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Applications Of Differential Subordination To Certain Subclasses Of Meromorphically Multivalent Functions Associated With Generalized Hypergeometric Function

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Abstract. By making use of the principle of differential subordination, we investigate several inclusion relationships and other interesting properties of certain subclasses of meromorphically multivalent functions which are defined by certain linear operator involving the generalized hypergeometric function.

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1. Introduction

For any integer $m > -p$, let $\sum_{p,m}$ denote the class of all meromorphic functions f of the form:

$$f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k \quad (p \in N = \{1, 2, \ldots\}), \tag{1}$$

which are analytic and p-valent in the punctured disc
$U^* = \{z : z \in C \text{ and } 0 < |z| < 1\} = U \backslash \{0\}$. For convenience, we write $\sum_{p,-p+1} = \sum_p$. If f and g are analytic in U, we say that f is subordinate to g, written symbolically as follows:

$$f \prec g \text{ or } f(z) \prec g(z),$$

if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1\,(z \in U)$ such that $f(z) = g(w(z))\,(z \in U)$.
In particular, if the function g is univalent in U, we have the equivalence (cf., e. g., [7]; see also [8, p. 4]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

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For functions $f \in \sum_{p,m}$, given by (1), and $g \in \sum_{p,m}$ defined by

$$g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k \quad (m > -p; p \in N), \tag{2}$$

then the Hadamard product (or convolution) of f and g is given by

$$(f * g) = z^{-p} + \sum_{k=m}^{\infty} a_k b_k z^k = (g * f)(z) \quad (m > -p; p \in N). \tag{3}$$

For complex parameters

$$\alpha_1, \ldots \alpha_q \ \beta_1, \ldots, \beta_s \quad (\beta_j \notin Z_0^- = \{0, -1, -2, \ldots\}; j = 1, 2, \ldots, s),$$

we now define the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ by (see, for example, [14, p.19])

$$\begin{gathered} {}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_q)_k}{(\beta_1)_k \ldots (\beta_s)_k} \cdot \frac{z^k}{k!} \\ (q \leq s+1; q, s \in N_0 = N \cup \{0\};\ z \in U), \end{gathered} \tag{4}$$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\theta)_\nu = \frac{\Gamma(\theta+\nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu = 0; \theta \in C \backslash \{0\}), \\ \theta(\theta-1) \ldots (\theta+\nu-1) & (\nu \in N; \theta \in C). \end{cases} \tag{5}$$

Corresponding to the function $h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$, defined by

$$h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = z^{-p} \, {}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z), \tag{6}$$

we consider a linear operator

$$H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) : \Sigma_p \to \Sigma_p,$$

which is defined by the following Hadamard product (or convolution):

$$H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) f(z) = h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) * f(z). \tag{7}$$

We observe that, for a function $f(z)$ of the form (1), we have

$$H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) f(z) = z^{-p} + \sum_{k=m}^{\infty} \frac{(\alpha_1)_{k+p} \ldots (\alpha_q)_{k+p}}{(\beta_1)_{k+p} \ldots (\beta_s)_{k+p}} \cdot \frac{a_k}{(k+p)!} z^k. \tag{8}$$

If, for convenience, we write

$$H_{p,q,s}(\alpha_1) = H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s), \tag{9}$$

then one can easily verify from the definition (7) that

$$z(H_{p,q,s}(\alpha_1)f(z))' = \alpha_1 H_{p,q,s}(\alpha_1+1)f(z) - (\alpha_1+p)H_{p,q,s}(\alpha_1)f(z). \tag{10}$$

For $m = -p+1\,(p \in N)$, the linear operator $H_{p,q,s}(\alpha_1)$ was investigated recently by Liu and Srivastava [5] and Aouf [1].

In particular, for $s = 1, q = 2, \alpha_1 > 0, \beta_1 > 0$ and $\alpha_2 = 1$, we obtain the linear operator $\ell_p(\alpha_1,\beta_1)f(z) = H_p(\alpha_1, 1; \beta_1)f(z)(f \in \sum_p)$, which was introduced and studied by Liu and Srivastava [4].

We note that, for any integer $n > -p$ and $f \in \sum_{p,m}$,

$$H_{p,2,1}(n+p,1;1)f(z) = D^{n+p-1}f(z) = \frac{1}{z^p(1-z)^{n+p}} * f(z)$$

where D^{n+p-1} is the differential operator studied by Uralegaddi and Somanatha [17].

Making use of the principle of differential subordination as well as the linear operator $H_{p,q,s}(\alpha_1)$, we now introduce a subclass of the function class $\sum_{p,m}$ as follows:

For fixed parameters A and $B(-1 \le B < A \le 1)$, we say that a function $f \in \sum_{p,m}$ is in the class $\sum_{p,q,s}^m(\alpha_1; A, B)$, if it satisfies the following subordination condition:

$$-\frac{z^{p+1}(H_{p,q,s}(\alpha_1)f(z))'}{p} \prec \frac{1+Az}{1+Bz}. \tag{11}$$

In view of the definition of subordination, (11) is equivalent to the following condition:

$$\left| \frac{z^{p+1}(H_{p,q,s}(\alpha_1)f(z))' + p}{Bz^{p+1}(H_{p,q,s}(\alpha_1)f(z))' + pA} \right| < 1 \quad (z \in U).$$

For convenience, we write

$$\Sigma_{p,q,s}^m(\alpha_1; 1 - \frac{2\zeta}{p}, 1) = \Sigma_{p,q,s}^m(\alpha_1; \zeta),$$

where $\Sigma_{p,q,s}^m(\alpha_1;\zeta)$ denotes the class of functions $f(z) \in \Sigma_{p,m}$ satisfying the following inequality:

$$\operatorname{Re}\left\{ -z^{p+1}(H_{p,q,s}(\alpha_1)f(z))' \right\} > \zeta \quad (0 \le \zeta < p; z \in U).$$

We note that $\Sigma_{p,q,s}^{-p+1}(\alpha_1; A + (B-A)\frac{\rho}{p}, B) = \Sigma_{p,q,s}(\alpha_1, A, B, \rho), 0 \le \rho < p; p \in N)$, where the class $\Sigma_{p,q,s}(\alpha_1, A, B, \rho)$ was introduced and studied by Aouf [1]. We also observe that:

(i) $\sum_{p,2,1}^{-p+1}(n+p,1;1;A,B) = C_{n,p}(A,B)$ $(n > -p; p \in N; -1 \le B < A \le 1)$, is the subclass of $\sum_p$ studied by Uralegaddi and Somanatha [17];

(ii) $\sum_{p,2,1}^{-p+1}(n+p,1;1;1-\frac{2\alpha}{p},-1) = \sum_{n,p}(\alpha)$ $(n > -p; p \in N; 0 \le \alpha < p)$, is the subclass of $\sum_p$ studied by Cho and Nunokawa [2];

(iii) For $q = 2$, $s = 1$, $\alpha_1 = a > 0$, $\beta_1 = c > 0$ and $\alpha_2 = 1$, we have

$$\Sigma_{a,c}(p;m,A,B) = \left\{ f(z) \in \Sigma_{p,m} : -\frac{z^{p+1}(\ell_p(a,c)f(z))'}{p} \prec \frac{1+Az}{1+Bz}, -1 \le B < A \le 1, z \in U \right\}, \tag{12}$$

where the class $\Sigma_{a,c}(p;m,A,B)$ was studied by Patel and Cho [12].

2. Preliminary Lemmas

To establish our main results, we need the following lemmas.

Lemma 1 ([3]). *Let the function h be analytic and convex (univalent) in U with $h(0) = 1$. Suppose also that the function φ given by*

$$\varphi(z) = 1 + c_{p+m}z^{p+m} + c_{p+m+1}z^{p+m+1} + \dots \tag{13}$$

in analytic in U. If

$$\varphi(z) + \frac{z\varphi'(z)}{\gamma} \prec h(z) \quad (\mathrm{Re}(\gamma) \ge 0; \gamma \ne 0), \tag{14}$$

then

$$\varphi(z) \prec \psi(z) = \frac{\gamma}{p+m} z^{\frac{-\gamma}{p+m}} \int_0^z t^{\frac{\gamma}{p+m}-1} h(t)dt \prec h(z),$$

and ψ is the best dominant of (14).

With a view to starting a well-known result (Lemma 2 below), we denote by $P(\gamma)$ the class of functions φ given by

$$\varphi(z) = 1 + b_1 z + b_2 z^2 + \dots, \tag{15}$$

which are analytic in U and satisfy the following inequality:

$$\mathrm{Re}\left\{\varphi(z)\right\} > \gamma \quad (0 \le \gamma < 1; z \in U).$$

Lemma 2 ([10]). *Let the function φ, given by (15), be in the class $P(\gamma)$. Then*

$$\mathrm{Re}\left\{\varphi(z)\right\} \ge 2\gamma - 1 + \frac{2(1-\gamma)}{1+|z|} \quad (0 \le \gamma < 1; z \in U).$$

Lemma 3 ([16]). *For $0 \le \gamma_1, \gamma_2 < 1$, we have*

$$P(\gamma_1) * P(\gamma_2) \subset P(\gamma_3) \quad (\gamma_3 = 1 - 2(1-\gamma_1)(1-\gamma_2)).$$

The result is the best possible.

For real or complex numbers a, b and c $(c \notin Z_0^-)$, the Gaussian hypergeometric function is defined by

$$ {}_2F_1(a,b;c;z) = 1 + \frac{ab}{c} \cdot \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \cdot \frac{z^2}{z!} + \ldots. $$

We note that the above series converges absolutely for $z \in U$ and hence represents an analytic function in U (see, for details [18, Chapter 14]).

Each of the identities (asserted by Lemma 4 below) is well-known (cf., e.g., [18, Chapter 14]).

Lemma 4 ([18]). *For real or complex parameters a, b and c $(c \notin Z_0^-)$,*

$$ \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\, {}_2\Gamma_1(a,b;c;z) \quad (\mathrm{Re}(c) > \mathrm{Re}(b) > 0); \tag{16} $$

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a}\, {}_2F_1(a,b;c;\frac{z}{z-1}); \tag{17} $$

$$ {}_2F_1(a,b;c;z) = {}_2F_1(a,b-1;c;z) + \frac{az}{c}\, {}_2F_1(a+1,b;c+1;z); \tag{18} $$

$$ {}_2F_1(a,b;\frac{a+b+1}{2};\frac{1}{2}) = \frac{\sqrt{\pi}\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})}. \tag{19} $$

Lemma 5 ([13]). *Let Φ be analytic in U with*

$$ \Phi(0) = 1 \text{ and } \mathrm{Re}\{\Phi(z)\} > \frac{1}{2} \quad (z \in U). $$

Then, for any function F analytic in U, $(\Phi * F)(U)$ is contained in the convex hull of $F(U)$.

3. Main Results

Remark 1. *Throughout our present paper, we assume that:*

$$ -1 \le B < A \le 1, \lambda > 0, p \in N \text{ and } \alpha_1 \in C\backslash\{0\}. $$

Theorem 1. *Let the function f defined by (1) satisfying the following subordination condition:*

$$ -\frac{(1-\lambda)z^{p+1}(H_{p,q,s}(\alpha_1)f(z))' + \lambda z^{p+1}(H_{p,q,s}(\alpha_1+1)f(z))'}{p} \prec \frac{1+Az}{1+Bz}. $$

Then

$$ -\frac{z^{p+1}(H_{p,q,s}(\alpha_1)f(z))'}{p} \prec Q(z) \prec \frac{1+Az}{1+Bz}, \tag{20} $$

where the function Q given by

$$Q(z)=\begin{cases} \frac{A}{B}+(1-\frac{A}{B})(1+Bz)^{-1}\,{}_2F_1(1,1;\frac{\alpha_1}{\lambda(p+m)}+1;\frac{Bz}{1+Bz}) & (B\neq 0)\\ 1+\frac{\alpha_1 A}{\lambda(p+m)+\alpha_1}z & (B=0)\end{cases}$$

is the best dominant of (20). Furthermore,

$$\operatorname{Re}\left\{-\frac{z^{p+1}(H_{p,q,s}(\alpha_1)f(z))'}{p}\right\}>\xi \quad (z\in U), \tag{21}$$

where

$$\xi=\begin{cases} \frac{A}{B}+(1-\frac{A}{B})(1-B)^{-1}\,{}_2F_1(1,1;\frac{\alpha_1}{\lambda(p+m)}+1;\frac{B}{B-1}) & (B\neq 0)\\ 1-\frac{\alpha_1 A}{\lambda(p+m)+\alpha_1} & (B=0).\end{cases}$$

The estimate in (21) is the best possible.

Proof. Consider the function φ defined by

$$\varphi(z)=-\frac{z^{p+1}(H_{p,q,s}(\alpha_1)f(z))'}{p} \quad (z\in U). \tag{22}$$

Then φ is of the form (13) and is analytic in U. Differentiating (22) with respect to z and using (10), we obtain

$$-\frac{(1-\lambda)z^{p+1}(H_{p,q,s}(\alpha_1)f(z))'+\lambda z^{p+1}(H_{p,q,s}(\alpha_1)f(z))'}{p}$$
$$=\varphi(z)+\frac{\lambda}{\alpha_1}z\varphi'(z)\prec\frac{1+Az}{1+Bz} \quad (z\in U).$$

Now, by using Lemma 1 for $\beta=\frac{\alpha_1}{\lambda}$, we obtain

$$-\frac{z^{p+1}(H_{p,q,s}(\alpha_1)f(z))'}{p}\prec Q(z)$$
$$=\frac{\alpha_1}{\lambda(p+m)}z^{-\frac{\alpha_1}{\lambda(p+m)}}\int_0^z t^{\frac{\alpha_1}{\lambda(p+m)}-1}\left(\frac{1+At}{1+Bt}\right)dt$$
$$=\begin{cases} \frac{A}{B}+(1-\frac{A}{B})(1+Bz)^{-1}\,{}_2F_1(1,1;\frac{\alpha_1}{\lambda(p+m)}+1;\frac{Bz}{1+Bz}) & (B\neq 0)\\ 1+\frac{\alpha_1 A}{\lambda(p+m)+\alpha_1}z & (B=0),\end{cases}$$

by change of variables followed by the use of the identities (16), (17) and (18) (with $a = 1$, $c = b + 1$, $b = \frac{\alpha_1}{\lambda(p+m)}$). This proves the assertion (20) of Theorem 1.

Next, in order to prove the assertion (21) of Theorem 1, it suffices to show that

$$\inf_{|z|<1} \{\mathrm{Re}(Q(z))\} = Q(-1). \tag{23}$$

Indeed we have, for $|z| \leq r < 1$,

$$\mathrm{Re}\left(\frac{1+Az}{1+Bz}\right) \geq \frac{1-Ar}{1-Br}.$$

Upon setting

$$g(\zeta,z) = \frac{1+A\zeta z}{1+B\zeta z} \quad \text{and} \quad d\nu(\zeta) = \frac{\alpha_1}{\lambda(p+m)}\zeta^{\frac{\alpha_1}{\lambda(p+m)}-1}d\zeta \quad (0 \leq \zeta \leq 1),$$

which is a positive measure on the closed interval $[0,1]$, we get

$$Q(z) = \int_0^1 g(\zeta,z)d\nu(\zeta),$$

so that

$$\mathrm{Re}\{Q(z)\} \geq \int_0^1 \left(\frac{1-A\zeta r}{1-B\zeta r}\right) d\nu(\zeta) = Q(-r) \quad (|z| \leq r < 1).$$

Letting $r \to 1^-$ in the above inequality, we obtain the assertion (21) of Theorem 1.

Finally, the estimate in (21) is the best possible as the function Q is the best dominant of (20).

Taking $\lambda = 1$, $A = 1 - \frac{2\sigma}{p}$ $(0 \leq \sigma < p)$ and $B = -1$ in Theorem 1, we obtain the following corollary.

Corollary 1. *The following inclusion property holds true for the function class $\Sigma^m_{p,q,s}(\alpha_1;\sigma)$:*

$$\Sigma^m_{p,q,s}(\alpha_1+1;\sigma) \subset \Sigma^m_{p,q,s}(\alpha_1;\beta(p,m,\alpha_1,\sigma)) \subset \Sigma^m_{p,q,s}(\alpha_1;\sigma),$$

where

$$\beta(p,m,\alpha_1,\sigma) = \sigma + (p-\sigma)\left\{{}_2F_1(1,1;\frac{\alpha_1}{p+m}+1;\frac{1}{2}) - 1\right\}.$$

The result is the best possible.

Taking $\lambda = 1$ and $m = 1 - p$ $(p \in N)$ in Theorem 1, we obtain the following corollary.

Corollary 2. *The following inclusion property holds true for the function class $\Sigma_{p,q,s}(\alpha_1;A,B)$:*

$$\Sigma_{p,q,s}(\alpha_1+1;A,B) \subset \Sigma_{p,q,s}(\alpha_1;1-\frac{2\sigma}{p},-1) \subset \Sigma_{p,q,s}(\alpha_1;A,B),$$

where

$$\sigma = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1+B)^{-1}\,{}_2F_1(1,1;\alpha_1+1;\frac{B}{B-1}) & (B \neq 0) \\ 1 - \frac{\alpha_1 A}{1+\alpha_1} & (B = 0). \end{cases}$$

The result is the best possible.

Remark 2.

(i) *Taking $\lambda = 1$, $q = 2$, $s = 1$, $\alpha_1 = a$, $\beta_1 = c\,(a > 0; c > 0)$ and $\alpha_2 = 1$ in Theorem 1, we obtain the result obtained by Patel and Cho [12, Theorem 1];*

(ii) *Taking $m = -p+1$, $\lambda = 1$, $q = 2$, $s = 1$, $\alpha_1 = n+p(n > -p)$, $\alpha_2 = \beta_1 = 1$ in Theorem 1, we obtain the result obtained by Patel and Cho [12, Corollary 2] which improves the corresponding result obtained by Uralegaddi and Somanatha [17];*

(iii) *Taking $q = 2$, $s = 1$, $\alpha_1 = a > 0$, $\beta_1 = c > 0$ and $\alpha_2 = 1$ in Corollary 2, we obtain the result obtained by Patel and Cho [12, Corollary 1].*

Theorem 2. *If $f \in \Sigma^m_{p,q,s}(\alpha_1;\theta)\,(0 \leq \theta < p)$, then*

$$\operatorname{Re}\left\{-z^{p+1}\left[(1-\lambda)(H_{p,q,s}(\alpha_1)f(z))^{'} + \lambda(H_{p,q,s}(\alpha_1+1)f(z))^{'}\right]\right\} > \theta \quad (|z| < R), \tag{24}$$

where

$$R = \left\{\frac{\sqrt{\alpha_1^2+\lambda^2(p+m)^2}-\lambda(p+m)}{\alpha_1}\right\}^{\frac{1}{p+m}}.$$

The result is the best possible.

Proof. Since $f \in \Sigma^m_{p,q,s}(\alpha_1;\theta)$, we write

$$-z^{p+1}(H_{p,q,s}(\alpha_1)f(z))^{'} = \theta + (p-\theta)u(z) \quad (z \in U). \tag{25}$$

Then, clearly, u is of the form (13), is analytic in U, and has a positive real part in U. Differentiating (25) with respect to z and using (10), we obtain

$$-\frac{z^{p+1}\left[(1-\lambda)(H_{p,q,s}(\alpha_1)f(z))^{'} + \lambda(H_{p,q,s}(\alpha_1+1)f(z))^{'}\right]+\theta}{p-\theta} = u(z) + \frac{\lambda}{\alpha_1}zu^{'}(z). \tag{26}$$

Now, by applying the well-known estimate [6]

$$\frac{\left|zu^{'}(z)\right|}{\operatorname{Re}\{u(z)\}} \leq \frac{2(p+m)r^{p+m}}{1-r^{2(p+m)}} \quad (|z| = r < 1)$$

in (26), we obtain

$$\operatorname{Re}\left\{-\frac{z^{p+1}\left[(1-\lambda)(H_{p,q,s}(\alpha_1)f(z))^{'}+\lambda(H_{p,q,s}(\alpha_1+1)f(z))^{'}\right]+\theta}{p-\theta}\right\}$$
$$\geq \operatorname{Re}\{u(z)\}\cdot\left(1-\frac{2\lambda(p+m)r^{p+m}}{\alpha_1(1-r^{2(p+m)})}\right). \tag{27}$$

It is easily seen that the right-hand side of (27) is positive provided that $r < R$, where R is given as in Theorem 2. This proves the assertion (24) of Theorem 2.

In order to show that the bound R is the best possible, we consider the function $f \in \Sigma_{p,m}$ defined by

$$-z^{p+1}(H_{p,q,s}(\alpha_1)f(z))^{'} = \theta + (p-\theta)\frac{1+z^{p+m}}{1-z^{p+m}}\ (0 \leq \theta < p; p \in N; z \in U).$$

Noting that

$$-\frac{z^{p+1}\left[(1-\lambda)(H_{p,q,s}(\alpha_1)f(z))^{'}+\lambda(H_{p,q,s}(\alpha_1+1)f(z))^{'}\right]+\theta}{p-\theta}$$
$$= \frac{\alpha_1-\alpha_1 z^{2(p+m)}+2\lambda(p+m)z^{p+m}}{\alpha_1(1-z^{p+m})^2} = 0$$

for $z = R^{\frac{1}{p+m}}\exp\left(\frac{i\pi}{p+m}\right)$, we complete the proof of Theorem 2.

Putting $\lambda = 1$ in Theorem 2, we obtain the following result.

Corollary 3. *If $f \in \Sigma^m_{p,q,s}(\alpha_1;\theta)(0 \leq \theta < p; p \in N)$, then $f \in \Sigma^m_{p,q,s}(\alpha_1+1;\theta)$ for $|z| < R^*$, where*

$$R^* = \left\{\frac{\sqrt{\alpha_1^2+(p+m)^2}-(p+m)}{\alpha_1}\right\}^{\frac{1}{p+m}}.$$

The result is the best possible.

Remark 3. *Taking $s = 1$, $q = 2$, $\alpha_1 = a$ and $\beta_1 = c\,(a > 0; c > 0)$ and $\alpha_2 = 1$ in Corollary 3, we obtain the result obtained by Patel and Cho [12, Theorem 2].*

Theorem 3. *Let* $f \in \Sigma_{p,q,s}^{m}(\alpha_1; A, B)$ *and let*

$$F_{\delta,p}(f)(z) = \frac{\delta}{z^{\delta+p}} \int_0^z t^{\delta+p-1} f(t)dt \quad (\delta > 0; z \in U). \tag{28}$$

Then

$$-\frac{z^{p+1}(H_{p,q,s}(\alpha_1)F_{\delta,p}f(z))'}{p} \prec \Phi(z) \prec \frac{1+Az}{1+Bz}, \tag{29}$$

where the function Φ *given by*

$$\Phi(z) = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1+Bz)^{-1}\,{}_2F_1(1,1;\frac{\delta}{p+m}+1;\frac{Bz}{Bz+1}) & (B \neq 0) \\ 1 + \frac{\delta}{\delta+p+m}Az & (B = 0), \end{cases}$$

is the best dominant of (29). Furthermore,

$$\mathrm{Re}\left\{-\frac{z^{p+1}(H_{p,q,s}(\alpha_1)F_{\delta,p}(f)(z))'}{p}\right\} > \xi^* \quad (z \in U), \tag{30}$$

where

$$\xi^* = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1-B)^{-1}\,{}_2F_1(1,1;\frac{\delta}{p+m}+1;\frac{B}{B-1}) & (B \neq 0) \\ 1 - \frac{\delta}{\delta+p+m}A & (B = 0). \end{cases}$$

The result is the best possible.

Proof. Defining the function φ by

$$\varphi(z) = -\frac{z^{p+1}(H_{p,q,s}(\alpha_1)F_{\delta,p}(f)(z))'}{p} \quad (z \in U), \tag{31}$$

we note that φ is of the form (13) and is analytic in U. Using the following operator identity:

$$z(H_{p,q,s}(\alpha_1)F_{\delta,p}(f)(z))' = \delta H_{p,q,s}(\alpha_1)f(z) - (\delta+p)H_{p,q,s}(\alpha_1)F_{\delta,p}(f)(z) \tag{32}$$

in (31) and differentiating the resulting equation with respect to z, we find that

$$-\frac{z^{p+1}(H_{p,q,s}(\alpha_1)f(z))'}{p} \prec \varphi(z) + \frac{z\varphi'(z)}{\delta} \prec \frac{1+Az}{1+Bz}.$$

Now the remaining part of Theorem 3 follows by employing the techniques that we used in proving Theorem 1 above.

Putting $m = 1-p$ $(p \in N)$ in Theorem 3, we obtain the following corollary.

Corollary 4. *If $\delta > 0$ and $f \in \Sigma_{p,q,s}(\alpha_1; A, B)$, then*

$$F_{\delta,p}(f)(z) \in \Sigma_{p,q,s}(\alpha_1; 1 - \frac{2\xi}{p}, -1) \subset \Sigma_{p,q,s}(\alpha_1; A, B),$$

where

$$\xi = \begin{cases} \frac{A}{B} + (1 - \frac{A}{B})(1+B)^{-1} {}_2F_1(1, 1; \delta + 1; \frac{B}{B-1}) & (B \neq 0) \\ 1 - \frac{\delta}{\delta+1} A & (B = 0). \end{cases}$$

The result is the best possible.

Remark 4. *By observing that*

$$z^{p+1}(H_{p,q,s}(\alpha_1)F_{\delta,p}(f)(z))' = \frac{\delta}{z^{\delta}} \int_0^z t^{\delta+p}(H_{p,q,s}(\alpha_1)f(t))' dt \, (f \in \Sigma_{p,m}; z \in U), \tag{33}$$

Corollary 4 can be restated as follows:
If $\delta > 0$ and $f \in \Sigma_{p,q,s}(\alpha_1; A, B)$, then

$$\operatorname{Re}\left\{ -\frac{\delta}{pz^{\delta}} \int_0^z t^{\delta+p}(H_{p,q,s}(\alpha_1)f(t))' dt \right\} > \xi \quad (z \in U).$$

where ξ is given as in Corollary 4.
In view of (33), Theorem 3 for $A = 1 - \frac{2\theta}{p}$ $(0 \leq \theta < p; p \in N)$ and $B = -1$ yields

Corollary 5. *If $\delta > 0$ and if $f \in \Sigma_{p,m}$ satisfies the following inequality:*

$$\operatorname{Re}\left\{ -z^{p+1}(H_{p,q,s}(\alpha_1)f(z))' \right\} > \theta \quad (0 \leq \theta < p; p \in N; z \in U),$$

then

$$\operatorname{Re}\left\{ \frac{-\delta}{z^{\delta}} \int_0^z (H_{p,q,s}(\alpha_1)f(t))' dt \right\} > \theta + (p - \theta)\left[{}_2F_1(1, 1; \frac{\delta}{p+m} + 1; \frac{1}{2}) - 1 \right] \quad (z \in U).$$

The result is the best possible.

Remark 5. *Putting $s = 1$, $q = 2$, $\alpha_1 = a$, $\beta_1 = c\,(a > 0; c > 0)$ and $\alpha_2 = 1$ in Theorem 3, we obtain the result obtained by Patel and Cho [12, Theorem 3].*

Theorem 4. *Let $f \in \Sigma_{p,m}$. Suppose also that $g \in \Sigma_{p,m}$ satisfies the following inequality:*

$$\operatorname{Re}\left\{z^p(H_{p,q,s}(\alpha_1)g(z))\right\} > 0 \quad (z \in U).$$

If

$$\left|\frac{H_{p,q,s}(\alpha_1)f(z)}{H_{p,q,s}(\alpha_1)g(z)} - 1\right| < 1 \quad (z \in U),$$

then

$$\operatorname{Re}\left\{-\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)}\right\} > 0 \quad (|z| < R_0),$$

where

$$R_0 = \frac{\sqrt{g(p+m)^2 + 4p(2p+m)} - 3(p+m)}{2(2p+m)}.$$

Proof. Letting

$$w(z) = \frac{H_{p,q,s}(\alpha_1)f(z)}{H_{p,q,s}(\alpha_1)g(z)} - 1 = t_{p+m}z^{p+m} + t_{p+m+1}z^{p+m+1} + \ldots \tag{34}$$

we note that w is analytic in U, with $w(0) = 0$ and $|w(z)| \leq |z|^{p+m}$ $(z \in U)$. Then, by applying the familiar Schwarz lemma [9], we obtain

$$w(z) = z^{p+m}\Psi(z),$$

where the functions Ψ is analytic in U and $|\Psi(z)| \leq 1\,(z \in U)$. Therefore, (34) leads us to

$$H_{p,q,s}(\alpha_1)f(z) = H_{p,q,s}(\alpha_1)g(z)\left(1 + z^{p+m}\Psi(z)\right) \quad (z \in U). \tag{35}$$

Differentiating (35) logarithmically with respect to z, we obtain

$$\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)} = \frac{z(H_{p,q,s}(\alpha_1)g(z))'}{H_{p,q,s}(\alpha_1)g(z)} + \frac{z^{p+m}\left\{(p+m)\Psi(z) + z\Psi'(z)\right\}}{1 + z^{p+m}\Psi(z)}. \tag{36}$$

Putting $\varphi(z) = z^p H_{p,q,s}(\alpha_1)g(z)$, we see that the function φ is of the form (13), is analytic in U, $\operatorname{Re}\{\varphi(z)\} > 0\,(z \in U)$ and

$$\frac{z(H_{p,q,s}(\alpha_1)g(z))'}{H_{p,q,s}(\alpha_1)g(z)} = \frac{z\varphi'(z)}{\varphi(z)} - p,$$

so that we find from (36) that

$$\operatorname{Re}\left\{-\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)}\right\} \geq p - \left|\frac{z\varphi'(z)}{\varphi(z)}\right| - \left|\frac{z^{p+m}\left\{(p+m)\Psi(z) + z\Psi'(z)\right\}}{1 + z^{p+m}\Psi(z)}\right| \quad (z \in U). \tag{37}$$

Now, by using the following known estimates [11] (see also [6]):

$$\left|\frac{\varphi'(z)}{\varphi(z)}\right| \leq \frac{2(p+m)r^{p+m-1}}{1-r^{2(p+m)}} \quad (|z|=r<1)$$

and

$$\left|\frac{(p+m)\Psi(z)+z\Psi'(z)}{1+z^{p+m}\Psi(z)}\right| \leq \frac{(p+m)}{1-r^{p+m}} \quad (|z|=r<1)$$

in (37), we obtain

$$\mathrm{Re}\left\{-\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)}\right\} \geq \frac{p-3(p+m)r^{p+m}-(2p+m)r^{2(p+m)}}{1-r^{2(p+m)}} \quad (|z|=r<1),$$

which is certainly positive, provided that $r<R_0$, R_0 being given as in Theorem 4.

Theorem 5. *Let $-1 \leq B_j < A_j \leq 1\,(j=1,2)$. If each of the functions $f_j \in \Sigma_p$ satisfies the following subordination condition:*

$$(1-\lambda)z^p H_{p,q,s}(\alpha_1)f_j(z)+\lambda z^p H_{p,q,s}(\alpha_1+1)f_j(z) \prec \frac{1+A_j z}{1+B_j z}\,(j=1,2;z\in U), \tag{38}$$

then

$$(1-\lambda)z^p H_{p,q,s}(\alpha_1)G(z)+\lambda z^p H_{p,q,s}(\alpha_1+1)G(z) \prec \frac{1+(1-2\zeta)z}{1-z} \quad (z\in U), \tag{39}$$

where

$$G(z)=H_{p,q,s}(\alpha_1) \quad (f_1 * f_2)(z)$$

and

$$\zeta = 1-\frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1-\frac{1}{2}\,{}_2F_1(1,1;\frac{\alpha_1}{\lambda}+1;\frac{1}{2})\right].$$

The result is the best possible when $B_1=B_2=-1$.

Proof. Suppose that each of the functions $f_j \in \Sigma_p\,(j=1,2)$ satisfies the condition (38). Then, by letting

$$\phi_j(z)=(1-\lambda)z^p H_{p,q,s}(\alpha_1)f_j(z)+\lambda z^p H_{p,q,s}(\alpha_1+1)f_j(z) \quad (j=1,2), \tag{40}$$

we have

$$\varphi_j(z)\in P(\gamma_j) \quad (\gamma_j=\frac{1-A_j}{1-B_j};\ j=1,2).$$

Using the identity (10) in (40), we observe that

$$H_{p,q,s}(\alpha_1)f_j(z)=\frac{\alpha_1}{\lambda}z^{-p-\frac{\alpha_1}{\lambda}}\int_0^z t^{\frac{\alpha_1}{\lambda}-1}\phi_j(t)dt \quad (j=1,2),$$

which, in view of the definition of G given already with (39), yields

$$H_{p,q,s}(\alpha_1)G(z)=\frac{\alpha_1}{\lambda}z^{-p-\frac{\alpha_1}{\lambda}}\int\limits_0^z t^{\frac{\alpha_1}{\lambda}-1}\varphi_0(t)dt, \tag{41}$$

where, for convenience,

$$\begin{aligned}\phi_0(z) &= (1-\lambda)z^p H_{p,q,s}(\alpha_1)G(z)+\lambda z^p H_{p,q,s}(\alpha_1+1)G(z)\\ &= \frac{\alpha_1}{\lambda}z^{-\frac{\alpha_1}{\lambda}}\int\limits_0^z t^{\frac{\alpha_1}{\lambda}-1}\,(\varphi_1*\varphi_2)(t)dt. \end{aligned}\tag{42}$$

Since $\varphi_1\in P(\gamma_1)$ and $\varphi_2\in P(\gamma_2)$, it follows from Lemma 3 that

$$(\varphi_1*\varphi_2)\in P(\gamma_3)\quad (\gamma_3=1-2(1-\gamma_1)(1-\gamma_2)). \tag{43}$$

Now, by using (43) in (42) and then appealing to Lemma 2 and Lemma 4, we obtain

$$\begin{aligned}\operatorname{Re}\{\varphi_0(z)\} &= \frac{\alpha_1}{\lambda}\int\limits_0^1 u^{\frac{\alpha_1}{\lambda}-1}\operatorname{Re}\{\varphi_1*\varphi_2\}(uz)du\\ &\geq \frac{\alpha_1}{\lambda}\int\limits_0^1 u^{\frac{\alpha_1}{\lambda}-1}(2\gamma_3-1+\frac{2(1-\gamma_3)}{1+u|z|})du\\ &> \frac{\alpha_1}{\lambda}\int\limits_0^1 u^{\frac{\alpha_1}{\lambda}-1}(2\gamma_3-1+\frac{2(1-\gamma_3)}{1+u})du\\ &= 1-\frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\quad(1-\frac{\alpha_1}{\lambda}\int\limits_0^1 u^{\frac{\alpha_1}{\lambda}-1}(1+u)^{-1}du)\\ &= 1-\frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1-\frac{1}{2}\,{}_2F_1(1,1;\frac{\alpha_1}{\lambda}+1;\frac{1}{2})\right]\\ &= \zeta\quad (z\in U).\end{aligned}$$

When $B_1=B_2=-1$, we consider the functions $f_j\in\Sigma_p\,(j=1,2)$, which satisfy the hypothesis (38) of Theorem 5 and are defined by

$$H_{p,q,s}(\alpha_1)f_j(z)=\frac{\alpha_1}{\lambda}z^{-\frac{\alpha_1}{\lambda}}\int\limits_0^z t^{\frac{\alpha_1}{\lambda}-1}(\frac{1+A_jt}{1-t})dt\quad (j=1,2).$$

Thus it follows from (42) and Lemma 4 that

$$\begin{aligned}\varphi_0(z) &= \frac{\alpha_1}{\lambda}\int_0^1 u^{\frac{\alpha_1}{\lambda}-1}\left\{1-(1+A_1)(1+A_2)+\frac{(1+A_1)(1+A_2)}{1-uz}\right\}du\\ &= 1-(1+A_1)(1+A_2)+(1+A_1)(1+A_2)(1-z)^{-1}\cdot{}_2F_1(1,1;\frac{\alpha_1}{\lambda}+1;\frac{z}{z-1})\\ &\to 1-(1+A_1)(1+A_2)+\frac{1}{2}(1+A_1)(1+A_2)\cdot{}_2F_1(1,1;\frac{\alpha_1}{\lambda}+1;\frac{1}{2}) \text{ as } z\to -1,\end{aligned}$$

which evidently completes the proof of Theorem 5.

Putting $A_j = 1-2\theta_j$, $B_j = -1\,(j=1,2; 0\le\theta_j<1)$, $s=1$, $q=2$, $\alpha_1 = a>0$, $\beta_1 = c>0$ and $\alpha_2 = 1$ in Theorem 5, we obtain the following corollary.

Corollary 6. *If the functions $f_j \in \Sigma_p\,(j=1,2)$ satisfy the following inequality:*

$$\begin{aligned}&\operatorname{Re}\left\{(1+\lambda p)z^p\ell_p(a,c)f_j(z)+\lambda z^{p+1}\left(\ell_p(a,c)f_j(z)\right)'\right\}\\ &\quad > \theta_j \quad (0\le\theta_j<1; j=1,2; z\in U),\end{aligned} \tag{44}$$

then

$$\operatorname{Re}\left\{(1+\lambda p)z^p\ell_p(a,c)(f_1*f_2)(z)+\lambda z^{p+1}(\ell_p(a,c)(f_1*f_2)(z))'\right\} > \eta_0 \quad (z\in U),$$

where

$$\eta_0 = 1-4(1-\theta_1)(1-\theta_2)\left[1-\frac{1}{2}\,{}_2F_1(1,1;\frac{a}{\lambda}+1;\frac{1}{2})\right].$$

The result is the best possible.

Choosing $A_j = 1-2\phi_j$, $B_j = 1\,(j=1,2; 0\le\phi_j<1)$, $q=s+1$, $\alpha_1=\beta_1=p$, $\alpha_j = 1\,(j=2,3,\ldots,s+1)$ and $\beta_j = 1\,(j=2,3,\ldots,s)$ in Theorem 5, we obtain the following result which refines the work of Yang [19, Theorem 4] and the work of Srivastava and Patel [15, Corollary 6].

Corollary 7. *If the functions $f_j\in\Sigma_p\,(j=1,2)$ satisfy the following inequality:*

$$\operatorname{Re}\left\{(1+\lambda)z^p f_j(z)+\frac{\lambda}{p}z^{p+1}f_j'(z)\right\} > \phi_j\,(0\le\phi_j<1; j=1,2; z\in U), \tag{45}$$

then

$$\operatorname{Re}\left\{(1+\lambda)z^p(f_1*f_2)(z)+\frac{\lambda}{p}z^{p+1}(f_1*f_2)(z))'\right\} > \rho_0 \quad (z\in U),$$

where

$$\rho_0 = 1-4(1-\phi_1)(1-\phi_2)\left[1-\frac{1}{2}\,{}_2F_1(1,1;\frac{p}{\lambda}+1;\frac{1}{2})\right].$$

The result is the best possible.

Theorem 6. *If $f \in \Sigma_{p,m}$ satisfies the following subordination condition:*

$$(1-\lambda)z^{p}H_{p,q,s}(\alpha_1)f(z)+\lambda z^{p}H_{p,q,s}(\alpha_1+1)f(z)\prec \frac{1+Az}{1+Bz},$$

then

$$\operatorname{Re}\left\{z^{p}H_{p,q,s}(\alpha_1)f(z)\right\}^{\frac{1}{d}}>\xi^{\frac{1}{d}}\quad (d\in N; z\in U),$$

where ξ is given as in Theorem 1. The result is the best possible.

Proof. Defining the function φ by

$$\varphi(z)=z^{p}H_{p,q,s}(\alpha_1)f(z)\quad (f\in\Sigma_{p,m}; z\in U), \tag{46}$$

we see that the function φ is of the form (13) and is analytic in U. Differentiating (46) with respect to z and using the identity (10), we obtain

$$(1-\lambda)z^{p}H_{p,q,s}(\alpha_1)f(z)+\lambda z^{p}H_{p,q,s}(\alpha_1+1)f(z)=\varphi(z)+\frac{\lambda}{\alpha_1}z\varphi^{'}(z)\prec\frac{1+Az}{1+Bz}.$$

Now, by following the lines of the proof of Theorem 1 mutates mutandis, and using the elementary inequality:

$$\operatorname{Re}\left(w^{\frac{1}{d}}\right)\geq(\operatorname{Re}w)^{\frac{1}{d}}\quad (\operatorname{Re}(w)>0; d\in N),$$

we arrive at the result asserted by Theorem 6.

Putting

$$A=\left[{}_2F_1(1,1;\frac{\alpha_1}{\lambda(p+m)}+1;\frac{1}{2})-1\right]\cdot\left[2-{}_2F_1(1,1;\frac{\alpha_1}{\lambda(p+m)}+1;\frac{1}{2})\right]^{-1},$$

$B=-1$, $s=1$, $q=2$, $\alpha_1=a>0, \beta_1=c>0$, $\alpha_2=1$ and $d=1$ in Theorem 6, we obtain the following corollary.

Corollary 8. *If $f\in\Sigma_{p,m}$ satisfies the following inequality:*

$$\operatorname{Re}\left\{(1+\lambda p)z^{p}\ell_p(a,c)f(z)+\lambda z^{p+1}(\ell_p(a,c)f(z))^{'}\right\}>\frac{3-2\,{}_2F_1(1,1;\frac{a}{\lambda(p+m)}+1;\frac{1}{2})}{2\left[2-{}_2F_1(1,1;\frac{a}{\lambda(p+m)}+1;\frac{1}{2})\right]}\quad z\in U), \tag{47}$$

then

$$\operatorname{Re}\left\{z^{p}\ell_p(a,c)f(z)\right\}>\frac{1}{2}\quad (z\in U).$$

The result is the best possible.

From Corollary 6 and Theorem 6 (for $m=-p+1, A=1-2\eta_0$, $B=-1$ and $d=1$), we obtain the following result.

Corollary 9. *If the function $f_j \in \Sigma_p\,(j=1,2)$ satisfy the inequality (44), then*

$$\operatorname{Re}\left\{z^p \ell_p(a,c)(f_1 * f_2)(z)\right\} > \eta_0 + (1-\eta_0)\left[{}_2F_1(1,1;\frac{a}{\lambda}+1;\frac{1}{2})-1\right] \quad (z \in U),$$

where η_0 is given as in Corollary 6. The result is the best possible.

Putting

$$A = \left[{}_2F_1(1,1;\frac{p}{\lambda(p+m)}+1;\frac{1}{2})-1\right]\cdot\left[2-{}_2F_1(1,1;\frac{p}{\lambda(p+m)}+1;\frac{1}{2})\right]^{-1},$$

$B=-1$, $q=s+1$, $\alpha_1=\beta_1=p$, $\alpha_j=1\,(j=2,3,\ldots,s+1)$, $\beta_j=1\,(j=2,2,\ldots,s)$ and $d=1$ in Theorem 6, we obtain the following result which refines the work of Srivastava and Patel [15, Corollary 7].

Corollary 10. *If $f \in \Sigma_{p,m}$ satisfies the following inequality:*

$$\operatorname{Re}\left\{(1+\lambda)z^p f(z)+\frac{\lambda}{p}z^{p+1}f'(z)\right\} > \frac{3-2\,{}_2F_1(1,1;\frac{p}{\lambda(p+m)}+1;\frac{1}{2})}{2\left[2-{}_2F_1(1,1;\frac{p}{\lambda(p+m)}+1;\frac{1}{2})\right]} \quad (z \in U), \tag{48}$$

then

$$\operatorname{Re}\left\{z^p f(z)\right\} > \frac{1}{2} \quad (z \in U).$$

The result is the best possible.

From Corollary 7 and Theorem 6 (for $m=-p+1$, $A=1-2\eta_0$, $B=-1$, $d=1$, $q=s+1$, $\alpha_1=\beta_1=p$, $\alpha_j=1\,(j=2,3,\ldots,s+1)$ and $\beta_j=1\,(j=2,3,\ldots,s)$), we deduce the following result.

Corollary 11. *If the functions $f_j \in \Sigma_p\,(j=1,2)$ satisfy the inequality (45), then*

$$\operatorname{Re}\left(z^p(f_1 * f_2)(z)\right) > \rho_0 + (1-\rho_0)\left[{}_2F_1\left(1,1;\frac{p}{\lambda}+1;\frac{1}{2}\right)-1\right] \quad (z \in U),$$

where ρ_0 is given as in Corollary 7. The result is the best possible.

Theorem 7. *Let $f \in \Sigma^m_{p,q,s}(\alpha_1;A,B)$ and let $g \in \Sigma_{p,m}$ satisfy the following inequality:*

$$\operatorname{Re}\left\{z^p g(z)\right\} > \frac{1}{2} \quad (z \in U).$$

Then

$$(f * g) \in \Sigma^m_{p,q,s}(\alpha_1;A,B).$$

Proof. We have

$$-\frac{z^{p+1}(H_{p,q,s}(\alpha_1)(f*g)(z))'}{p} = -\frac{z^{p+1}(H_{p,q,s}(\alpha_1)f(z))'}{p} * z^p g(z) \quad (z \in U).$$

Since

$$\mathrm{Re}\left\{z^p g(z)\right\} > \frac{1}{2} \quad (z \in U)$$

and the function

$$\frac{1+Az}{1+Bz}$$

is convex (univalent) in U, it follows from (11) and Lemma 5 that $(f*g)(z) \in \Sigma^m_{p,q,s}(\alpha_1; A, B)$. This completes the proof of Theorem 7.

In view of Corollary 10 and Theorem 7, we have Corollary 11 below.

Corollary 12. *If $f \in \Sigma^m_{p,q,s}(\alpha_1; A, B)$ and the function $g \in \Sigma_{p,m}$ satisfies the inequality (48), then $(f*g) \in \Sigma^m_{p,q,s}(\alpha_1; A, B)$.*

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