}) - 1 \otimes \left(\Im'(\mathcal{D})\mathcal{D}\right) \end{aligned}$$

and by (2.3) we derive the inequality of interest

Lemma 2. *Let $\mathcal{U}$ and $\mathcal{D}$ be adjoint operators, with their spectra denoted by σ_1 and σ_2, respectively. Suppose the functions $\Im$ and ϑ are defined on σ_1, while $\mathcal{D}$ and $\mathcal{Q}$ are defined on σ_2, and let φ be convex on σ. Then, the set $\vartheta(\sigma_1)+\Im(\sigma_2)$ satisfies the following equality:*

$$\begin{aligned}&(\Im(\mathcal{U}) \otimes 1 + 1 \otimes \mathcal{D}(\mathcal{D}))\varphi(\vartheta(\mathcal{U}) \otimes 1 + 1 \otimes \Im(\mathcal{D}))\\ &= \int_{\sigma_1}\int_{\sigma_2} (\Im(\nu_2) + \mathcal{D}(\nu_1))\varphi(\vartheta(\nu_2) + \mathcal{Q}(\nu_1))\mathtt{dE}_{\nu_2} \otimes \mathtt{dF}_{\nu_1},\end{aligned} \tag{2.4}$$

where $\mathcal{U}$ and $\mathcal{D}$ have the spectral resolutions

$$\mathcal{U} = \int_{\sigma_1} \nu_2 \mathtt{dE}(\nu_2) \text{ and } \mathcal{D} = \int_{\sigma_2} \nu_1 \mathtt{dF}(\nu_1).$$

Proof. According to Stone-Weierstrass, any continuous function can be represented in terms of polynomial sequence, hence simply checking its equivalence is adequate. Let

$$\mathcal{D}(\mu) = \begin{cases} \frac{1}{2}\mu^2\sigma_1, & -\sigma \le \mu \le \sigma \\ \sigma\left(|\mu|^{\sigma_1} - \frac{\sigma}{2}\right), & \sigma > \mu > -\sigma. \end{cases}$$

If σ_1 and σ_2 are integers, and $|\mu| \le \sigma$, then the following holds:

$$\begin{aligned}\Im &:= \int_{\sigma_1}\int_{\sigma_2} (\Im(\nu_2) + \mathcal{D}(\nu_1))\frac{1}{2}(\vartheta(\nu_2) + \mathcal{Q}(\nu_1))^{2\sigma_1}\mathtt{dE}_{\nu_2} \otimes \mathtt{dF}_{\nu_1}\\ &= \int_{\sigma_1}\int_{\sigma_2} (\Im(\nu_2) + \mathcal{D}(\nu_1)) \sum_{\sigma_1=0}^{\sigma_2} C_{\sigma_2}^{\sigma_1}\frac{1}{2}[\vartheta(\nu_2)]^{2\sigma_1}[\mathcal{Q}(\nu_1)]^{2\sigma_2-2\sigma_1}\mathtt{dE}_{\nu_2} \otimes \mathtt{dF}_{\nu_1}\\ &= \sum_{\sigma_1=0}^{\sigma_2} C_{\sigma_2}^{\sigma_1}\int_{\sigma_1}\int_{\sigma_2} (\Im(\nu_2) + \mathcal{D}(\nu_1))\frac{1}{2}[\vartheta(\nu_2)]^{2\sigma_1}[\mathcal{Q}(\nu_1)]^{2\sigma_2-2\sigma_1}\mathtt{dE}_{\nu_2} \otimes \mathtt{dF}_{\nu_1}\\ &= \sum_{\sigma_1=0}^{\sigma_2} C_{\sigma_2}^{\sigma_1}\left[\int_{\sigma_1}\int_{\sigma_2} \Im(\nu_2)\frac{1}{2}[\vartheta(\nu_2)]^{2\sigma_1}[\mathcal{Q}(\nu_1)]^{2\sigma_2-2\sigma_1}\mathtt{dE}_{\nu_2} \otimes \mathtt{dF}_{\nu_1}\right.\\ &\left.+ \int_{\sigma_1}\int_{\sigma_2} \mathcal{D}(\nu_1)\frac{1}{2}[\vartheta(\nu_2)]^{2\sigma_1}[\mathcal{Q}(\nu_1)]^{2\sigma_2-2\sigma_1}\mathtt{dE}_{\nu_2} \otimes \mathtt{dF}_{\nu_1}\right].\end{aligned}$$

Observe that

$$\begin{aligned}&\int_{\sigma_1}\int_{\sigma_2} \Im(\nu_2)\frac{1}{2}[\vartheta(\nu_2)]^{2\sigma_1}[\mathcal{Q}(\nu_1)]^{2\sigma_2-2\sigma_1}\mathtt{dE}_{\nu_2} \otimes \mathtt{dF}_{\nu_1}\\ &= \Im(\mathcal{U})\frac{1}{2}[\vartheta(\mathcal{U})]^{2\sigma_1} \otimes [\Im(\mathcal{D})]^{2\sigma_2-2\sigma_1} = (\Im(\mathcal{U}) \otimes 1)\big([\vartheta(\mathcal{U})]^{2\sigma_2} \otimes [\Im(\mathcal{D})]^{2\sigma_2-2\sigma_1}\big)\\ &= (\Im(\mathcal{U}) \otimes 1)\frac{1}{2}\big([\vartheta(\mathcal{U})]^{2\sigma_2} \otimes 1\big)\big(1 \otimes [\Im(\mathcal{D})]^{2\sigma_2-2\sigma_1}\big)\\ &= (\Im(\mathcal{U}) \otimes 1)\frac{1}{2}(\vartheta(\mathcal{U}) \otimes 1)^{2\sigma_1}(1 \otimes \Im(\mathcal{D}))^{2\sigma_2-2\sigma_1}\end{aligned}$$

and

$$\begin{aligned}
&\int_{\sigma_1}\int_{\sigma_2}\frac{1}{2}[\vartheta(\nu_2)]^{2\sigma_1}\mathcal{D}(\nu_1)[\mathcal{Q}(\nu_1)]^{2\sigma_2-2\sigma_1}\mathtt{dE}_{\nu_2}\otimes\mathtt{dF}_{\nu_1}\\
&=\frac{1}{2}[\vartheta(\mathcal{U})]^{2\sigma_1}\otimes\big(\mathcal{D}(\mathcal{D})[\Im(\mathcal{D})]^{2\sigma_2-2\sigma_1}\big)=(1\otimes\mathcal{D}(\mathcal{D}))\frac{1}{2}\big([\vartheta(\mathcal{U})]^{2\sigma_2}\otimes[\Im(\mathcal{D})]^{2\sigma_2-2\sigma_1}\big)\\
&=(1\otimes\mathcal{D}(\mathcal{D}))\frac{1}{2}\big([\vartheta(\mathcal{U})]^{2\sigma_2}\otimes 1\big)\big(1\otimes[\Im(\mathcal{D})]^{2\sigma_2-2\sigma_1}\big)\\
&=(1\otimes\mathcal{D}(\mathcal{D}))\frac{1}{2}(\vartheta(\mathcal{U})\otimes 1)^{2\sigma_1}(1\otimes\Im(\mathcal{D}))^{2\sigma_2-2\sigma_1}
\end{aligned}$$

where $\frac{1}{2}(\vartheta(\mathcal{U})\otimes 1)$ and $\frac{1}{2}(1\otimes\Im(\mathcal{D}))$ are commutative, so we have

$$\begin{aligned}
\Im&=(\Im(\mathcal{U})\otimes 1+1\otimes\mathcal{D}(\mathcal{D}))\sum_{\sigma_1=0}^{\sigma_2}C_{\sigma_2}^{\sigma_1}\frac{1}{2}(\vartheta(\mathcal{U})\otimes 1)^{2\sigma_1}(1\otimes\Im(\mathcal{D}))^{2\sigma_2-2\sigma_1}\\
&=(\Im(\mathcal{U})\otimes 1+1\otimes\mathcal{D}(\mathcal{D}))\frac{1}{2}(\vartheta(\mathcal{U})\otimes 1+1\otimes\Im(\mathcal{D}))^{2\sigma_2}.
\end{aligned}$$

Taking into account: if $|\mu|>\sigma$, then

$$\begin{aligned}
\Im&:=\int_{\sigma_1}\int_{\sigma_2}(\Im(\nu_2)+\mathcal{D}(\nu_1))\sigma\left(|(\vartheta(\nu_2)+\mathcal{Q}(\nu_1))|^{\sigma_1}-\frac{\sigma}{2}\right)\mathtt{dE}_{\nu_2}\otimes\mathtt{dF}_{\nu_1}\\
&=\int_{\sigma_1}\int_{\sigma_2}(\Im(\nu_2)+\mathcal{D}(\nu_1))\sum_{\sigma_1=0}^{\sigma_2}C_{\sigma_2}^{\sigma_1}\sigma\left(|(\vartheta(\nu_2))|^{\sigma_1}-\frac{\sigma}{2}\right)\sigma\left(|(\mathcal{Q}(\nu_1))|^{\sigma_2-\sigma_1}-\frac{\sigma}{2}\right)\mathtt{dE}_{\nu_2}\otimes\mathtt{dF}_{\nu_1}\\
&=\sum_{\sigma_1=0}^{\sigma_2}C_{\sigma_2}^{\sigma_1}\int_{\sigma_1}\int_{\sigma_2}(\Im(\nu_2)+\mathcal{D}(\nu_1))\sigma\left(|(\vartheta(\nu_2))|^{\sigma_1}-\frac{\sigma}{2}\right)\sigma\left(|(\mathcal{Q}(\nu_1))|^{\sigma_2-\sigma_1}-\frac{\sigma}{2}\right)\mathtt{dE}_{\nu_2}\otimes\mathtt{dF}_{\nu_1}\\
&=\sum_{\sigma_1=0}^{\sigma_2}C_{\sigma_2}^{\sigma_1}\left[\int_{\sigma_1}\int_{\sigma_2}\Im(\nu_2)\sigma\left(|(\vartheta(\nu_2))|^{\sigma_1}-\frac{\sigma}{2}\right)\sigma\left(|(\mathcal{Q}(\nu_1))|^{\sigma_2-\sigma_1}-\frac{\sigma}{2}\right)\mathtt{dE}_{\nu_2}\otimes\mathtt{dF}_{\nu_1}\right.\\
&\left.+\int_{\sigma_1}\int_{\sigma_2}\mathcal{D}(\nu_1)\sigma\left(|(\vartheta(\nu_2))|^{\sigma_1}-\frac{\sigma}{2}\right)\sigma\left(|(\mathcal{Q}(\nu_1))|^{\sigma_2-\sigma_1}-\frac{\sigma}{2}\right)\mathtt{dE}_{\nu_2}\otimes\mathtt{dF}_{\nu_1}\right].
\end{aligned}$$

Observe that

$$\begin{aligned}
&\int_{\sigma_1}\int_{\sigma_2}\Im(\nu_2)\sigma\left(|(\vartheta(\nu_2))|^{\sigma_1}-\frac{\sigma}{2}\right)\sigma\left(|(\mathcal{Q}(\nu_1))|^{\sigma_2-\sigma_1}-\frac{\sigma}{2}\right)\mathtt{dE}_{\nu_2}\otimes\mathtt{dF}_{\nu_1}\\
&=\Im(\mathcal{U})\sigma\left(|(\vartheta(\mathcal{U}))|^{\sigma_1}-\frac{\sigma}{2}\right)\otimes\sigma\left(|(\Im(\mathcal{D}))|^{\sigma_2-\sigma_1}-\frac{\sigma}{2}\right)\\
&=(\Im(\mathcal{U})\otimes 1)\left[\sigma\left(|(\vartheta(\mathcal{U}))|^{\sigma_1}-\frac{\sigma}{2}\right)\otimes\sigma\left(|(\Im(\mathcal{D}))|^{\sigma_2-\sigma_1}-\frac{\sigma}{2}\right)\right]\\
&=(\Im(\mathcal{U})\otimes 1)\left[\sigma\left(|(\vartheta(\mathcal{U}))|^{\sigma_1}\otimes 1-\frac{\sigma}{2}\right)\otimes\sigma\left(1\otimes|(\Im(\mathcal{D}))|^{\sigma_2-\sigma_1}-\frac{\sigma}{2}\right)\right]\\
&=(\Im(\mathcal{U})\otimes 1)\left[\sigma\left((|(\vartheta(\mathcal{U}))|\otimes 1)^{\sigma_1}-\frac{\sigma}{2}\right)\otimes\sigma\left((1\otimes|(\Im(\mathcal{D}))|)^{\sigma_2-\sigma_1}-\frac{\sigma}{2}\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\sigma_1}\int_{\sigma_2} \mathcal{D}(\nu_1)\sigma\left(|(\vartheta(\nu_2))|^{\sigma_1}-\frac{\sigma}{2}\right)\sigma\left(|(\mathcal{Q}(\nu_1))|^{\sigma_2-\sigma_1}-\frac{\sigma}{2}\right)\mathtt{dE}_{\nu_2}\otimes \mathtt{dF}_{\nu_1}\\
&=\mathcal{D}(\nu_1)\sigma\left(|(\vartheta(\mathcal{U}))|^{\sigma_1}-\frac{\sigma}{2}\right)\otimes\sigma\left(|(\Im(\mathcal{D}))|^{\sigma_2-\sigma_1}-\frac{\sigma}{2}\right)\\
&=(1\otimes\mathcal{D}(\nu_1))\left[\sigma\left(|(\vartheta(\mathcal{U}))|^{\sigma_1}-\frac{\sigma}{2}\right)\otimes\sigma\left(|(\Im(\mathcal{D}))|^{\sigma_2-\sigma_1}-\frac{\sigma}{2}\right)\right]\\
&=(1\otimes\mathcal{D}(\nu_1))\left[\sigma\left(|(\vartheta(\mathcal{U}))|^{\sigma_1}\otimes 1-\frac{\sigma}{2}\right)\otimes\sigma\left(1\otimes|(\Im(\mathcal{D}))|^{\sigma_2-\sigma_1}-\frac{\sigma}{2}\right)\right]\\
&=(1\otimes\mathcal{D}(\nu_1))\left[\sigma\left((|(\vartheta(\mathcal{U}))|\otimes 1)^{\sigma_1}-\frac{\sigma}{2}\right)\otimes\sigma\left((1\otimes|(\Im(\mathcal{D}))|)^{\sigma_2-\sigma_1}-\frac{\sigma}{2}\right)\right]
\end{aligned}$$

where $\sigma\left(|(\vartheta(\mathcal{U}))|\otimes 1)-\frac{\sigma}{2}\right)$ and $\sigma\left(1\otimes|(\Im(\mathcal{D}))|)-\frac{\sigma}{2}\right)$ are commute with each other. Therefore

$$\begin{aligned}
\Im&=(\Im(\mathcal{U})\otimes 1+1\otimes\mathcal{D}(\mathcal{D}))\sum_{\sigma_1=0}^{\sigma_2}C_{\sigma_2}^{\sigma_1}\sigma\left((|(\vartheta(\mathcal{U}))|\otimes 1)^{\sigma_1}-\frac{\sigma}{2}\right)\sigma\left((1\otimes|(\Im(\mathcal{D}))|)^{\sigma_2-\sigma_1}-\frac{\sigma}{2}\right)\\
&=(\Im(\mathcal{U})\otimes 1+1\otimes\mathcal{D}(\mathcal{D}))\left[\sigma\left((|(\vartheta(\mathcal{U}))|\otimes 1)-\frac{\sigma}{2}\right)+\sigma\left((1\otimes|(\Im(\mathcal{D}))|)-\frac{\sigma}{2}\right)\right]^{\sigma_2}
\end{aligned}$$

Lemma 3. *Let $\mathcal{U}$ and $\mathcal{D}$ be adjoint operators, with their spectra lying within the sets Δ_1 and Δ_2, respectively. Assume that the functions $\Im$ and ϑ are continuous on Δ_1, while $\mathcal{D}$ and $\Im$ are continuous on Δ_2, and φ is convex on Δ. Then, the product of the intervals $\vartheta(\Delta_1)+\Im(\Delta_2)$ satisfies the following equality:*

$$\varphi(\Im(\mathcal{U})\otimes\mathcal{D}(\mathcal{D}))\chi(\vartheta(\mathcal{U})\otimes\Im(\mathcal{D}))=\int_{\Delta_1}\int_{\Delta_2}\varphi(\Im(\nu_2)\mathcal{D}(\nu_1))\chi(\vartheta(\nu_2)\Im(\nu_1))\mathtt{dE}_{\nu_1}\otimes\mathtt{dF}_{\nu_2} \tag{2.5}$$

where $\mathcal{U}$ and $\mathcal{D}$ have the spectral resolutions

$$\mathcal{U}=\int_{\Delta_1}\nu_2\mathtt{dE}(\nu_2)\ \textit{and}\ \mathcal{D}=\int_{\Delta_2}\nu_1\mathtt{dF}(\nu_1).$$

Proof. *According to Weierstrass, any real-valued differentiable mapping can be represented in terms of polynomial sequence, hence simply checking its equivalence is adequate. Let two non-negative mappings $\varphi(\mu)=e^{\mu\sigma_1},\chi(\mu)=e^{\mu\sigma_2}$ with σ_1 and σ_2 for each natural numbers, one has*

$$\begin{aligned}
&\int_{\Delta_1}\int_{\Delta_2}(e^{\nu_1}e^{\nu_2})^{\sigma_2}(e^{\nu_1}e^{\nu_2})^{\sigma_1}\mathtt{dE}_{\nu_1}\otimes\mathtt{dF}_{\nu_2}=\int_{\Delta_1}\int_{\Delta_2}[e^{\nu_1}]^{\sigma_2}[e^{\nu_2}]^{\sigma_2}[e^{\nu_1}]^{\sigma_1}[e^{\nu_2}]^{\sigma_1}\mathtt{dE}_{\nu_1}\otimes\mathtt{dF}_{\nu_2}\\
&=\int_{\Delta_1}\int_{\Delta_2}[e^{\nu_1}]^{\sigma_2}[e^{\nu_1}]^{\sigma_1}[e^{\nu_2}]^{\sigma_2}[e^{\nu_2}]^{\sigma_1}\mathtt{dE}_{\nu_1}\otimes\mathtt{dF}_{\nu_2}=\left([e^{\mathcal{U}}]^{\sigma_2}[e^{\mathcal{U}}]^{\sigma_1}\right)\otimes\left([e^{\mathcal{D}}]^{\sigma_2}[e^{\mathcal{D}}]^{\sigma_1}\right)\\
&=\left([e^{\mathcal{U}}]^{\sigma_2}\otimes[e^{\mathcal{D}}]^{\sigma_2}\right)\left([e^{\mathcal{U}}]^{\sigma_1}\otimes[e^{\mathcal{D}}]^{\sigma_1}\right)=(e^{\mathcal{U}}\otimes e^{\mathcal{D}})^{\sigma_2}(e^{\mathcal{U}}\otimes e^{\mathcal{D}})^{\sigma_1}
\end{aligned}$$

and the equality (2.5) *is proven.*

3. The main results

First, we recall the following fractional operators that play a key role in our main findings:

Definition 5 (See [22]). *Let $\Im : [\nu_1, \nu_2] \to \mathsf{R}$ be a continuous mapping on $[\nu_1, \nu_2]$. For $\mathcal{S} > 0$ the associated integrals are defined as:*

$$\mathsf{J}^{\mathcal{S}}_{\nu_1+,k}\Im(\wp) = \frac{1}{\Gamma(\mathcal{S})}\int_{\nu_1}^{\wp}(\wp-\varepsilon)^{\frac{\mathcal{S}-k}{k}}\Im(\varepsilon)\mathtt{d}\varepsilon$$

for $\nu_1 < \wp \leqslant \nu_2$ and

$$\mathsf{J}^{\mathcal{S}}_{\nu_2-,k}\Im(\wp) = \frac{1}{\Gamma(\mathcal{S})}\int_{\wp}^{\nu_2}(\varepsilon-\wp)^{\frac{\mathcal{S}-k}{k}}\Im(\varepsilon)\mathtt{d}\varepsilon$$

for $\nu_1 \leqslant \wp < \nu_2$, where Γ is the gamma function.

We now have defined thek-Riemann-Liouville fractional integral operators and their corresponding generic identities.

Definition 6 (See [22]). *Let $\Im : [\nu_1, \nu_2] \to \mathsf{R}$ be a real-valued mapping on $[\nu_1, \nu_2]$. For $k, \mathcal{S} > 0$ the associated k-Riemann-Liouville integrals are represented as:*

$$\mathsf{J}^{\mathcal{S}}_{\nu_1+,k}\Im(\wp) = \frac{1}{k\Gamma_k(\mathcal{S})}\int_{\nu_1}^{\wp}(\wp-\varepsilon)^{\frac{\mathcal{S}-k}{k}}\Im(\varepsilon)\mathtt{d}\varepsilon$$

for $\nu_1 < \wp \leqslant \nu_2$ and

$$\mathsf{J}^{\mathcal{S}}_{\nu_2-,k}\Im(\wp) = \frac{1}{k\Gamma_k(\mathcal{S})}\int_{\wp}^{\nu_2}(\varepsilon-\wp)^{\frac{\mathcal{S}-k}{k}}\Im(\varepsilon)\mathtt{d}\varepsilon$$

for $\nu_1 \leqslant \wp < \nu_2$, where Γ_k is the k-Gamma function.

Lemma 4. *Let $\Im : [\nu_1, \nu_2] \to \mathsf{R}$ be a real-valued mapping on $[\nu_1, \nu_2]$. For any $\wp \in (\nu_1, \nu_2)$ we have*

$$\begin{aligned}
&\mathcal{J}^{\mathcal{S}}_{\nu_1+,k}\Im(\wp) + \mathsf{J}^{\mathcal{S}}_{\nu_2-,k}\Im(\wp)\\
&= \frac{1}{k\Gamma_k(\mathcal{S}+k)}\left[(\wp-\nu_1)^{\frac{\mathcal{S}}{k}}\Im(\nu_1) + (\nu_2-\wp)^{\frac{\mathcal{S}}{k}}\Im(\nu_2)\right]\\
&\quad + \frac{1}{k\Gamma_k(\mathcal{S}+k)}\left[\int_{\nu_1}^{\wp}(\wp-\varepsilon)^{\frac{\mathcal{S}}{k}}\Im'(\varepsilon)\mathtt{d}\varepsilon - \int_{\wp}^{\nu_2}(\varepsilon-\wp)^{\frac{\mathcal{S}}{k}}\Im'(\varepsilon)\mathtt{d}\varepsilon\right].
\end{aligned} \tag{3.1}$$

Proof. Let $\Im : [\nu_1, \nu_2] \to \mathbb{R}$ be a continuous function. In this case, the symmetry of the integrals is given by:

$$\int_{\nu_1}^{\wp}(\wp-\varepsilon)^{\frac{\mathcal{S}}{k}}\Im'(\varepsilon)\mathtt{d}\varepsilon \text{ and } \int_{\wp}^{\nu_2}(\varepsilon-\wp)^{\frac{\mathcal{S}}{k}}\Im'(\varepsilon)\mathtt{d}\varepsilon,$$

applying integration this follows

$$\begin{aligned}&\frac{1}{k\Gamma_k(\mathcal{S}+k)}\int_{\nu_1}^{\wp}(\wp-\varepsilon)^{\frac{\mathcal{S}}{k}}\Im'(\varepsilon)\mathtt{d}\varepsilon\\&=\frac{1}{\Gamma(\mathcal{S})}\int_{\nu_1}^{\wp}(\wp-\varepsilon)^{\frac{\mathcal{S}-k}{k}}\Im(\varepsilon)\mathtt{d}\varepsilon-\frac{1}{k\Gamma_k(\mathcal{S}+k)}(\wp-\nu_1)^{\frac{\mathcal{S}}{k}}\Im(\nu_1)\\&=\mathtt{J}^{\mathcal{S}}_{\nu_1+,k}\Im(\wp)-\frac{1}{k\Gamma_k(\mathcal{S}+k)}(\wp-\nu_1)^{\frac{\mathcal{S}}{k}}\Im(\nu_1)\end{aligned}\tag{3.2}$$

for $\nu_1<\wp\leqslant\nu_2$ and

$$\begin{aligned}&\frac{1}{k\Gamma_k(\mathcal{S}+k)}\int_{\wp}^{\nu_2}(\varepsilon-\wp)^{\frac{\mathcal{S}}{k}}\Im'(\varepsilon)\mathtt{d}\varepsilon\\&=\frac{1}{k\Gamma_k(\mathcal{S}+k)}(\nu_2-\wp)^{\frac{\mathcal{S}}{k}}\Im(\nu_2)-\frac{1}{\Gamma(\mathcal{S})}\int_{\wp}^{\nu_2}(\varepsilon-\wp)^{\frac{\mathcal{S}-k}{k}}\Im(\varepsilon)\mathtt{d}\varepsilon\\&=\frac{1}{k\Gamma_k(\mathcal{S}+k)}(\nu_2-\wp)^{\frac{\mathcal{S}}{k}}\Im(\nu_2)-\mathtt{J}^{\mathcal{S}}_{\nu_2-,k}\Im(\wp)\end{aligned}\tag{3.3}$$

for $\nu_1\leqslant\wp<\nu_2$. From (3.2), one has

$$\mathtt{J}^{\mathcal{S}}_{\nu_1+,k}\Im(\wp)=\frac{1}{k\Gamma_k(\mathcal{S}+k)}(\wp-\nu_1)^{\frac{\mathcal{S}}{k}}\Im(\nu_1)+\frac{1}{k\Gamma_k(\mathcal{S}+k)}\int_{\nu_1}^{\wp}(\wp-\varepsilon)^{\frac{\mathcal{S}}{k}}\Im'(\varepsilon)\mathtt{d}\varepsilon$$

for $\nu_1<\wp\leqslant\nu_2$ and from (3.3), one has

$$\mathtt{J}^{\mathcal{S}}_{\nu_2-,k}\Im(\wp)=\frac{1}{k\Gamma_k(\mathcal{S}+k)}(\nu_2-\wp)^{\frac{\mathcal{S}}{k}}\Im(\nu_2)-\frac{1}{k\Gamma_k(\mathcal{S}+k)}\int_{\wp}^{\nu_2}(\varepsilon-\wp)^{\frac{\mathcal{S}}{k}}\Im'(\varepsilon)\mathtt{d}\varepsilon.$$

Now, for any $\wp\in(\nu_1,\nu_2)$ we have

$$\begin{aligned}&\mathcal{J}^{\mathcal{S}}_{\wp-,k}\Im(\nu_1)+\mathtt{J}^{\mathcal{S}}_{\wp+,k}\Im(\nu_2)\\&=\frac{1}{k\Gamma_k(\mathcal{S}+k)}\left[(\wp-\nu_1)^{\frac{\mathcal{S}}{k}}+(\nu_2-\wp)^{\frac{\mathcal{S}}{k}}\right]\Im(\wp)\\&\quad+\frac{1}{k\Gamma_k(\mathcal{S}+k)}\left[\int_{\wp}^{\nu_2}(\nu_2-\varepsilon)^{\frac{\mathcal{S}}{k}}\Im'(\varepsilon)\mathtt{d}\varepsilon-\int_{\nu_1}^{\wp}(\varepsilon-\nu_1)^{\frac{\mathcal{S}}{k}}\Im'(\varepsilon)\mathtt{d}\varepsilon\right].\end{aligned}$$

Proof. Since we have

$$\mathtt{J}^{\mathcal{S}}_{\wp+,k}\Im(\nu_2)=\frac{1}{k\Gamma(\mathcal{S})}\int_{\wp}^{\nu_2}(\nu_2-\varepsilon)^{\frac{\mathcal{S}-k}{k}}\Im(\varepsilon)\mathtt{d}\varepsilon$$

for $\nu_1\leqslant\wp<\nu_2$ and

$$\mathtt{J}^{\mathcal{S}}_{\wp-,k}\Im(\nu_1)=\frac{1}{\Gamma(\mathcal{S})}\int_{\nu_1}^{\wp}(\varepsilon-\nu_1)^{\frac{\mathcal{S}-k}{k}}\Im(\varepsilon)\mathtt{d}\varepsilon$$

for $\nu_1 < \wp \leqslant \nu_2$. Since $\Im : [\nu_1, \nu_2] \to \mathsf{R}$ be an continuous function $[\nu_1, \nu_2]$, then the integrals

$$\int_{\nu_1}^{\wp} (\varepsilon - \nu_1)^{\mathcal{S}} \Im'(\varepsilon) \mathtt{d}\varepsilon \text{ and } \int_{\wp}^{\nu_2} (\nu_2 - \varepsilon)^{\mathcal{S}} \Im'(\varepsilon) \mathtt{d}\varepsilon,$$

holds, and by performing the integration, we obtain:

$$\begin{aligned}
&\frac{1}{k\Gamma_k(\mathcal{S}+k)} \int_{\nu_1}^{\wp} (\varepsilon - \nu_1)^{\mathcal{S}} \Im'(\varepsilon) \mathtt{d}\varepsilon \\
&= \frac{1}{k\Gamma_k(\mathcal{S}+k)} (\wp - \nu_1)^{\frac{\mathcal{S}}{k}} \Im(\wp) - \frac{1}{k\Gamma(\mathcal{S})} \int_{\nu_1}^{\wp} (\varepsilon - \nu_1)^{\frac{\mathcal{S}-k}{k}} \Im(\varepsilon) \mathtt{d}\varepsilon \\
&= \frac{1}{k\Gamma_k(\mathcal{S}+k)} (\wp - \nu_1)^{\frac{\mathcal{S}}{k}} \Im(\wp) - \mathtt{J}^{\mathcal{S}}_{\wp-,k} \Im(\nu_1)
\end{aligned} \tag{3.4}$$

for $\nu_1 < \wp \leqslant \nu_2$ and

$$\begin{aligned}
&\frac{1}{k\Gamma_k(\mathcal{S}+k)} \int_{\wp}^{\nu_2} (\nu_2 - \varepsilon)^{\frac{\mathcal{S}}{k}} \Im'(\varepsilon) \mathtt{d}\varepsilon \\
&= \frac{1}{k\Gamma(\mathcal{S})} \int_{\wp}^{\nu_2} (\nu_2 - \varepsilon)^{\frac{\mathcal{S}-k}{k}} \Im(\varepsilon) \mathtt{d}\varepsilon - \frac{1}{k\Gamma_k(\mathcal{S}+k)} (\nu_2 - \wp)^{\frac{\mathcal{S}}{k}} \Im(\wp) \\
&= \mathtt{J}^{\mathcal{S}}_{\wp+,k} \Im(\nu_2) - \frac{1}{k\Gamma_k(\mathcal{S}+k)} (\nu_2 - \wp)^{\frac{\mathcal{S}}{k}} \Im(\wp)
\end{aligned} \tag{3.5}$$

for $\nu_1 \leqslant \wp < \nu_2$. From (3.4) we have

$$\mathtt{J}^{\mathcal{S}}_{\wp-,k} \Im(\nu_1) = \frac{1}{k\Gamma_k(\mathcal{S}+k)} (\wp - \nu_1)^{\frac{\mathcal{S}}{k}} \Im(\wp) - \frac{1}{k\Gamma_k(\mathcal{S}+k)} \int_{\nu_1}^{\wp} (\varepsilon - \nu_1)^{\frac{\mathcal{S}}{k}} \Im'(\varepsilon) \mathtt{d}\varepsilon$$

for $\nu_1 < \wp \leqslant \nu_2$ and from (3.5)

$$\mathtt{J}^{\mathcal{S}}_{\wp+,k} \Im(\nu_2) = \frac{1}{k\Gamma_k(\mathcal{S}+k)} (\nu_2 - \wp)^{\frac{\mathcal{S}}{k}} \Im(\wp) + \frac{1}{k\Gamma_k(\mathcal{S}+k)} \int_{\wp}^{\nu_2} (\nu_2 - \varepsilon)^{\frac{\mathcal{S}}{k}} \Im'(\varepsilon) \mathtt{d}\varepsilon.$$

3.1. Some new fractional identities

We draw inspiration from section 2.1 in this section, which enables us to construct fractional identities at the interval's midpoint, which we utilize in important findings.

Lemma 5. *Let $\Im : [\nu_1, \nu_2] \to \mathsf{R}$ be a real-valued mapping on $[\nu_1, \nu_2]$. We have the following double equality*

$$\begin{aligned}
&\mathcal{J}^{\mathcal{S}}_{\nu_1+,k} \Im\left(\frac{\nu_1+\nu_2}{2}\right) + \mathtt{J}^{\mathcal{S}}_{\nu_2-,k} \Im\left(\frac{\nu_1+\nu_2}{2}\right) \\
&= \frac{1}{2^{\frac{\mathcal{S}-k}{k}} k\Gamma_k(\mathcal{S}+k)} \frac{\mathcal{Q}(\nu_1) + \Im(\nu_2)}{2} \\
&+ \frac{1}{k\Gamma_k(\mathcal{S}+k)} \left[\int_{\nu_1}^{\frac{\nu_1+\nu_2}{2}} \left(\frac{\nu_1+\nu_2}{2} - \varepsilon\right)^{\frac{\mathcal{S}}{k}} \Im'(\varepsilon) \mathtt{d}\varepsilon - \int_{\frac{\nu_1+\nu_2}{2}}^{\nu_2} \left(\varepsilon - \frac{\nu_1+\nu_2}{2}\right)^{\frac{\mathcal{S}}{k}} \Im'(\varepsilon) \mathtt{d}\varepsilon\right]
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{J}^{\mathcal{S}}_{\frac{\nu_1+\nu_2}{2}-,k}\Im(\nu_1) + \mathtt{J}^{\mathcal{S}}_{\frac{\nu_1+\nu_2}{2}+,k}\Im(\nu_2) \\
&= \frac{1}{2^{\frac{\mathcal{S}-k}{k}} k\Gamma_k(\mathcal{S}+k)}\Im\left(\frac{\nu_1+\nu_2}{2}\right)(\nu_2-\nu_1)^{\frac{\mathcal{S}}{k}} \\
&+ \frac{1}{k\Gamma_k(\mathcal{S}+k)}\left[\int_{\frac{\nu_1+\nu_2}{2}}^{\nu_2}(\varepsilon-\nu_2)^{\frac{\mathcal{S}}{k}}\Im'(\varepsilon)\mathtt{d}\varepsilon - \int_{\nu_1}^{\frac{\nu_1+\nu_2}{2}}(\varepsilon-\nu_1)^{\frac{\mathcal{S}}{k}}\Im'(\varepsilon)\mathtt{d}\varepsilon\right], \qquad (3.6)
\end{aligned}$$

for $\nu_1 \leqslant \frac{\nu_1+\nu_2}{2} < \nu_2$*. From (3.6) we have*

$$\begin{aligned}
&\mathcal{J}^{\mathcal{S}}_{\frac{\nu_1+\nu_2}{2}-,k}\Im(\nu_1) = \frac{1}{2^{\frac{\mathcal{S}-k}{k}} k\Gamma_k(\mathcal{S}+k)}\Im\left(\frac{\nu_1+\nu_2}{2}\right)(\nu_2-\nu_1)^{\frac{\mathcal{S}}{k}} \\
&- \frac{1}{k\Gamma_k(\mathcal{S}+k)}\left[\int_{\nu_1}^{\frac{\nu_1+\nu_2}{2}}(\varepsilon-\nu_1)^{\frac{\mathcal{S}}{k}}\Im'(\varepsilon)\mathtt{d}\varepsilon\right] = \frac{1}{2^{\frac{\mathcal{S}-k}{k}} k\Gamma_k(\mathcal{S}+k)}\Im\left(\frac{\nu_1+\nu_2}{2}\right)(\nu_2-\nu_1)^{\frac{\mathcal{S}}{k}} \\
&- \frac{\mathcal{W}^{\frac{\mathcal{S}}{k}}(\nu_2-\nu_1)^{\frac{\mathcal{S}+k}{k}}}{2^{\frac{\mathcal{S}+k}{k}} k\Gamma_k(\mathcal{S}+k)}\left[\int_0^1 \Im'\left((1-\mathcal{W})\nu_1 + \left(\frac{\nu_1+\nu_2}{2}\right)\mathcal{W}\right)\mathtt{d}\mathcal{W}\right], \qquad (3.7)
\end{aligned}$$

for $\nu_1 < \frac{\nu_1+\nu_2}{2} \leqslant \nu_2$ *and from (3.6) we have*

$$\begin{aligned}
&\mathcal{J}^{\mathcal{S}}_{\frac{\nu_1+\nu_2}{2}+,k}\Im(\nu_2) = \frac{1}{2^{\frac{\mathcal{S}-k}{k}} k\Gamma_k(\mathcal{S}+k)}\Im\left(\frac{\nu_1+\nu_2}{2}\right)(\nu_2-\nu_1)^{\frac{\mathcal{S}}{k}} \\
&+ \frac{1}{k\Gamma_k(\mathcal{S}+k)}\left[\int_{\frac{\nu_1+\nu_2}{2}}^{\nu_2}(\nu_2-\varepsilon)^{\mathcal{S}}\Im'(\varepsilon)\mathtt{d}\varepsilon\right] = \frac{1}{2^{\frac{\mathcal{S}-k}{k}} k\Gamma_k(\mathcal{S}+k)}\Im\left(\frac{\nu_1+\nu_2}{2}\right)(\nu_2-\nu_1)^{\frac{\mathcal{S}}{k}} \\
&- \frac{(1-\mathcal{W})^{\frac{\mathcal{S}}{k}}(\nu_2-\nu_1)^{\frac{\mathcal{S}+k}{k}}}{2^{\frac{\mathcal{S}+k}{k}} k\Gamma_k(\mathcal{S}+k)}\left[\int_0^1 \Im'\left((1-\mathcal{W})\left(\frac{\nu_1+\nu_2}{2}\right) + \nu_2\mathcal{W}\right)\mathtt{d}\mathcal{W}\right]. \qquad (3.8)
\end{aligned}$$

Lemma 6. *Assume* $\Im$ *is a convex mapping on* Δ*, and* $\mathcal{U}$*,* $\mathcal{D}$ *are adjoint operators whose spectra* $\mathcal{SP}(\mathcal{U}), \mathcal{SP}(\mathcal{D}) \subset \Delta$*, then*

$$\begin{aligned}
&\left[\frac{1}{6}(\Im(\mathcal{U})\otimes 1) + \frac{2}{3}\Im\left(\frac{\mathcal{U}\otimes 1 + 1\otimes\mathcal{D}}{2}\right) + \frac{1}{6}(1\otimes\Im(\mathcal{D}))\right] \\
&- \left[\Im\left(\frac{\mathcal{U}\otimes 1 + 1\otimes\mathcal{D}}{2}\right) - \frac{\mathcal{W}^{\frac{\mathcal{S}}{k}}(\nu_2-\nu_1)}{4}\left[\int_0^1 \Im'\left((1-\mathcal{W})\mathcal{U}\otimes 1 + \left(\frac{\mathcal{W}1\otimes\mathcal{D}}{2}\right)\right)\mathtt{d}\mathcal{W}\right]\right. \\
&+ \Im\left(\frac{\mathcal{U}\otimes 1 + 1\otimes\mathcal{D}}{2}\right) - \frac{(1-\mathcal{W})^{\frac{\mathcal{S}}{k}}(\nu_2-\nu_1)}{4}\left[\int_0^1 \Im'\left(\left(\frac{1-\mathcal{W}}{2}\right)\mathcal{U}\otimes 1 + \left(\frac{1+\mathcal{W}}{2}\right)1\otimes\mathcal{D}\right)\mathtt{d}\mathcal{W}\right]
\end{aligned}$$

$$=\frac{(1\otimes\mathcal{D}-\mathcal{U}\otimes 1)^2}{6}\left[\int_0^{\frac{1}{2}}\left(\mathcal{W}-\frac{3k.2^{\frac{\mathcal{S}}{k}}\mathcal{W}^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\left[\Im''\left(\mathcal{U}\otimes 1\mathcal{W}+1\otimes\mathcal{U}\left(1-\mathcal{W}\right)\right)\mathtt{d}\mathcal{W}\right]\right.$$
$$\left.+\int_{\frac{1}{2}}^1\left((1-\mathcal{W})-\frac{3k.2^{\frac{\mathcal{S}}{k}}\left(1-\mathcal{W}\right)^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\Im''\left(\mathcal{U}\otimes 1\mathcal{W}+1\otimes\mathcal{U}\left(1-\mathcal{W}\right)\right)\mathtt{d}\mathcal{W}\right]. \tag{3.9}$$

Proof. Considering the following result [13]. Let mapping $\Im:[\nu_1,\nu_2]\to\mathsf{R}$ be defined over interval (ν_1,ν_2) such that $\Im''\in\mathcal{L}([\nu_1,\nu_2])$. Then, we have

$$\frac{1}{6}\left[\Im(\nu_1)+4\Im\left(\frac{\nu_1+\nu_2}{2}\right)+\Im(\nu_2)\right]-\frac{2^{\frac{\mathcal{S}-k}{k}}\Gamma(\mathcal{S}+k)}{(\nu_2-\nu_1)^{\frac{\mathcal{S}}{k}}}\left[\mathtt{J}^{\mathcal{S}}_{\frac{\nu_1+\nu_2}{2}-,k}\Im(\nu_1)+\mathtt{J}^{\mathcal{S}}_{\frac{\nu_1+\nu_2}{2}+,k}\Im(\nu_2)\right]$$
$$=\frac{(\nu_2-\nu_1)^2}{6}\left[\int_0^{\frac{1}{2}}\left(\mathcal{W}-\frac{3k.2^{\frac{\mathcal{S}}{k}}\mathcal{W}^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\left[\Im''\left(\nu_2\mathcal{W}+\left(1-\mathcal{W}\right)\nu_2\right)\right]\mathtt{d}\mathcal{W}\right.$$
$$\left.+\int_{\frac{1}{2}}^1\left((1-\mathcal{W})-\frac{3k.2^{\frac{\mathcal{S}}{k}}\left(1-\mathcal{W}\right)^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\left[\Im''\left(\nu_2\mathcal{W}+\left(1-\mathcal{W}\right)\nu_2\right)\right]\mathtt{d}\mathcal{W}\right]. \tag{3.10}$$

By applying substitution from equations (3.7) and (3.8), we obtain:

$$\frac{1}{6}\left[\Im(\nu_1)+4\Im\left(\frac{\nu_1+\nu_2}{2}\right)+\Im(\nu_2)\right]-\frac{2^{\frac{\mathcal{S}-k}{k}}k\Gamma_k(\mathcal{S}+k)}{(\nu_2-\nu_1)^{\frac{\mathcal{S}}{k}}}\left[\frac{\Im\left(\frac{\nu_1+\nu_2}{2}\right)(\nu_2-\nu_1)^{\frac{\mathcal{S}}{k}}}{2^{\frac{\mathcal{S}-k}{k}}k\Gamma_k(\mathcal{S}+k)}\right.$$
$$-\frac{\mathcal{W}^{\frac{\mathcal{S}}{k}}(\nu_2-\nu_1)^{\frac{\mathcal{S}+k}{k}}}{2^{\frac{\mathcal{S}+k}{k}}k\Gamma_k(\mathcal{S}+k)}\left[\int_0^1\Im'\left((1-\mathcal{W})\nu_1+\left(\frac{\nu_1+\nu_2}{2}\right)\mathcal{W}\right)\mathtt{d}\mathcal{W}\right]$$
$$+\frac{\Im\left(\frac{\nu_1+\nu_2}{2}\right)(\nu_2-\nu_1)^{\frac{\mathcal{S}}{k}}}{2^{\frac{\mathcal{S}-k}{k}}k\Gamma_k(\mathcal{S}+k)}$$
$$\left.-\frac{(1-\mathcal{W})^{\frac{\mathcal{S}}{k}}(\nu_2-\nu_1)^{\frac{\mathcal{S}+k}{k}}}{2^{\frac{\mathcal{S}+k}{k}}k\Gamma_k(\mathcal{S}+k)}\left[\int_0^1\Im'\left((1-\mathcal{W})\left(\frac{\nu_1+\nu_2}{2}\right)+\nu_2\mathcal{W}\right)\mathtt{d}\mathcal{W}\right]\right]$$
$$=\frac{(\nu_2-\nu_1)^2}{6}\left[\int_0^{\frac{1}{2}}\left(\mathcal{W}-\frac{3k\cdot 2^{\frac{\mathcal{S}}{k}}\mathcal{W}^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\left[\Im''\left(\nu_2\mathcal{W}+(1-\mathcal{W})\nu_2\right)\right]\mathtt{d}\mathcal{W}\right.$$
$$\left.+\int_{\frac{1}{2}}^1\left((1-\mathcal{W})-\frac{3k\cdot 2^{\frac{\mathcal{S}}{k}}\left(1-\mathcal{W}\right)^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\left[\Im''\left(\nu_2\mathcal{W}+(1-\mathcal{W})\nu_2\right)\right]\mathtt{d}\mathcal{W}\right]. \tag{3.11}$$

By making several simplifications, we may have

$$\frac{1}{6}\left[\Im(\nu_1)+4\Im\left(\frac{\nu_1+\nu_2}{2}\right)+\Im(\nu_2)\right]$$
$$-\left[\Im\left(\frac{\nu_1+\nu_2}{2}\right)-\frac{\mathcal{W}^{\frac{\mathcal{S}}{k}}(\nu_2-\nu_1)}{4}\left[\int_0^1\Im'\left((1-\mathcal{W})\,\nu_1+\left(\frac{\nu_2}{2}\right)\mathcal{W}\right)\mathtt{d}\mathcal{W}\right]\right.$$

$$+\Im\left(\frac{\nu_1+\nu_2}{2}\right)-\frac{(1-\mathcal{W})^{\frac{\mathcal{S}}{k}}(\nu_2-\nu_1)}{4}\left[\int_0^1 \Im'\left(\left(\left(\frac{1-\mathcal{W}}{2}\right)\nu_1+\left(\left(\frac{1+\mathcal{W}}{2}\right)\nu_2\right)\mathtt{d}\mathcal{W}\right]\right]$$

$$=\frac{(\nu_2-\nu_1)^2}{6}\left[\int_0^{\frac{1}{2}}\left(\mathcal{W}-\frac{3k.2^{\frac{\mathcal{S}}{k}}\mathcal{W}^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\left[\Im''\left(\nu_2\mathcal{W}+(1-\mathcal{W})\,\nu_2\right)\right]\mathtt{d}\mathcal{W}\right.$$
$$\left.+\int_{\frac{1}{2}}^1\left((1-\mathcal{W})-\frac{3k.2^{\frac{\mathcal{S}}{k}}(1-\mathcal{W})^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\left[\Im''\left(\nu_2\mathcal{W}+(1-\mathcal{W})\,\nu_2\right)\right]\mathtt{d}\mathcal{W}\right]. \tag{3.12}$$

Assume that the spectral resolutions of $\mathcal{U}$ and $\mathcal{D}$

$$\mathcal{U}=\int_\Delta \nu_2\mathtt{dE}(\nu_2) \text{ and } \mathcal{D}=\int_\Delta \nu_1\mathtt{dF}(\nu_1).$$

$\int_\Delta\int_\Delta$ over $\mathtt{dE}_{\nu_1}\otimes\mathtt{dF}_{\nu_2}$ in (3.12), then we get

$$\int_\Delta\int_\Delta\frac{1}{6}\left[\Im(\nu_1)+4\Im\left(\frac{\nu_1+\nu_2}{2}\right)+\Im(\nu_2)\right]\mathtt{dE}_{\nu_1}\otimes\mathtt{dF}_{\nu_2}$$
$$-\left[\int_\Delta\int_\Delta\left(\Im\left(\frac{\nu_1+\nu_2}{2}\right)-\frac{\mathcal{W}^{\frac{\mathcal{S}}{k}}(\nu_2-\nu_1)}{4}\left[\int_0^1\Im'\left((1-\mathcal{W})\,\nu_1+\left(\frac{\nu_2}{2}\right)\mathcal{W}\right)\mathtt{d}\mathcal{W}\right)\right]\mathtt{dE}_{\nu_1}\otimes\mathtt{dF}_{\nu_2}$$
$$+\int_\Delta\int_\Delta\left(\Im\left(\frac{\nu_1+\nu_2}{2}\right)-\frac{(1-\mathcal{W})^{\frac{\mathcal{S}}{k}}(\nu_2-\nu_1)}{4}\right.$$
$$\times\left[\int_0^1\Im'\left(\left(\left(\frac{1-\mathcal{W}}{2}\right)\nu_1+\left(\left(\frac{1+\mathcal{W}}{2}\right)\nu_2\right)\mathtt{d}\mathcal{W}\right]\mathtt{dE}_{\nu_1}\otimes\mathtt{dF}_{\nu_2}\right]$$
$$=\frac{(\nu_2-\nu_1)^2}{6}\int_\Delta\int_\Delta\left[\int_0^{\frac{1}{2}}\left(\mathcal{W}-\frac{3k.2^{\frac{\mathcal{S}}{k}}\mathcal{W}^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\left[\Im''\left(\nu_2\mathcal{W}+(1-\mathcal{W})\,\nu_2\right)\right]\mathtt{d}\mathcal{W}\right.$$
$$\left.+\int_{\frac{1}{2}}^1\left((1-\mathcal{W})-\frac{3k.2^{\frac{\mathcal{S}}{k}}(1-\mathcal{W})^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\left[\Im''\left(\nu_2\mathcal{W}+(1-\mathcal{W})\,\nu_2\right)\right]\mathtt{d}\mathcal{W}\right]\mathtt{dE}_{\nu_1}\otimes\mathtt{dF}_{\nu_2}. \tag{3.13}$$

Taking into account Lemma 2 and Fubini's theorem, we obtain:

$$\int_\Delta\int_\Delta\Im(\nu_2)\mathtt{dE}_{\nu_1}\otimes\mathtt{dF}_{\nu_2}=(\Im(\mathcal{U})\otimes 1),$$
$$\int_\Delta\int_\Delta\Im\left(\frac{\nu_1+\nu_2}{2}\right)\mathtt{dE}_{\nu_1}\otimes\mathtt{dF}_{\nu_2}=\Im\left(\frac{\mathcal{U}\otimes 1+1\otimes\mathcal{D}}{2}\right),$$
$$\int_\Delta\int_\Delta\Im(\nu_1)\mathtt{dE}_{\nu_1}\otimes\mathtt{dF}_{\nu_2}=(1\otimes\Im(\mathcal{D})),$$
$$\int_\Delta\int_\Delta\int_0^1\Im'\left((1-\mathcal{W})\,\nu_1+\left(\frac{\nu_2}{2}\right)\mathcal{W}\right)\mathtt{d}\mathcal{W}\mathtt{dE}_{\nu_1}\otimes\mathtt{dF}_{\nu_2}$$

$$= \int_0^1 \int_\Delta \int_\Delta \Im' \left((1-\mathcal{W})\,\nu_1 + \left(\frac{\nu_2}{2}\right)\mathcal{W} \right) \mathtt{dE}_{\nu_1} \otimes \mathtt{dF}_{\nu_2} \mathtt{d}\mathcal{W}$$
$$= \int_0^1 \Im' \left((1-\mathcal{W})\,\mathcal{U} \otimes 1 + \left(\frac{\mathcal{W}1 \otimes \mathcal{D}}{2}\right) \right) \mathtt{d}\mathcal{W},$$
$$\int_\Delta \int_\Delta \int_0^1 \Im' \left(\left(\frac{1-\mathcal{W}}{2}\right)\nu_1 + \left(\frac{1+\mathcal{W}}{2}\right)\nu_2 \right) \mathtt{d}\mathcal{W}\mathtt{dE}_{\nu_1} \otimes \mathtt{dF}_{\nu_2}$$
$$= \int_0^1 \int_\Delta \int_\Delta \Im' \left(\left(\frac{1-\mathcal{W}}{2}\right)\nu_1 + \left(\frac{1+\mathcal{W}}{2}\right)\nu_2 \right) \mathtt{dE}_{\nu_1} \otimes \mathtt{dF}_{\nu_2} \mathtt{d}\mathcal{W}$$
$$= \int_0^1 \int_\Delta \int_\Delta \Im' \left(\left(\frac{1-\mathcal{W}}{2}\right)\mathcal{U} \otimes 1 + \left(\frac{1+\mathcal{W}}{2}\right) 1 \otimes \mathcal{D} \right) \mathtt{d}\mathcal{W},$$
$$\Im'' \left(\nu_1 \mathcal{W} + \nu_1 (1-\mathcal{W}) \right) \mathtt{d}\mathcal{W}\mathtt{dE}_{\nu_1} \otimes \mathtt{dF}_{\nu_2} = \Im'' \left(\mathcal{U} \otimes 1\mathcal{W} + 1 \otimes \mathcal{U}\,(1-\mathcal{W}) \right) \mathtt{d}\mathcal{W}. \tag{3.14}$$

A same technique has been taking into consideration we have

$$\int_\Delta \int_\Delta \frac{(\nu_2-\nu_1)^2}{6} \Bigg[\int_0^{\frac{1}{2}} \left(\mathcal{W} - \frac{3k.2^{\frac{\mathcal{S}}{k}} \mathcal{W}^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k} \right) \left[\Im'' \left(\nu_2 \mathcal{W} + (1-\mathcal{W})\,\nu_2 \right) \right] \mathtt{d}\mathcal{W}$$
$$+ \int_{\frac{1}{2}}^1 \left((1-\mathcal{W}) - \frac{3k.2^{\frac{\mathcal{S}}{k}} (1-\mathcal{W})^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k} \right) \left[\Im'' \left(\nu_2 \mathcal{W} + (1-\mathcal{W})\,\nu_2 \right) \right] \mathtt{d}\mathcal{W} \Bigg] \mathtt{dE}_{\nu_1} \otimes \mathtt{dF}_{\nu_2}$$
$$= \frac{(1 \otimes \mathcal{D} - \mathcal{U} \otimes 1)^2}{6} \Bigg[\int_0^{\frac{1}{2}} \left(\mathcal{W} - \frac{3k.2^{\frac{\mathcal{S}}{k}} \mathcal{W}^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k} \right) \left[\Im'' \left(\mathcal{U} \otimes 1\mathcal{W} + 1 \otimes \mathcal{U}\,(1-\mathcal{W}) \right) \mathtt{d}\mathcal{W} \right]$$
$$+ \int_{\frac{1}{2}}^1 \left((1-\mathcal{W}) - \frac{3k.2^{\frac{\mathcal{S}}{k}} (1-\mathcal{W})^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k} \right) \Im'' \left(\mathcal{U} \otimes 1\mathcal{W} + 1 \otimes \mathcal{U}\,(1-\mathcal{W}) \right) \mathtt{d}\mathcal{W} \Bigg]. \tag{3.15}$$

The desired result is obtained by incorporating (3.14) and (3.15) into (3.13).

Remark 1.
- *Choosing $\mathcal{S}$ and $k=1$ in Lemma 6 leads to a refinement of Lemma 2.1, in [57].*
- *By choosing $\mathcal{S}$ and $k=1$ in Lemma 6, we obtain a refined version of Lemma 2.3, in [2].*
- *By setting $\mathcal{S}$ and $k=1$ in Lemma 6, we enhance Lemma 3, in [58].*

Theorem 6. *Assume $\Im$ is a convex mapping on Δ, and $\mathcal{U}$, $\mathcal{D}$ are adjoint operators whose spectra $\mathcal{SP}(\mathcal{U}), \mathcal{SP}(\mathcal{D}) \subset \Delta$, then*

$$\Bigg\| \left[\frac{1}{6}(\Im(\mathcal{U}) \otimes 1) + \frac{2}{3}\Im\left(\frac{\mathcal{U} \otimes 1 + 1 \otimes \mathcal{D}}{2}\right) + \frac{1}{6}(1 \otimes \Im(\mathcal{D})) \right]$$

$$-\left[\Im\left(\frac{\mathcal{U}\otimes 1+1\otimes\mathcal{D}}{2}\right)-\frac{\mathcal{W}^{\frac{\mathcal{S}}{k}}(\nu_2-\nu_1)}{4}\left[\int_0^1\Im'\left((1-\mathcal{W})\mathcal{U}\otimes 1+\left(\frac{\mathcal{W}1\otimes\mathcal{D}}{2}\right)\right)\mathtt{d}\mathcal{W}\right]\right.$$
$$\left.+\Im\left(\frac{\mathcal{U}\otimes 1+1\otimes\mathcal{D}}{2}\right)-\frac{(1-\mathcal{W})^{\frac{\mathcal{S}}{k}}(\nu_2-\nu_1)}{4}\left[\int_0^1\Im'\left(\left(\frac{1-\mathcal{W}}{2}\right)\mathcal{U}\otimes 1+\left(\frac{1+\mathcal{W}}{2}\right)1\otimes\mathcal{D}\right)\mathtt{d}\mathcal{W}\right]\right\|$$
$$\leq\frac{\|(1\otimes\mathcal{D}-\mathcal{U}\otimes 1)^2\|}{6}\left[\left(\frac{\mathcal{S}(\mathcal{S}+k)^{\frac{a+2k}{\mathcal{S}}}+3^{\frac{\mathcal{S}+2k}{a}}k^{\frac{2\mathcal{S}+2k}{\mathcal{S}}}}{4\cdot 3^{\frac{2k}{\mathcal{S}}}k^{\frac{2k}{\mathcal{S}}}(\mathcal{S}+k)(\mathcal{S}+2k)}-\frac{1}{8}\right)\left\|\left|\Im''(\mathcal{U})\right|+\left|\Im''(\mathcal{D})\right|\right\|\right].$$

Proof.
As $|\Im''|$ is convex over Δ, one has

$$\left|\Im''\left(\nu_1\mathcal{W}+\nu_1\left(1-\mathcal{W}\right)\right)\right|\leq\mathcal{W}\left|\Im''\left(\nu_1\right)\right|+\left(1-\mathcal{W}\right)\left|\Im''(\nu_1)\right|$$

Similarly, we get

$$\left|\Im''\left(\nu_2\mathcal{W}+\nu_2\left(1-\mathcal{W}\right)\right)\right|\leq\mathcal{W}\left|\Im''\left(\nu_2\right)\right|+\left(\left(1-\mathcal{W}\right)\right)\left|\Im''(\nu_2)\right|$$

for all for $\tau\in[0,1]$ and $\nu_1,\nu_2\in\Delta$.

Taking $\int_\Delta\int_\Delta$ over $\mathtt{dE}_{\nu_1}\otimes\mathtt{dF}_{\nu_2}$, then we get

$$\begin{aligned}&\left|\Im''\left(1\otimes\mathcal{U}\mathcal{W}+1\otimes\mathcal{U}\left(1-\mathcal{W}\right)\right)\right|=\int_\Delta\int_\Delta\left|\Im''\left(\nu_1\mathcal{W}+\nu_1\left(1-\mathcal{W}\right)\right)\right|\mathtt{dE}_{\nu_1}\otimes\mathtt{dF}_{\nu_2}\\&\leq\int_\Delta\int_\Delta\mathcal{W}\left|\Im''\left(\nu_1\right)\right|+\left(1-\mathcal{W}\right)\left|\Im''(\nu_1)\right|\mathtt{dE}_{\nu_1}\otimes\mathtt{dF}_{\nu_2}\\&\leq\mathcal{W}1\otimes\left|\Im''\left(\mathcal{U}\right)\right|+\left(1-\mathcal{W}\right)\left|\Im''(\mathcal{U})\right|\otimes 1.\end{aligned}\tag{3.16}$$

If we apply norm in (3.16), then we have

$$\begin{aligned}&\left\|\Im''\left(1\otimes\mathcal{U}\mathcal{W}+1\otimes\mathcal{U}\left(1-\mathcal{W}\right)\right)\right\|\\&\leq\left\|\mathcal{W}1\otimes\left|\Im''\left(\mathcal{U}\right)\right|+\left(1-\mathcal{W}\right)\left|\Im''(\mathcal{U})\right|\otimes 1\right\|\leq\mathcal{W}\left\|\Im''(\mathcal{U})\right\|+\left(1-\mathcal{W}\right)\left\|\Im''(\mathcal{U})\right\|.\end{aligned}$$

Similarly, we get

$$\begin{aligned}&\left\|\Im''\left(1\otimes\mathcal{D}\mathcal{W}+1\otimes\mathcal{D}\left(1-\mathcal{W}\right)\right)\right\|\\&\leq\left\|\mathcal{W}1\otimes\left|\Im''\left(\mathcal{D}\right)\right|+\left(\left(1-\mathcal{W}\right)\right)\left|\Im''(\mathcal{D})\right|\otimes 1\right\|\leq\mathcal{W}\left\|\Im''(\mathcal{D})\right\|+\left(\left(1-\mathcal{W}\right)\right)\left\|\Im''(\mathcal{D})\right\|.\end{aligned}$$

Using the norm in (3.12) and considering triangle inequality, we have

$$\left\|\left[\frac{1}{6}(\Im(\mathcal{U})\otimes 1)+\frac{2}{3}\Im\left(\frac{\mathcal{U}\otimes 1+1\otimes\mathcal{D}}{2}\right)+\frac{1}{6}(1\otimes\Im(\mathcal{D}))\right]\right.$$
$$-\left[\Im\left(\frac{\mathcal{U}\otimes 1+1\otimes\mathcal{D}}{2}\right)-\frac{\mathcal{W}^{\frac{\mathcal{S}}{k}}(\nu_2-\nu_1)}{4}\left[\int_0^1\Im'\left((1-\mathcal{W})\mathcal{U}\otimes 1+\left(\frac{\mathcal{W}1\otimes\mathcal{D}}{2}\right)\right)\mathtt{d}\mathcal{W}\right]\right.$$

$$
\begin{aligned}
&+\Im\left(\frac{\mathcal{U}\otimes 1+1\otimes\mathcal{D}}{2}\right)-\frac{(1-\mathcal{W})^{\frac{\mathcal{S}}{k}}(\nu_2-\nu_1)}{4}\left[\int_0^1\Im'\left(\left(\frac{1-\mathcal{W}}{2}\right)\mathcal{U}\otimes 1+\left(\frac{1+\mathcal{W}}{2}\right)1\otimes\mathcal{D}\right)\mathtt{d}\mathcal{W}\right]\Bigg\|\\
&\leq\frac{\|(1\otimes\mathcal{D}-\mathcal{U}\otimes 1)^2\|}{6}\left(\Bigg\|\int_0^{\frac{1}{2}}\left(\mathcal{W}-\frac{3k.2^{\frac{\mathcal{S}}{k}}\mathcal{W}^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\left[\Im''\left(\mathcal{U}\otimes 1\mathcal{W}+1\otimes\mathcal{U}\left(1-\mathcal{W}\right)\right)\mathtt{d}\mathcal{W}\right]\right.\\
&\left.+\int_{\frac{1}{2}}^1\left((1-\mathcal{W})-\frac{3k.2^{\frac{\mathcal{S}}{k}}(1-\mathcal{W})^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\Im''\left(\mathcal{U}\otimes 1\mathcal{W}+1\otimes\mathcal{U}\left(1-\mathcal{W}\right)\right)\Bigg\|\right)\\
&\leq\frac{\|(1\otimes\mathcal{D}-\mathcal{U}\otimes 1)^2\|}{6}\left(\Bigg\|\int_0^{\frac{1}{2}}\left(\mathcal{W}-\frac{3k.2^{\frac{\mathcal{S}}{k}}\mathcal{W}^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\left[\mathcal{W}1\otimes\left|\Im''\left(\mathcal{U}\right)\right|+(1-\mathcal{W})\left|\Im''(\mathcal{U})\right|\otimes 1\right]\mathtt{d}\mathcal{W}\right.\\
&\left.+\int_{\frac{1}{2}}^1\left((1-\mathcal{W})-\frac{3k.2^{\frac{\mathcal{S}}{k}}(1-\mathcal{W})^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\mathcal{W}1\otimes\left|\Im''\left(\mathcal{D}\right)\right|+(1-\mathcal{W})\left|\Im''(\mathcal{D})\right|\otimes 1\Bigg\|\right)\\
&\leq\frac{\|(1\otimes\mathcal{D}-\mathcal{U}\otimes 1)^2\|}{6}\left(\Bigg\|\int_0^{\frac{1}{2}}\left(\mathcal{W}^2-\frac{3.2^{\mathcal{S}}\mathcal{W}^{\mathcal{S}+2}}{\mathcal{S}+1}\right)\otimes\left|\Im''\left(\mathcal{U}\right)\right|\right.\\
&\left.+\int_{\frac{1}{2}}^1\left((\mathcal{W}-\mathcal{W}^2)-\frac{3.2^{\mathcal{S}}\mathcal{W}(1-\mathcal{W})^{\mathcal{S}+1}}{\mathcal{S}+1}\right)\otimes\left|\Im''\left(\mathcal{D}\right)\right|\Bigg\|\right)\mathtt{d}\mathcal{W}\\
&\leq\frac{\|(1\otimes\mathcal{D}-\mathcal{U}\otimes 1)^2\|}{6}\left[\left(\frac{k}{4\left(\mathcal{S}+2\,k\right)}\left(\frac{\mathcal{S}}{k}\left(\frac{\mathcal{S}+k}{3\,k}\right)^{\frac{2\,k}{\mathcal{S}}}+\frac{3\,k}{\mathcal{S}+k}\right)-\frac{1}{8}\right)\Big\|\left|\Im''\left(\mathcal{U}\right)\right|+\left|\Im''\left(\mathcal{D}\right)\right|\Big\|\right]\\
&=\frac{\|(1\otimes\mathcal{D}-\mathcal{U}\otimes 1)^2\|}{6}\left[\left(\frac{\mathcal{S}(\mathcal{S}+k)^{\frac{a+2k}{\mathcal{S}}}+3^{\frac{\mathcal{S}+2k}{a}}k^{\frac{2\mathcal{S}+2k}{\mathcal{S}}}}{4\cdot 3^{\frac{2k}{\mathcal{S}}}k^{\frac{2k}{\mathcal{S}}}(\mathcal{S}+k)(\mathcal{S}+2k)}-\frac{1}{8}\right)\Big\|\left|\Im''\left(\mathcal{U}\right)\right|+\left|\Im''\left(\mathcal{D}\right)\right|\Big\|\right].
\end{aligned}
\tag{3.17}
$$

Remark 2.

- *Setting $\mathcal{S}$ and $k=1$, and the tensor operations are vanished in Theorem 6, then Theorem 6 reduces to Theorem 2.2 in [50].*
- *If the norm structure and tensor operations are vanished in Theorem 6, then Theorem 6 simplifies to Corollary 2 in [13].*
- *If the tensor operations are vanished in Theorem 6, then Theorem 6 simplifies to Theorem 2.3 in [32].*
- *Setting $\mathcal{S},k=1$ in Theorem 6, it strained Theorem 2.3 in [57].*
- *Setting $\mathcal{S},k=1$ in Theorem 6, it strained Theorem 2.3 in [2].*
- *If we choose $\mathcal{S},k=1$ in Theorem 6, it strained Theorem 9 in [58].*

Theorem 7. *Let $\mathcal{U}$ and $\mathcal{D}$ be adjoint operators whose spectra lies in Δ_1 and Δ_2 respectively. Let $\Im$ be a continuous over Δ with $\|\Im''\|_{\Delta,\infty}:=\sup_{\mathcal{S}\in\Delta}|\Im''(\mathcal{S})|<\infty$, then we* *have*

$$\begin{aligned}
&\Bigg\| \left[\frac{1}{6}(\Im(\mathcal{U})\otimes 1)+\frac{2}{3}\Im\left(\frac{\mathcal{U}\otimes 1+1\otimes\mathcal{D}}{2}\right)+\frac{1}{6}(1\otimes\Im(\mathcal{D}))\right]\\
&-\left[\Im\left(\frac{\mathcal{U}\otimes 1+1\otimes\mathcal{D}}{2}\right)-\frac{\mathcal{W}^{\frac{\mathcal{S}}{k}}(\nu_2-\nu_1)}{4}\left[\int_0^1\Im'\left((1-\mathcal{W})\mathcal{U}\otimes 1+\left(\frac{\mathcal{W}1\otimes\mathcal{D}}{2}\right)\right)\mathtt{d}\mathcal{W}\right]\right.\\
&+\Im\left(\frac{\mathcal{U}\otimes 1+1\otimes\mathcal{D}}{2}\right)-\frac{(1-\mathcal{W})^{\frac{\mathcal{S}}{k}}(\nu_2-\nu_1)}{4}\left[\int_0^1\Im'\left(\left(\frac{1-\mathcal{W}}{2}\right)\mathcal{U}\otimes 1+\left(\frac{1+\mathcal{W}}{2}\right)1\otimes\mathcal{D}\right)\mathtt{d}\mathcal{W}\right]\Bigg\|\\
&\leq\frac{\|(1\otimes\mathcal{D}-\mathcal{U}\otimes 1)^2\|}{6}\Bigg[\Bigg(\left(\frac{\mathcal{W}^2}{2}+\frac{6\ln^{-\frac{\mathcal{S}}{k}-1}(2)\,k\Gamma_k(\mathcal{S}+k,\ln(2)(1-\mathcal{W}))}{\mathcal{S}+k}\right.\\
&\left.+\frac{\mathcal{W}^2}{2}+\mathcal{W}+\frac{6\ln^{-\mathcal{S}-1}(2)\,k\Gamma_k(\mathcal{S}+k,\ln(2)(1-\mathcal{W}))}{\mathcal{S}+k}\right)\|\Im'\|_{\Delta,+\infty}\Bigg)\Bigg].
\end{aligned}$$

Proof. Taking into account lemma 6 and applying the triangle result, we have

$$\begin{aligned}
&\Bigg\| \left[\frac{1}{6}(\Im(\mathcal{U})\otimes 1)+\frac{2}{3}\Im\left(\frac{\mathcal{U}\otimes 1+1\otimes\mathcal{D}}{2}\right)+\frac{1}{6}(1\otimes\Im(\mathcal{D}))\right]\\
&-\left[\Im\left(\frac{\mathcal{U}\otimes 1+1\otimes\mathcal{D}}{2}\right)-\frac{\mathcal{W}^{\frac{\mathcal{S}}{k}}(\nu_2-\nu_1)}{4}\left[\int_0^1\Im'\left((1-\mathcal{W})\mathcal{U}\otimes 1+\left(\frac{\mathcal{W}1\otimes\mathcal{D}}{2}\right)\right)\mathtt{d}\mathcal{W}\right]\right.\\
&+\Im\left(\frac{\mathcal{U}\otimes 1+1\otimes\mathcal{D}}{2}\right)-\frac{(1-\mathcal{W})^{\frac{\mathcal{S}}{k}}(\nu_2-\nu_1)}{4}\left[\int_0^1\Im'\left(\left(\frac{1-\mathcal{W}}{2}\right)\mathcal{U}\otimes 1+\left(\frac{1+\mathcal{W}}{2}\right)1\otimes\mathcal{D}\right)\mathtt{d}\mathcal{W}\right]\Bigg\|\\
&\leq\frac{\|(1\otimes\mathcal{D}-\mathcal{U}\otimes 1)^2\|}{6}\Bigg\|\int_0^{\frac{1}{2}}\left(\mathcal{W}-\frac{3k.2^{\frac{\mathcal{S}}{k}}\mathcal{W}^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\left[\Im''(\mathcal{U}\otimes 1\mathcal{W}+1\otimes\mathcal{U}(1-\mathcal{W}))\,\mathtt{d}\mathcal{W}\right]\\
&+\int_{\frac{1}{2}}^1\left((1-\mathcal{W})-\frac{3k.2^{\frac{\mathcal{S}}{k}}(1-\mathcal{W})^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\Im''(\mathcal{U}\otimes 1\mathcal{W}+1\otimes\mathcal{U}(1-\mathcal{W}))\Bigg\|\\
&\leq\frac{\|(1\otimes\mathcal{D}-\mathcal{U}\otimes 1)^2\|}{6}\left(\Bigg\|\int_0^{\frac{1}{2}}\left(\mathcal{W}-\frac{3k.2^{\frac{\mathcal{S}}{k}}\mathcal{W}^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\left[\Im''(\mathcal{U}\otimes 1\mathcal{W}+1\otimes\mathcal{U}(1-\mathcal{W}))\,\mathtt{d}\mathcal{W}\right]\right.\\
&\left.+\int_{\frac{1}{2}}^1\left((1-\mathcal{W})-\frac{3k.2^{\frac{\mathcal{S}}{k}}(1-\mathcal{W})^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\Im''(\mathcal{U}\otimes 1\mathcal{W}+1\otimes\mathcal{U}(1-\mathcal{W}))\Bigg\|\right). \qquad (3.18)
\end{aligned}$$

Observe that, by Lemma 2

$$\Big|\left(\Im''(\mathcal{U}1\otimes\mathcal{W}+1\otimes\mathcal{U}(1-\mathcal{W})\right)\Big|=\int_\Delta\int_\Delta\Big|\left(\Im''(\nu_1\mathcal{W}+\nu_1(1-\mathcal{W})\right)\Big|\mathtt{dE}_{\nu_1}\otimes\mathtt{dF}_{\nu_2}.$$

Since

$$\Big|\left(\Im''(\nu_1\mathcal{W}+\nu_1(1-\mathcal{W})\right)\Big|\leqslant\|\Im'\|_{\Delta,+\infty}$$

for all $\nu_1, \nu_2 \in \Delta$.

Taking $\int_\Delta \int_\Delta$ over $\mathtt{dE}_{\nu_1} \otimes \mathtt{dF}_{\nu_2}$, then we get

$$\begin{aligned}
&\left|\Im''\left(1 \otimes \mathcal{U}\mathcal{W} + 1 \otimes \mathcal{U}\left(1-\mathcal{W}\right)\right)\right| = \int_\Delta \int_\Delta \left|\Im''\left(\nu_1 \mathcal{W} + \nu_1\left(1-\mathcal{W}\right)\right)\right| \mathtt{dE}_{\nu_1} \otimes \mathtt{dF}_{\nu_2} \\
&\leqslant \left\|\Im'\right\|_{\Delta,+\infty} \int_\Delta \int_\Delta \mathtt{dE}_{\nu_1} \otimes \mathtt{dF}_{\nu_2} = \left\|\Im'\right\|_{\Delta,+\infty}. \qquad (3.19)
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\left|\Im''\left(1 \otimes \mathcal{D}\mathcal{W} + 1 \otimes \mathcal{D}\left(1-\mathcal{W}\right)\right)\right| = \int_\Delta \int_\Delta \left|\Im''\left(\nu_2 \mathcal{W} + \nu_2\left(1-\mathcal{W}\right)\right)\right| \mathtt{dE}_{\nu_1} \otimes \mathtt{dF}_{\nu_2} \\
&\leqslant \left\|\Im'\right\|_{\Delta,+\infty} \int_\Delta \int_\Delta \mathtt{dE}_{\nu_1} \otimes \mathtt{dF}_{\nu_2} = \left\|\Im'\right\|_{\Delta,+\infty}. \qquad (3.20)
\end{aligned}$$

Taking into account equation (3.18), this follows

$$\begin{aligned}
&\frac{\left\|(1 \otimes \mathcal{D} - \mathcal{U} \otimes 1)^2\right\|}{6}\left(\left\| \int_0^{\frac{1}{2}} \left(\mathcal{W} - \frac{3k.2^{\frac{\mathcal{S}}{k}} \mathcal{W}^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\left[\Im''\left(\mathcal{U} \otimes 1\mathcal{W} + 1 \otimes \mathcal{U}\left(1-\mathcal{W}\right)\right) \mathtt{d}\mathcal{W}\right]\right.\right. \\
&\left.\left. + \int_{\frac{1}{2}}^1 \left(\left(1-\mathcal{W}\right) - \frac{3k.2^{\frac{\mathcal{S}}{k}}\left(1-\mathcal{W}\right)^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\Im''\left(\mathcal{U} \otimes 1\mathcal{W} + 1 \otimes \mathcal{U}\left(1-\mathcal{W}\right)\right)\right\|\right) \\
&\leq \frac{\left\|(1 \otimes \mathcal{D} - \mathcal{U} \otimes 1)^2\right\|}{6}\left(\left\| \int_0^{\frac{1}{2}} \left(\mathcal{W} - \frac{3k.2^{\frac{\mathcal{S}}{k}} \mathcal{W}^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\right\| \left\|\Im'''\left(1 \otimes \mathcal{U}\mathcal{W} + 1 \otimes \mathcal{U}\left(1-\mathcal{W}\right)\right)\right\|\right. \\
&\left. + \left\| \int_{\frac{1}{2}}^1 \left(\left(1-\mathcal{W}\right) - \frac{3k.2^{\frac{\mathcal{S}}{k}}\left(1-\mathcal{W}\right)^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\right\| \left\|\Im''\left(1 \otimes \mathcal{U}\mathcal{W} + 1 \otimes \mathcal{U}\left(1-\mathcal{W}\right)\right) \mathtt{d}\mathcal{W}\right\|\right) \\
&\leq \frac{\left\|(1 \otimes \mathcal{D} - \mathcal{U} \otimes 1)^2\right\|}{6}\left(\left\|\frac{\mathcal{W}^2}{2} - \frac{6\ln^{-\frac{\mathcal{S}}{k}-1}(2)\, k\Gamma_k\left(\mathcal{S}+k, \ln(2)\left(1-\mathcal{W}\right)\right)}{\mathcal{S}+k}\right\| \left\|\Im'''\left(1 \otimes \mathcal{U}\mathcal{W} + 1 \otimes \mathcal{U}\left(1-\mathcal{W}\right)\right)\right\|\right. \\
&\left. + \left\| -\frac{\mathcal{W}^2}{2} + \mathcal{W} - \frac{6\ln^{-\frac{\mathcal{S}}{k}-1}(2)\, k\Gamma_k\left(\mathcal{S}+k, \ln(2)\left(1-\mathcal{W}\right)\right)}{\mathcal{S}+k}\right\| \left\|\Im''\left(1 \otimes \mathcal{U}\mathcal{W} + 1 \otimes \mathcal{U}\left(1-\mathcal{W}\right)\right) \mathtt{d}\mathcal{W}\right\|\right) \\
&\leq \frac{\left\|(1 \otimes \mathcal{D} - \mathcal{U} \otimes 1)^2\right\|}{6}\left(\left\|\frac{\mathcal{W}^2}{2} + \frac{6\ln^{-\frac{\mathcal{S}}{k}-1}(2)\, k\Gamma_k\left(\mathcal{S}+k, \ln(2)\left(1-\mathcal{W}\right)\right)}{\mathcal{S}+k}\right\| \left\|\Im'\right\|_{\Delta,+\infty}\right. \\
&\left. + \left\|\frac{\mathcal{W}^2}{2} + \mathcal{W} + \frac{6\ln^{-\mathcal{S}-1}(2)\, k\Gamma_k\left(\mathcal{S}+1, \ln(2)\left(1-\mathcal{W}\right)\right)}{\mathcal{S}+1}\right\| \left\|\Im'\right\|_{\Delta,+\infty}\right) \\
&\leq \frac{\left\|(1 \otimes \mathcal{D} - \mathcal{U} \otimes 1)^2\right\|}{6}\left[\left(\left(\frac{\mathcal{W}^2}{2} + \frac{6\ln^{-\frac{\mathcal{S}}{k}-1}(2)\, k\Gamma_k\left(\mathcal{S}+1, \ln(2)\left(1-\mathcal{W}\right)\right)}{\mathcal{S}+1}\right.\right.\right. \\
&\left.\left.\left. + \frac{\mathcal{W}^2}{2} + \mathcal{W} + \frac{6\ln^{-\mathcal{S}-1}(2)\, k\Gamma_k\left(\mathcal{S}+k, \ln(2)\left(1-\mathcal{W}\right)\right)}{\mathcal{S}+k}\right) \left\|\Im'\right\|_{\Delta,+\infty}\right)\right]. \qquad (3.21)
\end{aligned}$$

Using equation (3.21) in (3.18), we get required result.

Theorem 8. *Let $\Im$ be a twice differentiable as well as quasi convex $|\Im''|$ on Δ, then the following inequality holds true:*

$$\Bigg\| \left[\frac{1}{6}(\Im(\mathcal{U})\otimes 1)+\frac{2}{3}\Im\left(\frac{\mathcal{U}\otimes 1+1\otimes\mathcal{D}}{2}\right)+\frac{1}{6}(1\otimes\Im(\mathcal{D}))\right]$$
$$-\left[\Im\left(\frac{\mathcal{U}\otimes 1+1\otimes\mathcal{D}}{2}\right)-\frac{\mathcal{W}^{\frac{\mathcal{S}}{k}}(\nu_2-\nu_1)}{4}\left[\int_0^1\Im'\left((1-\mathcal{W})\mathcal{U}\otimes 1+\left(\frac{\mathcal{W}1\otimes\mathcal{D}}{2}\right)\right)\mathtt{d}\mathcal{W}\right]\right.$$
$$\left.+\Im\left(\frac{\mathcal{U}\otimes 1+1\otimes\mathcal{D}}{2}\right)-\frac{(1-\mathcal{W})^{\frac{\mathcal{S}}{k}}(\nu_2-\nu_1)}{4}\left[\int_0^1\Im'\left(\left(\frac{1-\mathcal{W}}{2}\right)\mathcal{U}\otimes 1+\left(\frac{1+\mathcal{W}}{2}\right)1\otimes\mathcal{D}\right)\mathtt{d}\mathcal{W}\right]\right\|$$
$$\leq \frac{\|(1\otimes\mathcal{D}-\mathcal{U}\otimes 1)^2\|}{6}\left(\frac{k}{(4\mathcal{S}+8\,k)}\left(\frac{\mathcal{S}}{k}\left(\frac{\mathcal{S}+k}{3\,k}\right)^{\frac{2\,k}{\mathcal{S}}}+\frac{3\,k}{\mathcal{S}+k}\right)-\frac{1}{8}\right)$$
$$\times\left\|\frac{1}{2}\left(\left|\Im''(\mathcal{U})\right|\otimes 1+1\otimes\left|\Im''(\mathcal{D})\right|+\left||\Im''(\mathcal{U})|\otimes 1-1\otimes|\Im''(\mathcal{D})|\right|\right)\right\|.$$

Proof.
As $|\Im''|$ is convex in a quasi sense over Δ, then one has

$$\begin{aligned}&\left|\left(\Im''\left(\nu_1\mathcal{W}+\nu_1(1-\mathcal{W})\right)-\Im''\left(\nu_2\mathcal{W}+\nu_2(1-\mathcal{W})\right)\right)\right|\\ &\leq\left|\left(\Im''\left(\nu_1\mathcal{W}+\nu_1(1-\mathcal{W})\right)+\Im''\left(\nu_2\mathcal{W}+\nu_2(1-\mathcal{W})\right)\right)\right|\\ &\leq\frac{1}{2}\left(\left|\Im''(\nu_2)\right|+\left|\Im''(\nu_1)\right|+\left||\Im''(\nu_2)|-|\Im''(\nu_1)|\right|\right)\end{aligned}$$

$\forall\,\tau\in[0,1]$ and $\nu_1,\nu_2\in\Delta$.
Taking $\int_\Delta\int_\Delta$ over $\mathtt{dE}_{\nu_1}\otimes\mathtt{dF}_{\nu_2}$ yields:

$$\begin{aligned}&\left|\left(\Im''\left(1\otimes\mathcal{U}\mathcal{W}+1\otimes\mathcal{U}(1-\mathcal{W})\right)-\Im''\left(1\otimes\mathcal{D}\mathcal{W}+1\otimes\mathcal{D}(1-\mathcal{W})\right)\right)\right|\\ &\quad=\int_\Delta\int_\Delta\left|\left(\Im''\left(\nu_1\mathcal{W}+\nu_1(1-\mathcal{W})\right)-\Im''\left(\nu_2\mathcal{W}+\nu_2(1-\mathcal{W})\right)\right)\right|\mathtt{dE}_{\nu_1}\otimes\mathtt{dF}_{\nu_2}\\ &\leq\frac{1}{2}\int_\Delta\int_\Delta\left(\left|\Im''(\nu_2)\right|+\left|\Im''(\nu_1)\right|+\left||\Im''(\nu_2)|-|\Im''(\nu_1)|\right|\right)\mathtt{dE}_{\nu_1}\otimes\mathtt{dF}_{\nu_2}\\ &=\frac{1}{2}\left(\left|\Im''(\mathcal{U})\right|\otimes 1+1\otimes\left|\Im''(\mathcal{D})\right|+\left||\Im''(\mathcal{U})|\otimes 1-1\otimes|\Im''(\mathcal{D})|\right|\right).\end{aligned}$$

Applying the norm this follows as

$$\begin{aligned}&\left\|\left(\Im''\left(1\otimes\mathcal{U}\mathcal{W}+1\otimes\mathcal{U}(1-\mathcal{W})\right)-\Im''\left(1\otimes\mathcal{D}\mathcal{W}+1\otimes\mathcal{D}(1-\mathcal{W})\right)\right)\right\|\\ &\leq\left\|\frac{1}{2}\left(\left|\Im''(\mathcal{U})\right|\otimes 1+1\otimes\left|\Im''(\mathcal{D})\right|+\left||\Im''(\mathcal{U})|\otimes 1-1\otimes|\Im''(\mathcal{D})|\right|\right)\right\|\end{aligned}$$

$$\leqslant \frac{1}{2}\left(\left\|\left|\Im''(\mathcal{U})\right|\otimes 1+1\otimes\left|\Im''(\mathcal{D})\right|\right\|+\left\|\left|\Im''(\mathcal{U})\right|\otimes 1-1\otimes\left|\Im''(\mathcal{D})\right|\right\|\right).$$

Involving the norm in (3.12) and, we have

$$\begin{aligned}
&\Bigg\|\left[\frac{1}{6}(\Im(\mathcal{U})\otimes 1)+\frac{2}{3}\Im\left(\frac{\mathcal{U}\otimes 1+1\otimes\mathcal{D}}{2}\right)+\frac{1}{6}(1\otimes\Im(\mathcal{D}))\right]\\
&-\left[\Im\left(\frac{\mathcal{U}\otimes 1+1\otimes\mathcal{D}}{2}\right)-\frac{\mathcal{W}^{\frac{\mathcal{S}}{k}}(\nu_2-\nu_1)}{4}\left[\int_0^1\Im'\left((1-\mathcal{W})\mathcal{U}\otimes 1+\left(\frac{\mathcal{W}1\otimes\mathcal{D}}{2}\right)\right)\mathtt{d}\mathcal{W}\right]\right.\\
&\left.+\Im\left(\frac{\mathcal{U}\otimes 1+1\otimes\mathcal{D}}{2}\right)-\frac{(1-\mathcal{W})^{\frac{\mathcal{S}}{k}}(\nu_2-\nu_1)}{4}\left[\int_0^1\Im'\left(\left(\frac{1-\mathcal{W}}{2}\right)\mathcal{U}\otimes 1+\left(\frac{1+\mathcal{W}}{2}\right)1\otimes\mathcal{D}\right)\mathtt{d}\mathcal{W}\right]\right\|\\
&\leq\frac{\|(1\otimes\mathcal{D}-\mathcal{U}\otimes 1)^2\|}{6}\left(\left\|\int_0^{\frac{1}{2}}\left(\mathcal{W}-\frac{3k.2^{\frac{\mathcal{S}}{k}}\mathcal{W}^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\left[\Im''\left(\mathcal{U}\otimes 1\mathcal{W}+1\otimes\mathcal{U}\left(1-\mathcal{W}\right)\right)\mathtt{d}\mathcal{W}\right]\right.\right.\\
&\left.\left.+\int_{\frac{1}{2}}^1\left((1-\mathcal{W})-\frac{3k.2^{\frac{\mathcal{S}}{k}}(1-\mathcal{W})^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\Im''\left(\mathcal{U}\otimes 1\mathcal{W}+1\otimes\mathcal{U}\left(1-\mathcal{W}\right)\right)\right\|\right)\\
&\leq\frac{\|(1\otimes\mathcal{D}-\mathcal{U}\otimes 1)^2\|}{6}\\
&\times\left(\left\|\int_0^{\frac{1}{2}}\left(\mathcal{W}-\frac{3k.2^{\frac{\mathcal{S}}{k}}\mathcal{W}^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\mathtt{d}\mathcal{W}\frac{1}{2}\left(\left|\Im''(\mathcal{U})\right|\otimes 1+1\otimes\left|\Im''(\mathcal{D})\right|+\left|\left|\Im''(\mathcal{U})\right|\otimes 1-1\otimes\left|\Im''(\mathcal{D})\right|\right|\right)\right.\right.\\
&+\int_{\frac{1}{2}}^1\left((1-\mathcal{W})-\frac{3k.2^{\frac{\mathcal{S}}{k}}(1-\mathcal{W})^{\frac{\mathcal{S}+k}{k}}}{\mathcal{S}+k}\right)\mathtt{d}\mathcal{W}\\
&\left.\left.\times\frac{1}{2}\left(\left|\Im''(\mathcal{U})\right|\otimes 1+1\otimes\left|\Im''(\mathcal{D})\right|+\left|\left|\Im''(\mathcal{U})\right|\otimes 1-1\otimes\left|\Im''(\mathcal{D})\right|\right|\right)\right\|\right)\\
&\leq\frac{\|(1\otimes\mathcal{D}-\mathcal{U}\otimes 1)\|}{6}\left\|\left(\frac{1}{(4\mathcal{S}+4)}\left(\mathcal{S}\left(\frac{\mathcal{S}+4}{2}\right)^{\frac{3}{2\mathcal{S}}}+\frac{3}{2\mathcal{S}+2}\right)-\frac{1}{8}\right)\right\|\\
&\times\left\|\frac{1}{2}\left(\left|\Im''(\mathcal{U})\right|\otimes 1+1\otimes\left|\Im''(\mathcal{D})\right|+\left|\left|\Im''(\mathcal{U})\right|\otimes 1-1\otimes\left|\Im''(\mathcal{D})\right|\right|\right)\right\|\\
&\leq\frac{\|(1\otimes\mathcal{D}-\mathcal{U}\otimes 1)^2\|}{12}\left\|\left(\frac{k}{(4\mathcal{S}+8\,k)}\left(\frac{\mathcal{S}}{k}\left(\frac{\mathcal{S}+k}{3\,k}\right)^{\frac{2\,k}{\mathcal{S}}}+\frac{3\,k}{\mathcal{S}+k}\right)+\frac{1}{8}\right)\right\|\\
&\times\left\|\left(\left|\Im''(\mathcal{U})\right|\otimes 1+1\otimes\left|\Im''(\mathcal{D})\right|+\left|\left|\Im''(\mathcal{U})\right|\otimes 1+1\otimes\left|\Im''(\mathcal{D})\right|\right|\right)\right\|\\
&=\frac{\|(1\otimes\mathcal{D}-\mathcal{U}\otimes 1)^2\|}{12}\left(\frac{\mathcal{S}(\mathcal{S}+k)^{\frac{a+2k}{\mathcal{S}}}+3^{\frac{\mathcal{S}+2k}{a}}k^{\frac{2\mathcal{S}+2k}{\mathcal{S}}}}{4\cdot 3^{\frac{2k}{\mathcal{S}}}k^{\frac{2k}{\mathcal{S}}}(\mathcal{S}+k)(\mathcal{S}+2k)}+\frac{1}{8}\right)\\
&\times\left\|\left(\left|\Im''(\mathcal{U})\right|\otimes 1+1\otimes\left|\Im''(\mathcal{D})\right|+\left|\left|\Im''(\mathcal{U})\right|\otimes 1+1\otimes\left|\Im''(\mathcal{D})\right|\right|\right)\right\|. \qquad (3.22)
\end{aligned}$$

Remark 3. • *Setting $\mathcal{S}=1$ in Theorem 8, then it strained Theorem 2.4 in [57].*

- *Setting $\mathcal{S}=1$ in Theorem 8, then it strained Theorem 2.4 in [2].*
- *Setting $\mathcal{S}=1$ in Theorem 8, then it strained Theorem 10 in [58].*

4. Examples and consequences

For an exponential function, if self-adjoint operators $\mathcal{U}$ and $\mathcal{D}$ are commute, then we have

$$e^{\mathcal{U}}e^{\mathcal{D}}=e^{\mathcal{D}}e^{\mathcal{U}}=e^{(\mathcal{U}+\mathcal{D})}.$$

Further, if $\mathcal{U}$ is invertible and $\nu_1,\nu_2\in\mathsf{R}$ with $\nu_1<\nu_2$, then

$$\int_{\nu_1}^{\nu_2} e^{\mathcal{W}\mathcal{U}}\mathtt{d}\mathcal{W}=\frac{[e^{\nu_2\mathcal{U}}-e^{\nu_1\mathcal{U}}]}{\mathcal{U}}.$$

Further if $\mathcal{D}-\mathcal{U}$ is invertible, then we have

$$\begin{aligned}\int_0^1 e^{((1-\nu_2)\mathcal{U}+\mathcal{S}\mathcal{D})}\mathtt{d}\mathcal{S}&=\int_0^1 e^{(\mathcal{S}(\mathcal{D}-\mathcal{U}))}e^{\mathcal{U}}\mathtt{d}\mathcal{S}=\left(\int_0^1 e^{(\mathcal{S}(\mathcal{D}-\mathcal{U}))}\mathtt{d}\mathcal{S}\right)e^{\mathcal{U}}\\&=\frac{[e^{(\mathcal{D}-\mathcal{U})}-\mathtt{I}]e^{\mathcal{U}}}{\mathcal{D}-\mathcal{U}}=\frac{[e^{\mathcal{D}}-e^{\mathcal{U}}]}{\mathcal{D}-\mathcal{U}}.\end{aligned}$$

Corollary 4.1. *Assume the identical hypothesis of Theorem 7 with $\Im(\mu)=\ln\mu$ over Δ, and $\mathcal{S}=\frac{1}{5}, k=\frac{1}{2}$, then*

$$\begin{aligned}&\Bigg\|\left[\frac{1}{6}(\ln\mu(\mathcal{U})\otimes 1)+\frac{2}{3}\ln\mu\left(\frac{\mathcal{U}\otimes 1+1\otimes\mathcal{D}}{2}\right)+\frac{1}{6}(1\otimes\ln\mu(\mathcal{D}))\right]\\&-\left[\ln\mu\left(\frac{\mathcal{U}\otimes 1+1\otimes\mathcal{D}}{2}\right)-\frac{\mathcal{W}^{\frac{2}{5}}(\nu_2-\nu_1)}{4}\left[\int_0^1\ln\mu'\left((1-\mathcal{W})\mathcal{U}\otimes 1+\left(\frac{\mathcal{W}1\otimes\mathcal{D}}{2}\right)\right)\mathtt{d}\mathcal{W}\right]\right.\\&\left.+\ln\mu\left(\frac{\mathcal{U}\otimes 1+1\otimes\mathcal{D}}{2}\right)-\frac{(1-\mathcal{W})^{\frac{2}{3}}(\nu_2-\nu_1)}{4}\left[\int_0^1\ln\mu'\left(\left(\frac{1-\mathcal{W}}{2}\right)\mathcal{U}\otimes 1+\left(\frac{1+\mathcal{W}}{2}\right)1\otimes\mathcal{D}\right)\mathtt{d}\mathcal{W}\right]\right\|\\&\le\frac{\|(1\otimes\mathcal{D}-\mathcal{U}\otimes 1)^2\|}{6}\left[\frac{1}{5\left(\frac{1}{5}+2\right)}\left(\frac{1}{5}\left(\frac{\frac{1}{3}+1}{3}\right)\frac{2}{\frac{2}{5}}+\frac{3}{\frac{1}{5}+1}\right)-\frac{1}{8}\right)\Big\|\left|\ln\mu''(\mathcal{U})\right|+\left|\ln\mu''(\mathcal{D})\right|\Big\|.\end{aligned}$$

5. Conclusion and future remarks

Tensor Hilbert spaces allow for the extension and decomposition of operators in higher dimensions, useful in spectral theory. Through the use of tensor operations for continuous differentiable mappings, we introduced gradient type result in the setup of function spaces, offering improvements and extensions of earlier findings that take fractional operators into account. We also point out important implications and comments that relate our results to previous research. In addition to promoting further study of Simpson-type inequalities and other quantum, fractional, and stochastic integrals, this publication advances mathematical inequality theory in tensor Hilbert spaces, an area that has received little attention. operators.

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