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Higher-order $(F, \alpha, \beta, \rho, d)$-Convexity and its Application in Fractional Programming

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Abstract. In this paper we introduce the concept of higher-order $(F, \alpha, \beta, \rho, d)$-convexity with respect to a differentiable function K. Based on this generalized convexity, sufficient optimality conditions for a nonlinear programming problem (NP) are obtained. Duality relations for Mond-Weir and Wolfe duals of (NP) have also been discussed. These duality results are then applied to nonlinear fractional programming problems.

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1. Introduction

Optimality conditions and duality in nonlinear programming were first investigated under convexity assumptions. As they have played an important role in the development of mathematical programming, several authors have generalized the concept of convexity under which sufficient optimality conditions and duality theorems holds. Hanson [2] defined invex functions.

The concept of (F, ρ)-convexity was introduced by Pareda [7] as an extension of F-convexity [3] and ρ-convexity [8]. Liang et al. [4] introduced a unified formulation of generalized convexity called (F, α, ρ, d)-convexity and obtained some optimality conditions and duality results for nonlinear fractional programming problems.

Recently, Yuan et al. [9] introduced the concept of (C, α, ρ, d)-convexity which is the generalization of (F, α, ρ, d)-convexity, and proved optimality conditions and duality theorems for

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non-differentiable minimax fractional programming problems.

This paper is organized as follows. In Section 2, we define higher-order $(F, \alpha, \beta, \rho, d)$-convex functions. Under this generalized convexity, we obtain sufficient optimality conditions for a nonlinear programming problem (NP) in Section 3. In Section 4 we establish weak and strong duality for Mond-Weir dual program for (NP). An application for a fractional programming problem (FP) has been discussed in Section 5. In the last Section we present Wolfe duality for (NP) and (FP).

2. Definitions and Preliminaries

We consider the following nonlinear programming problem:

(NP) Minimize $\phi(x)$,
Subject to $h(x) \leqq 0,\ x \in X$,

where X is an open subset of R^n and the functions $\phi : X \mapsto R$ and $h = (h_1, h_2, \ldots, h_m) : X \mapsto R^m$ are differentiable on X. Let $S = \{x \in X : \ h(x) \leqq 0\}$ denote the set of all feasible solutions for (NP).

Definition 1. *A functional $F : X \times X \times R^n \mapsto R$ is said to be sublinear in the third variable, if for all $x, \bar{x} \in X$,*

(i) $F(x, \bar{x}; \xi_1 + \xi_2) \leqq F(x, \bar{x}; \xi_1) + F(x, \bar{x}; \xi_2)$, for all $\xi_1, \xi_2 \in R^n$; and

(ii) $F(x, \bar{x}; \alpha a) = \alpha F(x, \bar{x}; a)$ for all $\alpha \in R_+$, and $a \in R^n$.

From (ii), it is clear that $F(x, \bar{x}; 0) = 0$.

Based on the concept of the sublinear functional, we now introduce the class of higher-order $(F, \alpha, \beta, \rho, d)$-convex functions as follows:

Let $X \subseteq R^n$ be an open set. Let $\phi : X \mapsto R,\ K : X \times R^n \mapsto R$ be differentiable functions, $F : X \times X \times R^n \mapsto R$ be a sublinear functional in the third variable and $d : X \times X \mapsto R$. Further, let $\alpha,\ \beta : X \times X \mapsto R_+ \setminus \{0\}$ and $\rho \in R$.

Definition 2. *The function ϕ is said to be higher-order $(F, \alpha, \beta, \rho, d)$-convex at $\bar{x}$ with respect to K, if for all $x \in X$ and $p \in R^n$,*

$$\begin{aligned} \phi(x) - \phi(\bar{x}) &\geqq F(x, \bar{x}; \alpha(x, \bar{x})\{\nabla\phi(\bar{x}) + \nabla_p K(\bar{x}, p)\}) \\ &+ \beta(x, \bar{x})\{K(\bar{x}, p) - p^T \nabla_p K(\bar{x}, p)\} + \rho d^2(x, \bar{x}). \end{aligned}$$

Remark 1. *Let $K(\bar{x}, p) = 0$.*

(i) Then the above definition becomes that of (F, α, ρ, d)-convex function introduced by Liang et al. [4].

(ii) *If $\alpha(x,\bar{x}) = 1$, we obtain the definition of (F,ρ)-convex function given by Pareda [7].*

(iii) *If $\alpha(x,\bar{x}) = 1, \rho = 0$ and $F(x,\bar{x};\nabla\phi(\bar{x})) = \eta^T(x,\bar{x})\nabla\phi(\bar{x})$ for a certain map $\eta : X \times X \longrightarrow R^n$, then (F,α,β,ρ,d)-convexity reduces to the invexity in Hanson [2].*

(iv) *If F is convex with respect to the third argument, then we obtain the definition of (F,α,ρ,d)-convex function introduced by Yuan et al. [9].*

Remark 2. *Let $\beta(x,\bar{x}) = 1$.*

(i) *If $K(\bar{x},p) = \frac{1}{2}p^T\nabla^2\phi(\bar{x})p$, then the above inequality reduces to the definition of second order (F,α,ρ,d)-convex function given by Ahmad and Husain [1].*

(ii) *If $\alpha(x,\bar{x}) = 1, \rho = 0$, $K(\bar{x},p) = \frac{1}{2}p^T\nabla^2\phi(\bar{x})p$ and $F(x,\bar{x};a) = \eta^T(x,\bar{x})a$, where $\eta : X \times X \longrightarrow R^n$, the above definition becomes that of η-bonvexity introduced by Pandey [6].*

Proposition 1 (Kuhn-Tucker Necessary Optimality Conditions [see 5])**.** *Let $\bar{x} \in S$ be an optimal solution of (NP) and let h satisfy a constraint qualification [Theorem 7.3.7 in 5]. Then there exists a $\bar{v} \in R^m$ such that*

$$\nabla\phi(\bar{x}) + \nabla h(\bar{x})\bar{v} = 0, \tag{1}$$

$$\bar{v}^T h(\bar{x}) = 0, \tag{2}$$

$$\bar{v} \geqq 0,\ h(\bar{x}) \leqq 0, \tag{3}$$

where $\nabla h(\bar{x})$ denotes the $n \times m$ matrix $[\nabla h_1(\bar{x}), \nabla h_2(\bar{x}), \ldots, \nabla h_m(\bar{x})]$.

3. Sufficient Optimality Conditions

In this section, we establish Kuhn-Tucker sufficient optimality conditions for (NP) under (F,α,β,ρ,d)-convexity assumptions.

Theorem 1. *Let $\bar{x} \in S$ and $\bar{v} \in R^m$ satisfy (1)-(3). If*

(i) *ϕ is higher-order $(F,\alpha,\beta,\rho_1,d)$-convex at $\bar{x}$ with respect to K,*

(ii) *$\bar{v}^T h$ is higher-order $(F,\alpha,\beta,\rho_2,d)$-convex at $\bar{x}$ with respect to $-K$, and*

(iii) *$\rho_1 + \rho_2 \geqq 0$,*

then $\bar{x}$ is an optimal solution of the problem (NP).

Proof. Let $\bar{x} \in S$. Since ϕ is higher-order $(F,\alpha,\beta,\rho_1,d)$-convex at $\bar{x}$ with respect to K, for all $x \in S$, we have

$$\begin{aligned} \phi(x) - \phi(\bar{x}) \quad &\geqq \quad F\left(x,\bar{x};\alpha(x,\bar{x})[\nabla\phi(\bar{x}) + \nabla_p K(\bar{x},p)]\right) \\ &+ \quad \beta(x,\bar{x})\left(K(\bar{x},p) - p^T\nabla_p K(\bar{x},p)\right) + \rho_1 d^2(x,\bar{x}). \end{aligned} \tag{4}$$

Using (1) , we get

$$\begin{aligned}\phi(x)-\phi(\bar{x}) &\geqq F\left(x,\bar{x};\alpha(x,\bar{x})[-\nabla h(\bar{x})\bar{v}+\nabla_p K(\bar{x},p)]\right)\\ &+ \beta(x,\bar{x})\left(K(\bar{x},p)-p^T\nabla_p K(\bar{x},p)\right)+\rho_1 d^2(x,\bar{x}).\end{aligned} \tag{5}$$

Also, $\bar{v}^T h$ is higher-order $(F,\alpha,\beta,\rho_2,d)$-convex at $\bar{x}$ with respect to $-K$. Therefore

$$\begin{aligned}\bar{v}^T h(x)-\bar{v}^T h(\bar{x}) &\geqq F\left(x,\bar{x};\alpha(x,\bar{x})[\nabla \bar{v}^T h(\bar{x})-\nabla_p K(\bar{x},p)]\right)\\ &- \beta(x,\bar{x})\left(K(\bar{x},p)-p^T\nabla_p K(\bar{x},p)\right)+\rho_2 d^2(x,\bar{x}).\end{aligned} \tag{6}$$

Since $\bar{v}^T h(\bar{x})=0,\ \bar{v}\geqq 0$ and $h(x)\leqq 0$, we get

$$\begin{aligned}0 &\geqq F\left(x,\bar{x};\alpha(x,\bar{x})[\nabla \bar{v}^T h(\bar{x})-\nabla_p K(\bar{x},p)]\right)\\ &- \beta(x,\bar{x})\left(K(\bar{x},p)-p^T\nabla_p K(\bar{x},p)\right)+\rho_2 d^2(x,\bar{x}).\end{aligned} \tag{7}$$

Adding the inequalities (5) and (7) , we obtain

$$\phi(x)-\phi(\bar{x})\geqq(\rho_1+\rho_2)d^2(x,\bar{x}),$$

which by hypothesis (iii) implies,

$$\phi(x)\geqq\phi(\bar{x}).$$

Hence $\bar{x}$ is an optimal solution of the problem (NP).

4. Mond Weir Duality

In this section, we establish weak and strong duality theorems for the following Mond Weir dual (MD) for (NP):

(MD) Maximize $\phi(u)$,

Subject to $\nabla\phi(u)+\nabla h(u)v=0, \quad (8)$

$v^T h(u)\geqq 0, \quad (9)$

$u\in X, v\geqq 0, v\in R^m. \quad (10)$

Theorem 2 (Weak Duality)**.** *Let x and (u,v) be feasible solutions of (NP) and (MD) respectively. Let*

(i) ϕ be higher-order $(F,\alpha,\beta,\rho_1,d)$-convex at u with respect to K,

(ii) $v^T h$ be higher-order $(F,\alpha,\beta,\rho_2,d)$-convex at u with respect to $-K$, and

(iii) $\rho_1+\rho_2\geqq 0$.

Then

$$\phi(x)\geqq\phi(u).$$

Proof. By hypothesis (i), we have

$$\begin{aligned}\phi(x)-\phi(u) &\geqq F\left(x,u;\alpha(x,u)[\nabla\phi(u)+\nabla_p K(u,p)]\right)\\ &+ \beta(x,u)\left(K(u,p)-p^T\nabla_p K(u,p)\right)+\rho_1 d^2(x,u).\end{aligned} \tag{11}$$

Also hypothesis (ii) yields

$$\begin{aligned}v^T h(x)-v^T h(u) &\geqq F\left(x,u;\alpha(x,u)[\nabla v^T h(u)-\nabla_p K(u,p)]\right)\\ &- \beta(x,u)\left(K(u,p)-p^T\nabla_p K(u,p)\right)+\rho_2 d^2(x,u).\end{aligned}$$

By (9), (10) and $h(x)\leqq 0$, it follows that

$$\begin{aligned}0 &\geqq F\left(x,u;\alpha(x,u)[\nabla v^T h(u)-\nabla_p K(u,p)]\right)\\ &- \beta(x,u)\left(K(u,p)-p^T\nabla_p K(u,p)\right)+\rho_2 d^2(x,u).\end{aligned} \tag{12}$$

Adding the inequalities (11), (12) and applying the properties of sublinear functional, we obtain

$$\phi(x)-\phi(u) \geqq F\left(x,u;\alpha(x,u)[\nabla\phi(u)+\nabla v^T h(u)]\right)+\rho_1 d^2(x,u)+\rho_2 d^2(x,u)$$

which in view of (8) implies

$$\phi(x)-\phi(u)\geqq(\rho_1+\rho_2)d^2(x,u).$$

Using hypothesis (iii) in the above inequality, we get

$$\phi(x)\geqq\phi(u).$$

Remark 3. *A constraint qualification is not required to establish weak duality. It has been erroneously assumed in Theorem 3.4 in [4].*

Theorem 3 (Strong Duality)**.** *Let $\bar{x}$ be an optimal solution of the problem (NP) and let h satisfy a constraint qualification. Further, let Theorem 2 hold for the feasible solution $\bar{x}$ of (NP) and all feasible solutions (u,v) of (MD). Then there exists a $\bar{v}\in R^m_+$ such that $(\bar{x},\bar{v})$ is an optimal solution of (MD).*

Proof. Since $\bar{x}$ is an optimal solution for the problem (NP) and h satisfies a constraint qualification, by Proposition 1 there exists a $\bar{v}\in R^m_+$ such that the Kuhn-Tucker conditions, (1) - (3) hold. Hence $(\bar{x},\bar{v})$ is feasible for (MD).

Now let (u,v) be any feasible solution of (MD). Then by weak duality (Theorem 2), we have

$$\phi(\bar{x})\geqq\phi(u).$$

Therefore $(\bar{x},\bar{v})$ in an optimal solution of (MD).

5. Application in Fractional Programming

If $\phi : X \to R$ is defined by

$$\phi(x) = \frac{f(x)}{g(x)},$$

where $f, g : X \to R$, $f(x) \geqq 0$ and $g(x) > 0$ on X, then the nonlinear programming problem (NP) becomes the following fractional programming problem (FP):

$$\textbf{(FP)} \qquad \text{Minimize } \frac{f(x)}{g(x)}$$
$$\text{Subject to } h(x) \leqq 0, \ x \in X.$$

We now prove the following result, which gives higher-order $(F, \bar{\alpha}, \bar{\beta}, \rho, \bar{d})$-convexity of the ratio function $f(x)/g(x)$.

Theorem 4. *Let $f(x)$ and $-g(x)$ be higher-order $(F, \alpha, \beta, \rho, d)$-convex at $\bar{x}$ with respect to the same function K. Then the fractional function $\frac{f(x)}{g(x)}$ is higher-order $(F, \bar{\alpha}, \bar{\beta}, \rho, \bar{d})$-convex at $\bar{x}$ with respect to $\bar{K}$, where*

$$\bar{\alpha}(x, \bar{x}) = \alpha(x, \bar{x}) \frac{g(\bar{x})}{g(x)}$$
$$\bar{\beta}(x, \bar{x}) = \beta(x, \bar{x}) \frac{g(\bar{x})}{g(x)},$$
$$\bar{K}(\bar{x}, p) = \left[\frac{1}{g(\bar{x})} + \frac{f(\bar{x})}{g^2(\bar{x})} \right] K(\bar{x}, p),$$
$$\bar{d}(x, \bar{x}) = \left[\frac{1}{g(x)} + \frac{f(\bar{x})}{g(x)g(\bar{x})} \right]^{\frac{1}{2}} d(x, \bar{x}).$$

Proof. Since $f(x)$ and $-g(x)$ are higher-order $(F, \alpha, \beta, \rho, d)$-convex at $\bar{x}$ with respect to the same function K, we have

$$\begin{aligned} f(x) - f(\bar{x}) \ &\geqq F(x, \bar{x}; \alpha(x, \bar{x})\{\nabla f(\bar{x}) + \nabla_p K(\bar{x}, p)\}) \\ &\quad + \beta(x, \bar{x})\{K(\bar{x}, p) - p^T \nabla_p K(\bar{x}, p)\} + \rho d^2(x, \bar{x}) \end{aligned}$$

and

$$\begin{aligned} -g(x) + g(\bar{x}) &\geqq F(x, \bar{x}; \alpha(x, \bar{x})\{-\nabla g(\bar{x}) + \nabla_p K(\bar{x}, p)\}) \\ &\quad + \beta(x, \bar{x})\{K(\bar{x}, p) - p^T \nabla_p K(\bar{x}, p)\} + \rho d^2(x, \bar{x}). \end{aligned}$$

Also

$$\frac{f(x)}{g(x)} - \frac{f(\bar{x})}{g(\bar{x})} = \frac{1}{g(x)} \left[f(x) - f(\bar{x}) \right] + \frac{f(\bar{x})}{g(x)g(\bar{x})} \left[-g(x) + g(\bar{x}) \right].$$

Using the above inequalities and sublinearity of F, we get

$$
\begin{aligned}
\frac{f(x)}{g(x)}-\frac{f(\bar{x})}{g(\bar{x})} \geqq\; & \frac{1}{g(x)}F(x,\bar{x};\alpha(x,\bar{x})\{\nabla f(\bar{x})+\nabla_p K(\bar{x},p)\}) \\
& +\frac{1}{g(x)}\left(\beta(x,\bar{x})\{K(\bar{x},p)-p^T\nabla_p K(\bar{x},p)\}+\rho d^2(x,\bar{x})\right) \\
& +\frac{f(\bar{x})}{g(x)g(\bar{x})}F(x,\bar{x};\alpha(x,\bar{x})\{-\nabla g(\bar{x})+\nabla_p K(\bar{x},p)\}) \\
& +\frac{f(\bar{x})}{g(x)g(\bar{x})}\left(\beta(x,\bar{x})\{K(\bar{x},p)-p^T\nabla_p K(\bar{x},p)\}+\rho d^2(x,\bar{x})\right). \\
=\; & F(x,\bar{x};\frac{\alpha(x,\bar{x})}{g(x)}\{\nabla f(\bar{x})+\nabla_p K(\bar{x},p)\}) \\
& +F(x,\bar{x};\alpha(x,\bar{x})\frac{f(\bar{x})}{g(x)g(\bar{x})}\{-\nabla g(\bar{x})+\nabla_p K(\bar{x},p)\}) \\
& +\beta(x,\bar{x})\left[\frac{1}{g(x)}+\frac{f(\bar{x})}{g(x)g(\bar{x})}\right]\{K(\bar{x},p)-p^T\nabla_p K(\bar{x},p)\} \\
& +\rho\left[\frac{1}{g(x)}+\frac{f(\bar{x})}{g(x)g(\bar{x})}\right]d^2(x,\bar{x}). \\
=\; & F(x,\bar{x};\alpha(x,\bar{x})\frac{g(\bar{x})}{g(x)}\{\nabla\frac{f(\bar{x})}{g(\bar{x})}+\left[\frac{1}{g(\bar{x})}+\frac{f(\bar{x})}{g^2(\bar{x})}\right]\nabla_p K(\bar{x},p)\}) \\
& +\beta(x,\bar{x})\frac{g(\bar{x})}{g(x)}\left[\frac{1}{g(\bar{x})}+\frac{f(\bar{x})}{g^2(\bar{x})}\right]\{K(\bar{x},p)-p^T\nabla_p K(\bar{x},p)\} \\
& +\rho\left[\frac{1}{g(x)}+\frac{f(\bar{x})}{g(x)g(\bar{x})}\right]d^2(x,\bar{x}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{f(x)}{g(x)}-\frac{f(\bar{x})}{g(\bar{x})} \geqq F\left(x,\bar{x};\bar{\alpha}(x,\bar{x})\left[\nabla\frac{f(\bar{x})}{g(\bar{x})}+\nabla_p\bar{K}(\bar{x},p)\right]\right) \\
+\bar{\beta}(x,\bar{x})\{\bar{K}(\bar{x},p)-p^T\nabla_p\bar{K}(\bar{x},p)\}+\rho\bar{d}^2(x,\bar{x}),
\end{aligned}
$$

i.e., $\frac{f(x)}{g(x)}$ is higher-order $(F,\bar{\alpha},\bar{\beta},\rho,\bar{d})$-convex at $\bar{x}$ with respect to $\bar{K}$.

In view of Theorem 4, the results of Section 4 lead to the following duality relations between (FP) and its Mond-Weir dual (MFD).

(MFD) Maximize $\frac{f(u)}{g(u)}$

Subject to $\nabla\left(\frac{f(u)}{g(u)}\right)+\nabla h(u)v=0,$

$v^T h(u)\geqq 0,$

$u\in X, v\geqq 0, v\in R^m.$

Theorem 5 (Weak Duality). *Let x and (u,v) be feasible solutions of (FP) and (MFD) respectively. Let*

(i) f *and* $-g$ *be higher-order* $(F,\alpha,\beta,\rho_1,d)$*-convex at* u *with respect to* K*,*

(ii) $v^T h$ *be higher-order* $(F,\bar{\alpha},\bar{\beta},\rho_2,\bar{d})$*-convex at* u *with respect to* $-\bar{K}$*, where* $\bar{\alpha},\bar{\beta},\bar{K}$ *and* $\bar{d}$ *are as given in Theorem 4, and*

(iii) $\rho_1+\rho_2 \geqq 0$.

Then

$$\frac{f(x)}{g(x)} \geqq \frac{f(u)}{g(u)}.$$

Theorem 6 (Strong Duality). *Let $\bar{x}$ be an optimal solution of the problem (FP) and let h satisfy a constraint qualification. Further, let Theorem 5 hold for the feasible solution $\bar{x}$ of (FP) and all feasible solutions (u,v) of (MFD). Then there exists a $\bar{v}\in R^m_+$ such that $(\bar{x},\bar{v})$ is an optimal solution of (MFD).*

6. Wolfe Duality

The Wolfe dual of (NP) and (FP) are respectively

(WD)
$$\begin{aligned}&\text{Maximize } \phi(u)+v^T h(u)\\&\text{Subject to } \nabla\phi(u)+\nabla h(u)v=0,\\&u\in X, v\geqq 0, v\in R^m,\end{aligned}$$

(WFD)
$$\begin{aligned}&\text{Maximize } \frac{f(u)}{g(u)}+v^T h(u)\\&\text{Subject to } \nabla\left(\frac{f(u)}{g(u)}\right)+\nabla h(u)v=0,\\&u\in X, v\geqq 0, v\in R^m.\end{aligned}$$

Now we state duality relations for the primal problems (NP) and (FP) and their Wolfe duals (WD) and (WFD) respectively. Their proofs follow as in Section 4.

Theorem 7 (Weak Duality). *Let x and (u,v) be feasible solutions of (NP) and (WD) respectively. Let*

(i) ϕ *be higher-order* $(F,\alpha,\beta,\rho_1,d)$*-convex at* u *with respect to* K*,*

(ii) $v^T h$ *be higher-order* $(F,\alpha,\beta,\rho_2,d)$*-convex at* u *with respect to* $-K$*, and*

(iii) $\rho_1+\rho_2 \geqq 0$.

Then

$$\phi(x) \geqq \phi(u)+v^T h(u).$$

Theorem 8 (Strong Duality). *Let $\bar{x}$ be an optimal solution of the problem (NP) and let h satisfy a constraint qualification. Further, let Theorem 7 hold for the feasible solution $\bar{x}$ of (NP) and all feasible solutions (u,v) of (WD). Then there exists a $\bar{v} \in R^m_+$ such that $(\bar{x},\bar{v})$ is an optimal solution of (WD) and the optimal objective function values of (NP) and (WD) are equal.*

Theorem 9 (Weak Duality). *Let x and (u,v) be feasible solutions of (FP) and (WFD) respectively. Let*

(i) f *and* $-g$ *be higher-order* $(F,\alpha,\beta,\rho_1,d)$*-convex at u with respect to K,*

(ii) $v^T h$ *be higher-order* $(F,\bar{\alpha},\bar{\beta},\rho_2,\bar{d})$*-convex at u with respect to* $-\bar{K}$*, where* $\bar{\alpha},\bar{\beta},\bar{K}$ *and* $\bar{d}$ *are as given in Theorem 4, and*

(iii) $\rho_1+\rho_2 \geqq 0$.

Then

$$\frac{f(x)}{g(x)} \geqq \frac{f(u)}{g(u)} + v^T h(u).$$

Theorem 10 (Strong Duality). *Let $\bar{x}$ be an optimal solution of the problem (FP) and let h satisfy a constraint qualification. Further, let Theorem 9 hold for the feasible solution $\bar{x}$ of (FP) and all feasible solutions (u,v) of (WFD). Then there exists a $\bar{v} \in R^m_+$ such that $(\bar{x},\bar{v})$ is an optimal solution of (WFD) and the optimal objective function values of (FP) and (WFD) are equal.*

7. Conclusion

In this paper a new concept of generalized convexity has been introduced. Under this generalized convexity we establish sufficient optimality conditions and duality results for a nonlinear programming problem.These duality relations lead to duality in nonlinear fractional programming.

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