Certain Results for a Subclass of Meromorphic Multivalent Functions Associated with the Wright Function

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Abstract. In this paper, we introduce a new subclass of meromorphic multivalent functions associated with Wright generalized hypergeometric function and obtain new results for this class by the application of Briot-Bouquet differential subordination.

Key Words and Phrases: Analytic functions, Wright generalized hypergeometric function, The Briot-Bouquet differential subordination.

1. Introduction

Let $\Sigma_p$ denote the class of meromorphic function of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p} \quad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \quad (1)$$

which are analytic in the punctured open unit disk

$$\mathbb{D} := \{z \in \mathbb{C} | 0 < |z| < 1\} = \mathcal{U} \setminus \{0\},$$

where $\mathcal{U} := \{z \in \mathbb{C} | |z| < 1\}$. Also, we denote $\Sigma = \Sigma_1$.

If $f(z)$ and $F(z)$ are analytic in $\mathcal{U}$, we say that $f(z)$ is subordinate to a function $F(z)$ written symbolically as $f \prec F$ or $f(z) \prec F(z), (z \in \mathcal{U})$, if there exists a Schwarz function $w(z)$ which (by definition) is analytic in $\mathcal{U}$ with

$$w(0) = 0, \quad |w(z)| < 1 \quad (z \in \mathcal{U}),$$

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Let defined by the following Hadamard product

where the Hadamard product (or convolution) of $f$ and $g$ is defined by

$$ (f * g)(z) := z^{-p} + \sum_{k=1}^{\infty} d_k b_k z^{k-p}, $$

(3)

In particular, if the function $F(z)$ is univalent in $\mathcal{U}$, then we have the following equivalence [cf. 7]:

$$ f(z) \prec F(z)(z \in \mathcal{U}) \iff f(0) = F(0) \text{ and } f(\mathcal{U}) \subset F(\mathcal{U}). $$

For functions $f(z) \in \Sigma_p$ given by (1) and $g(z) \in \Sigma_p$ given by

$$ g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p}, $$

(2)

the Hadamard product (or convolution) of $f$ and $g$ is defined by

$$ (f * g)(z) := z^{-p} + \sum_{k=1}^{\infty} d_k b_k z^{k-p} =: (g * f)(z) \quad (p \in \mathbb{N}; z \in \mathbb{D}). $$

Let $l, s \in \mathbb{N}$. For positive real parameters $\alpha_j, A_j \ (j = 1, \ldots, q); \beta_j, B_j > 0 \ (j = 1, \ldots, s)$, with

$$ 1 + \sum_{j=1}^{s} B_j = \sum_{j=1}^{q} A_j \geq 0, $$

the Fox-Wright function $\psi$ is defined by [see 8]

$$ \psi_{\lambda}(\alpha, \beta; z) := \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{l} \Gamma(\alpha_j + nA_j)\Gamma(\beta_j + nB_j)n!}{\prod_{j=1}^{s} \Gamma(\beta_j + nB_j)n!} (z \in \mathcal{U}). $$

(4)

In particular, when $A_i = B_j = 1 \ (i = 1, \ldots, l; j = 1, \ldots, s)$, we have the following relationship:

$$ \psi_{\lambda}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_s; z) = \prod_{j=1}^{l} \Gamma(\alpha_j + nA_j)\Gamma(\beta_j + nB_j)n! (z \in \mathcal{U}). $$

(5)

where

$$ \Omega := \frac{\Gamma(\beta_1)\ldots\Gamma(\beta_s)}{\Gamma(\alpha_1)\ldots(\alpha_l)}. $$

(6)

Let

$$ \phi_p[(\alpha_j, A_j)_{1,l}; (\beta_j, B_j)_{1,s}; z] = \Omega z^{-p} \psi_{\lambda}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_s; z) \quad (z \in \mathbb{D}). $$

(7)

Due to Dziok and Raina [2] (see also [1] and [3]) we consider a linear operator

$$ \theta_{\lambda}^{l,s} \{ (\alpha_1, A_1) \} f(z) = \theta_p[(\alpha_1, A_1), \ldots, (\alpha_1, A_1); (\beta_1, B_1), \ldots, (\beta_s, B_s)] : \Sigma_p \mapsto \Sigma_p $$

defined by the following Hadamard product

$$ \theta_{\lambda}^{l,s} \{ (\alpha_1, A_1) \} f(z) := \phi_p[(\alpha_j, A_j)_{1,l}; (b_j, \beta_j)_{1,s}; z] \ast f(z). $$

(8)
If \( f \in \Sigma_p \) is given by the equation (1), then we have

\[
\theta^l_s \left\{ \left( \alpha_1, A_1 \right) \right\} f(z) = z^{-p} + \Omega \sum_{n=1}^{\infty} \frac{\Pi_{j=1}^{l} \Gamma(\alpha_j + nA_j)z^{n-p}}{\Pi_{j=1}^{l} \Gamma(\beta_j + nB_j)n!} a_n (z \in \mathbb{D}).
\]  

(9)

In particular, for \( A_i = B_j = 1 \) (\( i = 1, \ldots, l, j = 1, \ldots, s \)), we get the linear operator

\[
\mathcal{H}^p_{\alpha_1} \left[ \alpha_1 \right] f(z) = z^{-p} + \sum_{n=1}^{\infty} \frac{\Pi_{j=1}^{l} (\alpha_j)_n}{\Pi_{j=1}^{s} (\beta_j)_n n!} a_n z^{n-p} \quad (z \in \mathcal{U}),
\]  

(10)

studied by Liu and Srivastava [6]. Obviously, for \( l = 2, s = p = 1 \) and \( \alpha_2 = 1 \), we get

\[
\mathcal{L}(\alpha_1, \beta_1) f(z) = z^{-1} + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n}{(\beta_1)_n} a_n z^{n-1} \quad (z \in \mathcal{U}).
\]

It is easy to verify that

\[
z \left[ \frac{\theta^l_s \left\{ \left( \alpha_1, A_1 \right) \right\} f(z)}{A_1} \right]' = \frac{\alpha_1}{A_1} \frac{\theta^l_s \left\{ \left( \alpha_1 + 1, A_1 \right) \right\} f(z)}{\theta^l_s \left\{ \left( \alpha_1, A_1 \right) \right\} f(z)} - \left( \frac{\alpha_1}{A_1} + p \right) \frac{\theta^l_s \left\{ \left( \alpha_1, A_1 \right) \right\} f(z)}{\theta^l_s \left\{ \left( \alpha_1, A_1 \right) \right\} f(z)} - \frac{1 + Az}{1 + Bz}. \]

(11)

Also, for \(-1 \leq B < A \leq 1\) we denote by

\[V((\alpha_1, A_1); A, B) = V((\alpha_1, A_1), \ldots, (\alpha_l, A_l); A, B)\]

the class of functions \( f \in \Sigma_p \) which satisfy the following condition:

\[
\left( \frac{\alpha_1}{A_1} + p \right) - \left( \frac{\alpha_1}{A_1} \right) \frac{\theta^l_s \left\{ \left( \alpha_1 + 1, A_1 \right) \right\} f(z)}{\theta^l_s \left\{ \left( \alpha_1, A_1 \right) \right\} f(z)} \prec \frac{1 + Az}{1 + Bz}.
\]

(12)

Let \( h \) and \( q \) be analytic functions in \( \mathcal{U} \) with \( h(0) = q(0) = p \) and let \( q \) be univalent convex function. The first-order differential subordination

\[
h(z) + \frac{zh'(z)}{\beta h(z) + \gamma} \prec q(z),
\]

(13)

is called the Briot-Bouquet differential subordination. This particular differential subordination has a surprising number of important applications in the theory of analytic functions (for details see [7]). In this paper we present one more application of the Briot-Bouquet differential subordination.
2. Main result

To prove our main results we need the following lemmas:

**Lemma 1** ([7], see also [4]). Let $\beta, \gamma \in \mathcal{C}$ and suppose $q(z)$ is convex univalent in $\mathcal{U}$ with $q(0) = p$ and

$$\text{Re}\left\{\beta q(z) + \gamma\right\} > 0 \quad (z \in \mathcal{U})$$

If $h(z)$ is analytic in $\mathcal{U}$ with $h(0) = p$, and:

$$h(z) + \frac{zh'(z)}{\beta h(z) + \gamma} \prec q(z) \quad (z \in \mathcal{U}), \quad (14)$$

then

$$h(z) \prec q(z).$$

**Lemma 2** ([7]). Let the function $w(z)$ be (nonconstant) analytic in $\mathcal{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathcal{U}$, then

$$z_0 w'(z_0) = kw(z_0), \quad (15)$$

where $k$ is real and $k \geq 1$.

Making use of Lemma 1, we get the following theorem:

**Theorem 1.** If

$$\alpha_1 (1 + B) > p A_1 (A - B),$$

then

$$V((\alpha_1 + m, A_1); A, B) \subset V((\alpha_1, A_1); A, B) \quad (m \in \mathbb{N}).$$

**Proof.** Obviously, it is sufficient to prove the theorem for $m = 1$. Let a function $f$ belong to the class $V((\alpha_1 + 1, A_1); A, B)$ or equivalently

$$-z \left[\frac{\theta_p^{1,s}(\alpha_1 + 1, A_1)f(z)}{\theta_p^{1,s}(\alpha_1, A_1)f(z)}\right]' \prec p \frac{1 + Az}{1 + Bz}. \quad (16)$$

Then the function

$$h(z) = -z \left[\frac{\theta_p^{1,s}(\alpha_1, A_1)f(z)}{\theta_p^{1,s}(\alpha_1, A_1)f(z)}\right]', \quad (17)$$

is analytic in $\mathcal{U}$ and $h(0) = p$. Using equation (11) the equation (17) can be rewritten as

$$-h(z) + \left(\frac{\alpha_1}{A_1} + p\right) = \frac{\alpha_1}{A_1} \frac{\theta_p^{1,s}(\alpha_1 + 1, A_1)f(z)}{\theta_p^{1,s}(\alpha_1, A_1)f(z)}. \quad (18)$$
Taking the logarithmic derivative of equation (18), we get

\[
\frac{-zh'(z)}{A_1^2 + p} - h(z) = z \left[ \frac{\theta_{p}^{1,1}(\alpha_1 + 1, A_1)f(z)}{\theta_{p}^{1,1}(\alpha_1 + 1, A_1)f(z)} \right]' - z \left( \frac{\partial_s^{1,1}(\alpha_1, A_1)f(z)}{\theta_{p}^{1,1}(\alpha_1, A_1)f(z)} \right)'.
\]

(19)

Using (17) in the above equation we have,

\[
\frac{-zh'(z)}{A_1^2 + p} - h(z) = z \left[ \frac{\theta_{p}^{1,1}(\alpha_1 + 1, A_1)f(z)}{\theta_{p}^{1,1}(\alpha_1 + 1, A_1)f(z)} \right]' + h(z),
\]

(20)

\[
h(z) + \frac{zh'(z)}{A_1^2 + p} - h(z) = -z \left[ \frac{\partial_s^{1,1}(\alpha_1 + 1, A_1)f(z)}{\theta_{p}^{1,1}(\alpha_1 + 1, A_1)f(z)} \right]'.
\]

(21)

Thus by (16) we have

\[
h(z) + \frac{zh'(z)}{A_1^2 + p} - h(z) < \frac{1 + Ax}{1 + Bx}.
\]

(22)

Lemma 1 now yields

\[
h(z) < \frac{1 + Ax}{1 + Bx}.
\]

Thus, by (17) and (11) we conclude that \( f(z) \in V((\alpha_1, A_1); A, B) \). This completes the proof of the Theorem 1.

Using Lemma 2 we now show the following sufficient conditions for functions to belong to the class \( V((\alpha_1, A_1); A, B) \).

**Theorem 2.** Let \( m \in \mathbb{N} \) and

\[
\alpha_1 (1 + B) > pA_1 (A - B), \ 2 \left( \alpha_1 + m - 1 \right) B^2 \leq A_1 p [(A - B)(2B + 1)].
\]

(23)

If a function \( f \in \Sigma_p \) satisfies the inequality

\[
\left( \frac{\alpha_1 + m}{A_1} \right) \left| \frac{\theta_{p}^{1,1}(\alpha_1 + m + 1; A_1)f(z)}{\theta_{p}^{1,1}(\alpha_1 + m; A_1)f(z)} - 1 \right| < \frac{A - B - \frac{\alpha_1}{pA_1} B}{A - B + \frac{\alpha_1}{pA_1} (1 - B) + \frac{B + p (A - B)}{1 + B}}, \ (z \in U),
\]

(24)

then \( f \in V((\alpha_1, A_1); A, B) \).

**Proof.** It is sufficient to consider the case \( m = 1 \). Let a function \( f \) belong to the class \( \Sigma_p \).

On putting

\[
h(z) = p \frac{1 + Ax}{1 + Bx} \ (z \in U).
\]

(25)
Now, suppose that there exists a point $z \in \mathcal{U}$ such that (24), we conclude that
\[
\left(\frac{\alpha_1 + 1}{A_1} + p\right) - \left(\frac{\alpha_1 + 1}{A_1}\right) \theta_p^{\epsilon,\alpha_1 + 2, A_1} f(z) = \frac{(A - B - \frac{\alpha_1}{pA_1} B)zw'(z)}{\theta_p^{\epsilon,\alpha_1 + 1, A_1} f(z)} = \frac{(A - B - \frac{\alpha_1}{pA_1} B)zw'(z)}{\theta_p^{\epsilon,\alpha_1 + 1, A_1} f(z)} + \frac{\alpha_1}{pA_1} + \left\{\frac{\alpha_1}{pA_1} B + B - A\right\}w(z)Bzw'(z) + \frac{1 + Aw(z)}{1 + Bw(z)} + \frac{p(A - B)}{1 + Bw(z)}.
\]

Consequently, we have
\[
F(z) = w(z) \left\{ \frac{zw'(z)}{w(z)} \left( \frac{A - B - \frac{\alpha_1}{pA_1} B}{\frac{\alpha_1}{pA_1} + \left\{\frac{\alpha_1}{pA_1} B + B - A\right\}w(z)} + \frac{B}{1 + Bw(z)} \right) + \frac{p(A - B)}{1 + Bw(z)} \right\},
\]

where
\[
F(z) = \left(\frac{\alpha_1 + 1}{A_1} + p\right) - \left(\frac{\alpha_1 + 1}{A_1}\right) \left[ \theta_p^{\epsilon,\alpha_1 + 2, A_1} f(z) \right] = -p.
\]

By (12), (17) and (25), it is sufficient to verify that $w$ is analytic in $\mathcal{U}$ and
\[
|w(z)| < 1 \quad (z \in \mathcal{U}).
\]

Now, suppose that there exists a point $z_0 \in \mathcal{U}$ such that
\[
|w(z_0)| = 1, \quad |w(z)| < 1 \quad (|z| < |z_0|).
\]

Then, applying Lemma 2, we can write
\[
z_0w'(z_0) = kw(z_0), \quad w(z_0) = e^{i\theta} \quad (k \geq 1).
\]

Combining these with (26), we obtain
\[
|F(z_0)| \geq kRe \left( \frac{A - B - \frac{\alpha_1}{pA_1} B}{\frac{\alpha_1}{pA_1} + \left\{\frac{\alpha_1}{pA_1} B + B - A\right\}e^{i\theta}} + \frac{B}{1 + Be^{i\theta}} \right) + \frac{p(A - B)}{1 + B} \geq k \left( \frac{A - B - \frac{\alpha_1}{pA_1} B}{\frac{\alpha_1}{pA_1} + A - B - \frac{\alpha_1}{pA_1} B} + \frac{B}{1 + B} \right) + \frac{p(A - B)}{1 + B} \geq \frac{A - B - \frac{\alpha_1}{pA_1} B}{A - B + \frac{\alpha_1}{pA_1} (1 - B) + B + p(A - B)} \cdot
\]

Since this results contradicts (24), we conclude that $w$ is the analytic function in $\mathcal{U}$ and
\[
|w(z)| < 1 \quad (z \in \mathcal{U}),
\]

which completes the proof of the Theorem 2.

Putting $p = 1, A = 1 - \alpha$, and $B = 0$ in Theorem 2, we obtain the following result.
Corollary 1. Let \( m \in \mathbb{N} \), \( 0 \leq \alpha < 1 \) and \( \alpha_1 > A_1 \). If a function \( f \in \Sigma \) satisfies the following inequality:

\[
\left| \frac{\theta_p^{l,s}(\alpha_1 + m + 1; A_1)f(z)}{\theta_p^{l,s}(\alpha_1 + m; A_1)f(z)} - 1 \right| < \frac{(1 - \alpha) \left( 2 + \frac{\alpha_1}{A_1} - \alpha \right)}{\frac{\alpha_1 + m}{A_1} \left( \frac{\alpha_1}{A_1} + 1 - \alpha \right)},
\]

then

\[
\left| \frac{\alpha_1}{A_1} \left( 1 - \frac{\theta_p^{l,s}(\alpha_1 + 1, A_1)f(z)}{\theta_p^{l,s}(\alpha_1 + A_1)f(z)} \right) \right| < 1 - \alpha.
\]

Putting \( A_i = B_j = 1 \) \( (i = 1, \ldots, l, j = 1, \ldots, s) \) in Corollary 1, we obtain the following result.

Corollary 2. Let \( m \in \mathbb{N} \), \( 0 \leq \alpha < 1 \) and \( \alpha_1 + \alpha > 1 \). If a function \( f \in \Sigma \) satisfies the following inequality:

\[
\left| \frac{\mathcal{H}(\alpha_1 + m + 1)f(z)}{\mathcal{H}(\alpha_1 + m)f(z)} - 1 \right| < \frac{(1 - \alpha) \left( 2 + \alpha_1 - \alpha \right)}{(\alpha_1 + m) \left( \alpha_1 + 1 - \alpha \right)},
\]

then

\[
\left| \alpha_1 + 1 - \alpha \frac{\mathcal{H}(\alpha_1 + 1)f(z)}{\mathcal{H}(\alpha_1)f(z)} \right| < 1 - \alpha.
\]

Putting \( l = 2, s = p = A_1 = A_2 = B_1 = 1 \), and \( \alpha_2 = 1 \) in Theorems 1 and 2, we get the following two results:

Corollary 3. Let \( m \in \mathbb{N} \), \( \alpha_1 (1 + B) > (A - B) \). If a function \( f \in \Sigma \) satisfies the following condition:

\[
(\alpha_1 + m + 1) - (\alpha_1 + m) \frac{\mathcal{L}(\alpha_1 + m + 1, \beta_1)f(z)}{\mathcal{L}(\alpha_1 + m, \beta_1)f(z)} < \frac{1 + Az}{1 + Bz},
\]

then

\[
(\alpha_1 + 1) - \alpha_1 \frac{\mathcal{L}(\alpha_1 + 1, \beta_1)f(z)}{\mathcal{L}(\alpha_1, \beta_1)f(z)} < \frac{1 + Az}{1 + Bz}.
\]

Corollary 4. Let \( m \in \mathbb{N} \), \( \alpha_1 (1 + B) > A - B \) and \( 2 (\alpha_1 + m - 1) B^2 \leq (A - B)(2B + 1) \). If a function \( f \in \Sigma \) satisfies the inequality:

\[
(\alpha_1 + m) \left| \frac{\mathcal{L}(\alpha_1 + m + 1, \beta_1)f(z)}{\mathcal{L}(\alpha_1 + m, \beta_1)f(z)} - 1 \right| < \frac{A - B - \alpha_1 B}{A - B + \alpha_1 (1 - B)} + \frac{A}{1 + B} \quad (z \in U),
\]

then

\[
\alpha_1 + 1 - \alpha_1 \frac{\mathcal{L}(\alpha_1 + 1, \beta_1)f(z)}{\mathcal{L}(\alpha_1, \beta_1)f(z)} < \frac{1 + Az}{1 + Bz}.
\]

Putting \( \alpha_1 = \beta_1 = m = 1 \) in Corollary 4 we obtain the sufficient conditions for starlikeness.
Corollary 5. Let \( 2B^2 \leq (A - B)(2B + 1) \). If a function \( f \in \Sigma \) satisfies the inequality:

\[
\left| \frac{z^2 f''(z) + 4zf'(z) + 2f(z)}{2 (zf'(z) + 2f(z))} \right| < \frac{A - 2B}{1 + A - 2B} + \frac{A}{1 + B} \quad (z \in \mathcal{U}),
\]

then

\[
-\frac{zf'(z)}{f(z)} < \frac{1 + Az}{1 + Bz},
\]

i.e., the function \( f \) is starlike in \( \mathcal{U} \).

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References


