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Generalised Hyers-Ulam Product-Sum Stability of a Cauchy Type Additive Functional Equation

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Abstract. In 1940 (and 1964) S.M. Ulam proposed the well-known Ulam stability problem. In 1941 D.H. Hyers solved the Hyers-Ulam problem for linear mappings. In 2008, J. M. Rassias introduced the generalised Hyers-Ulam "product-sum" stability. In this paper we introduce a Cauchy type additive functional equation and investigate the generalised Hyers-Ulam "product-sum" stability of this equation.

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1. Introduction and Preliminaries

In 1940 (and 1964) Stanislaw M. Ulam [9] proposed the following stability problem, well-known as *Ulam stability problem*:

> "When is true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?"

In particular he stated the stability question:

> "Let G_1 be a group and G_2 a metric group with the metric $\rho(.,.)$. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies $\rho(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then a unique homomorphism $h : G_1 \to G_2$ exists with $\rho(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In 1941 D.H. Hyers [2] solved this problem for linear mappings as follows:

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Theorem 1 (D.H. Hyers, 1941: [2]). *If a mapping $f : E \to E'$ satisfies the approximately additive inequality*

$$||f(x+y) - f(x) - f(y)|| \leq \varepsilon,$$

for some fixed $\varepsilon > 0$ and all $x, y \in E$, where E and E' are Banach spaces, then there exists a unique additive mapping $A : E \to E'$, satisfying the formula

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x),$$

and inequality

$$||f(x) - A(x)|| \leq \varepsilon$$

for some fixed $\varepsilon > 0$ and all $x \in E$.

No continuity conditions are required for this result.

Theorem 2 (T. Aoki, 1950: [1]). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$||f(x+y) - f(x) - f(y)|| \leq \varepsilon(||x||^p + ||y||^p), \tag{1}$$

for all $x, y \in E$, where $\varepsilon > 0$ and $p < 1$ constants. Then the limit

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x),$$

exists for all $x \in E$ and $A : E \to E'$ is the unique additive mapping which satisfies

$$||f(x) - A(x)|| \leq \frac{2\varepsilon}{2 - 2^p}||x||^p \tag{2}$$

for all $x \in E$. If $p < 0$ then the inequality (1) holds for $x, y \neq 0$ and (2) for $x \neq 0$.

Theorem 3 (Th. M. Rassias, 1978: [6]). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$||f(x+y) - f(x) - f(y)|| \leq \varepsilon(||x||^p + ||y||^p), \tag{3}$$

for all $x, y \in E$, where $\varepsilon > 0$ and $p < 1$ constants. Then the limit

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x),$$

exists for all $x \in E$ and $A : E \to E'$ is the unique additive mapping which satisfies

$$||f(x) - A(x)|| \leq \frac{2\varepsilon}{2 - 2^p}||x||^p \tag{4}$$

for all $x \in E$. If $p < 0$ then the inequality (3) holds for $x, y \neq 0$ and (4) for $x \neq 0$.
If, moreover, $f(tx)$ is continuous in $t \in R$ for each fixed $x \in E$, then $A(tx) = tA(x)$ for all $x \in E$ and $t \in R$. $A : E \to E'$ is a unique linear additive mapping satisfying equation

$$A(x+y) = A(x) + A(y).$$

Theorem 4 (J. M. Rassias, 1982-1989: [3, 4, 5]). *Let X be a real normed linear space and Y a real Banach space. Assume that $f : X \to Y$ is a mapping for which there exist constants $\theta \geq 0$ and $p, q \in R$ such that $r = p + q \neq 1$ and f satisfies the functional inequality*

$$||f(x+y) - f(x) - f(y)|| \leq \theta ||x||^p ||y||^q,$$

for all $x, y \in X$. Then the limit

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x),$$

exists for all $x \in X$ and $A : X \to Y$ is the unique additive mapping which satisfies

$$||f(x) - A(x)|| \leq \frac{\theta}{|2^r - 2|} ||x||^r$$

for all $x \in X$.
If, moreover, $f(tx)$ is continuous in $t \in R$ for each fixed $x \in X$, then $A(tx) = tA(x)$ for all $x \in X$ and $t \in R$. $A : X \to Y$ is a unique linear additive mapping satisfying equation

$$A(x+y) = A(x) + A(y).$$

For the theorem that follows, let $(E, \perp)$ denote an orthogonality normed space with norm $||.||_E$ and $(F, ||.||_F)$ is a Banach space.

Theorem 5 (Ravi, K., Arunkumar, M. and Rassias, J. M., 2008: [7]). *Let $f : E \to F$ be a mapping which satisfies the inequality*

$$\begin{aligned} ||f(mx+y) + f(mx-y) - 2f(x+y) - 2f(x-y) - 2(m^2-2)f(x) + 2f(y)||_F \\ \leq \varepsilon\{||x||_E^p ||y||_E^p + (||x||_E^{2p} + ||y||_E^{2p})\} \end{aligned} \tag{5}$$

for all $x, y \in E$ with $x \perp y$, where ε and p are constants with $\varepsilon, p > 0$ and either $m > 1$; $p < 1$ or $m < 1$; $p > 1$ with $m \neq 0$; $m \neq \pm 1$; $m \neq \pm\sqrt{2}$ and $-1 \neq |m|^{p-1} < 1$.
Then the limit

$$Q(x) = \lim_{n\to\infty} \frac{f(m^n x)}{m^{2n}}$$

exists for all $x \in E$ and $Q : E \to F$ is the unique orthogonally Euler-Lagrange quadratic mapping such that

$$||f(x) - Q(x)||_F \leq \frac{\varepsilon}{2|m^2 - m^{2p}|} ||x||_E^{2p}$$

for all $x \in E$.

Note that the mixed type product-sum function

$$(x, y) \to \varepsilon\left[||x||_E^p ||y||_E^p + (||x||_E^{2p} + ||y||_E^{2p})\right]$$

was introduced by J. M. Rassias ([7, 8]).

In this paper we introduce a Cauchy type additive functional equation and investigate the generalised Hyers-Ulam "product-sum" stability of this equation.

2. Cauchy Type Additive Functional Equation

Let X be a real normed linear space and Y a real Banach space.

Definition 1. *A mapping $f : X \to Y$ is called approximately Cauchy type additive, if the approximately Cauchy additive functional inequality*

$$||f(x+y)+f(x-y)+f(y-x)-f(x)-f(y)|| \le \varepsilon\left(||x||^{\frac{\alpha}{2}}||y||^{\frac{\alpha}{2}}+||x||^{\alpha}+||y||^{\alpha}\right) \qquad (6)$$

holds for every $x, y \in X$ with $\varepsilon \ge 0$ and $\alpha \ne 1$.

Lemma 1. *Mapping $A : X \to Y$ satisfies the Cauchy-type additive equation*

$$A(x+y)+A(x-y)+A(y-x)=A(x)+A(y)$$

for all $x, y \in X$ if and only if there exists a mapping $C : X \to Y$ satisfying the Cauchy additive equation

$$C(x+y)=C(x)+C(y)$$

for all $x, y \in X$ such that $A(x)=C(x)$ for all $x \in X$.

Proof. (⇒) Let mapping $A : X \to Y$ satisfy the Cauchy-type additive equation

$$A(x+y)+A(x-y)+A(y-x)=A(x)+A(y) \qquad (7)$$

for all $x, y \in X$. Assume that there exists a mapping $C : X \to Y$ such that $A(x)=C(x)$ for all $x \in X$. Observe that for $x=y=0$ and $x=x$, $y=x$ from (7) we obtain respectively

$$C(0)=A(0)=0$$

and

$$C(-x)=A(-x)=-A(x)=-C(x), \text{ for } x \in X. \qquad (8)$$

From (7) and (8) it is obvious that

$$\begin{aligned} C(x+y)+C(x-y)+C(y-x) &= C(x)+C(y), \text{or} \\ C(x+y)+C(x-y)+C(-(x-y)) &= C(x)+C(y), \text{or} \\ C(x+y) &= C(x)+C(y). \end{aligned}$$

Hence, C satisfies the Cauchy additive equation.

(⇐) Let mapping $C : X \to Y$ satisfy the Cauchy additive equation

$$C(x+y)=C(x)+C(y) \qquad (9)$$

for all $x, y \in X$. Assume that there exists a mapping $A : X \to Y$ such that $A(x)=C(x)$ for all $x \in X$. Observe that for $x=y=0$, from (9) we obtain

$$A(0)= C(0)= 0. \qquad (10)$$

Thus, from (9) and (10) one gets

$$\begin{aligned} A(x)+A(y) &= C(x)+C(y)=C(x+y)=A(x+y) \\ &= A(x+y)+A(0)=A(x+y)+A((x-y)+(y-x)) \\ &= A(x+y)+A(x-y)+A(y-x). \end{aligned}$$

Hence, A satisfies the Cauchy type additive equation.

Thus the proof of Lemma 1 is complete.

Theorem 6. *Assume that $f : X \to Y$ is an approximately Cauchy type additive mapping satisfying (6).*
Then, there exists a unique Cauchy type additive mapping $A : X \to Y$ which satisfies the formula

$$A(x) = \lim_{n\to\infty} f_n(x),$$

where

$$f_n(x) = \begin{cases} 2^{-n} f(2^n x), & -\infty<\alpha<1 \\ 2^n f(2^{-n}x), & \alpha>1 \end{cases}$$

for all $x \in X$ and $n \in N = \{0,1,2,\dots\}$, which is the set of natural numbers and

$$||f(x)-A(x)|| \le \frac{3\varepsilon}{|2-2^{\alpha}|}||x||^{\alpha}$$

for some fixed $\varepsilon > 0$, $\alpha \ne 1$ and all $x \in X$.
If, moreover, $f(tx)$ is continuous in $t \in R$ for each fixed $x \in X$, then $A(tx) = tA(x)$ for all $t \in R$ and $x \in X$. $A : X \to Y$ is a unique linear Cauchy type additive mapping satisfying equation

$$A(x+y)+A(x-y)+A(y-x)=A(x)+A(y). \tag{11}$$

Proof. We start our proof considering: $-\infty < \alpha < \mathbf{1}$.

Step 1 By substituting $x = y = 0$ and $x = y$ in (6), respectively, we can observe that

$$f(0)=0$$

and

$$||f(x)-2^{-1}f(2x)|| \le \frac{3}{2}\varepsilon||x||^{\alpha}.$$

Hence, for $n \in N - \{0\}$

$$\begin{aligned} ||f(x)-2^{-n}f(2^n x)|| &\le ||f(x)-2^{-1}f(2x)||+||2^{-1}f(2x)-2^{-2}f(2^2x)||+\dots \\ &+ ||2^{-(n-1)}f(2^{n-1}x)-2^{-n}f(2^n x)|| \\ &\le \frac{3}{2}(1+2^{\alpha-1}+\dots+2^{(n-1)(\alpha-1)})\varepsilon||x||^{\alpha} \\ &= \frac{3}{2-2^{\alpha}}(1-2^{n(\alpha-1)})\varepsilon||x||^{\alpha}. \end{aligned}$$

Thus,

$$||f(x) - 2^{-n} f(2^n x)|| \leq \frac{3}{2-2^\alpha}(1 - 2^{n(\alpha-1)})\varepsilon ||x||^\alpha,$$

for $n \in N - \{0\}$ and $-\infty < \alpha < 1$.

Step 2 Following, we need to show that if there is a sequence $\{f_n\} : f_n(x) = 2^{-n} f(2^n x)$, then $\{f_n\}$ converges.

For every $n > m > 0$, we can obtain

$$\begin{aligned} ||f_n(x) - f_m(x)|| &= ||2^{-n} f(2^n x) - 2^{-m} f(2^m x)|| \\ &= 2^{-m} ||f(2^m x) - 2^{-(n-m)} f(2^{(n-m)} 2^m x)|| \\ &\leq 2^{-m} \frac{3\varepsilon}{2-2^\alpha}(1 - 2^{(n-m)(\alpha-1)})||x||^\alpha \\ &< 2^{-m} \frac{3\varepsilon}{2-2^\alpha}||x||^\alpha \to 0, \end{aligned}$$

for $m \to \infty$, as $\alpha < 1$. Therefore, $\{f_n\}$ is a Cauchy sequence. Since Y is *complete* we can conclude that $\{f_n\}$ is convergent. Thus, there is a well-defined $A : X \to Y$ such that $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$, for $-\infty < \alpha < 1$.

Step 3 Observe that

$$||f(x) - f_n(x)|| = ||f(x) - 2^{-n} f(2^n x)|| \leq \frac{3\varepsilon}{2-2^\alpha}(1 - 2^{n(\alpha-1)})||x||^\alpha,$$

from which by letting $n \to \infty$ we obtain

$$||f(x) - A(x)|| \leq \frac{3\varepsilon}{2-2^\alpha}||x||^\alpha. \tag{12}$$

Step 4 Claim that mapping $A : X \to Y$ satisfies (11). In fact, by letting $x \to 2^n x$ and $y \to 2^n y$, from (6), we have:

$$\begin{aligned} &||f\left(2^n(x+y)\right) + f\left(2^n(x-y)\right) + f\left(2^n(y-x)\right) - f\left(2^n x\right) - f\left(2^n y\right)|| \\ &\qquad \leq \varepsilon\left(||2^n x||^{\frac{\alpha}{2}} ||2^n y||^{\frac{\alpha}{2}} + ||2^n x||^\alpha + ||2^n y||^\alpha\right). \end{aligned}$$

Next, by multiplying with 2^{-n} we obtain

$$\begin{aligned} 0 \leq\ &||2^{-n} f\left(2^n(x+y)\right) + 2^{-n} f\left(2^n(x-y)\right) + 2^{-n} f\left(2^n(y-x)\right) - 2^{-n} f\left(2^n x\right) - 2^{-n} f\left(2^n y\right)|| \\ &\leq 2^{n(\alpha-1)}\varepsilon\left(||x||^{\frac{\alpha}{2}} ||y||^{\frac{\alpha}{2}} + ||x||^\alpha + ||y||^\alpha\right) \end{aligned}$$

and by letting $n \to \infty$, for $-\infty < \alpha < 1$ we can conclude that an $A : X \to Y$ truly exists such that: $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ satisfies the *Cauchy-type additivity property*

$$A(x+y) + A(x-y) + A(y-x) = A(x) + A(y). \tag{13}$$

Therefore, existence of Theorem holds.

Step 5 We need to prove that A is *unique*.
Observe, from (13), that

$$A(0)=0 \quad \text{and} \quad A(2x)=2A(x).$$

Therefore, by *induction* we can show that

$$A(2^n x)=2A(2^{n-1}x)=2^n A(x)$$

or equivalently

$$A(x)=2^{-n}A(2^n x). \tag{14}$$

Assume, now, the existence of another $A' : X \to Y$, such that $A'(x)=2^{-n}A'(2^n x)$. With the aid of the (12)-(14) and the triangular inequality, one gets

$$\begin{aligned} 0 \le ||A(x)-A'(x)|| &= ||2^{-n}A(2^n x)-2^{-n}A'(2^n x)|| \\ &\le ||2^{-n}A(2^n x)-2^{-n}f(2^n x)||+||2^{-n}f(2^n x)-2^{-n}A'(2^n x)|| \\ &\le 2^{n(\alpha-1)}\frac{3\varepsilon}{2-2^{\alpha}}||x||^{\alpha} \\ &\to 0, \end{aligned}$$

as $n \to \infty$, $(-\infty < \alpha < 1)$. Thus, the *uniqueness* of A is proved and the stability of *Cauchy-type additive mapping* $A : X \to Y$ is established.

Step 6 To complete the proof of Theorem 6, we only need to examine whether $A : X \to Y$ is a *linear Cauchy-type mapping*. To be more precise, we need to show that:
(1) $A(x+y)+A(x-y)+A(y-x)=A(x)+A(y)$, and
(2) $A(rx)=rA(x), \quad \forall r \in R$.

Recall that we have shown already that **(1)** holds.
Therefore, we only need to show that **(2)** is valid $\forall r \in R$.
For that we will study four cases.

Case 1: Let $r=k \in N=\{0,1,2,\ldots\}$.
For $k=0$, from **(2)**, we have $A(0)=0$. This is verified if we substitute $x=y=0$ in (13).
Assume, that $A\big((k-1)x\big)=(k-1)A(x)$ is true $\forall k$.
Then, we need to prove that $A(kx)=kA(x)$.
Note that for $x=x$, and $y=0$ from (13), we can easily obtain $A(-x)=(-1)A(x)$.
Let $x=x$ and $y=(k-1)x$ in (13). Then,

$$A(kx)+A\big(-(k-2)x\big)+A\big((k-2)x\big)=A(x)+A\big((k-1)x\big),$$

or

$$A(kx)=kA(x), \quad \forall k \in N=\{0,1,2,\ldots\}.$$

Case 2: Let $r = k \in Z$.
We only need to observe that A is *odd*. Since, we have already proved that **(2)** is valid $\forall k \in N = \{0, 1, 2, \ldots\}$ we can then conclude that

$$A(kx) = kA(x), \quad \forall k \in Z.$$

Case 3: Let $r = \frac{k}{l} \in Q$, for $k \in Z$, $l \in Z - \{0\}$.
Then, $A(x) = A\left(l\frac{1}{l}x\right) = lA\left(\frac{1}{l}x\right)$, for $l \in Z - \{0\}$. Hence, $A\left(\frac{1}{l}x\right) = \frac{1}{l}A(x)$.
Besides, for $k \in Z$, $A\left(\frac{k}{l}x\right) = A\left(k\frac{1}{l}x\right) = kA(\frac{1}{l}x)$, from *Case 2*.
Thus, $A\left(\frac{k}{l}x\right) = \frac{k}{l}A(x)$, or $A(rx) = rA(x)$ for $r \in Q$.

Case 4: Let $r \in R$, where $r = q_n$: rational numbers.
Since R is a *complete* space, every sequence $\{q_n\}$ converges in R, i.e. $\lim_{n\to\infty} q_n = q \in R$.
Recall that $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ and $f(tx)$ is continuous in t for each fixed x in X. Therefore, $A(tx)$ is continuous in t for each fixed x in X. Besides,

$$\lim_{n\to\infty} A(q_n x) = A\left(\lim_{n\to\infty} q_n x\right) = A(qx) \tag{15}$$

and

$$\lim_{n\to\infty} A(q_n x) = \lim_{n\to\infty} q_n A(x) = qA(x). \tag{16}$$

From (15) and (16) *Case 4.* is now proved, which completes *Step 6.* and thus the proof of our Theorem 6 for the case of $-\infty < \alpha < 1$.

The proof for the case of $\alpha > 1$ is similar to the proof for $-\infty < \alpha < 1$.
In fact, we can find the general inequality

$$||f(x) - 2^n f(2^{-n}x)|| \le \frac{3\varepsilon}{2^\alpha - 2}(1 - 2^{n(1-\alpha)})||x||^\alpha, \tag{17}$$

for all $n \in N - \{0\}$. Thus from this inequality (17) and the formula

$$A(x) = \lim_{n\to\infty} 2^n f(2^{-n}x),$$

for $n \to \infty$, we get the inequality

$$||f(x) - A(x)|| \le \frac{3\varepsilon}{2^\alpha - 2}||x||^\alpha, \text{ for } \alpha > 1.$$

The rest of the proof for $\alpha > 1$ is omitted as similar to the above mentioned proof for $-\infty < \alpha < 1$.

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