EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 4, No. 1, 2011, 59-66
ISSN 1307-5543 – www.ejpam.com

On New Subclasses of Analytic Functions Involving Generalized Differential and Integral Operators

Maslina Darus*, Rabha W. Ibrahim

School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi 43600, Selangor Darul Ehsan, Malaysia

Abstract. We define a generalized differential and integral operators on the class $\mathscr{A}$ of analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the unit disk $U := \{z \in \mathbb{C} : |z| < 1\}$ involving $k-$th Hadamard product (convolution) as follows

$$D^k_{\alpha,\lambda} f(z) = z + \sum_{n=2}^{\infty} [(n-1)(\lambda - \alpha) + n]^k a_n z^n, \quad (z \in U).$$

These operators are generalized for some of well known operators for example Sălăgean operator. New classes containing these operators are investigated. Characterization and other properties of these classes are studied.
2000 Mathematics Subject Classifications: 30C45
Key Words and Phrases: Hadamard product; Integral operator; Differential operator; Sălăgean operator.

1. Introduction and Preliminaries

Let $\mathscr{H}$ be the class of functions analytic in $U := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathscr{H}[a,n]$ be the subclass of $\mathscr{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$. Let $\mathscr{A}$ be the subclass of $\mathscr{H}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U). \tag{1}$$

Given two functions $f, g \in \mathscr{A}$, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ their convolution or Hadamard product $f(z) * g(z)$ is defined by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad (z \in U).$$

*Corresponding author.

Email addresses: maslina@ukm.my (M. Darus), rabhaibrahim@yahoo.com (R. Ibrahim)

And for several functions $f_1(z),\dots,f_m(z)\in\mathscr{A}$

$$f_1(z)*\dots*f_m(z)=z+\sum_{n=2}^{\infty}(a_{1n}...a_{mn})z^n,\quad (z\in U).$$

Our aim is to use the Hadamard product of $k-$th order to define generalized differential and integral operators.

For a function f in $\mathscr{A}$ of the form (1) first, we define the following generalized differential operator:

$$\begin{aligned} D^0f(z)&=f(z)\\ &=z+\sum_{n=2}^{\infty}a_nz^n,\\ D^1_{\alpha,\lambda}f(z)&=(\alpha-\lambda)f(z)+(\lambda-\alpha+1)zf'(z)\\ &=z+\sum_{n=2}^{\infty}[(n-1)(\lambda-\alpha)+n]a_nz^n,\\ &\vdots\\ D^k_{\alpha,\lambda}f(z)&=D^1_{\alpha,\lambda}\left(D^{k-1}_{\alpha,\lambda,}f(z)\right)\\ &=z+\sum_{n=2}^{\infty}[(n-1)(\lambda-\alpha)+n]^ka_nz^n \end{aligned} \tag{2}$$

for $\alpha\geq 0,\lambda\geq 0$ and $k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ with $D^k_{\alpha,\lambda}f(0)=0$. Note that when $\alpha=\lambda$ we get Sălăgean's differential operator [see 11].

Let $\mathscr{M}(\mu)$ be the subclass of the class $\mathscr{A}$ consisting of functions $f(z)$ which satisfy the inequality

$$\Re\{\frac{zf'(z)}{f(z)}\}<\mu,\quad (z\in U)$$

for some $\mu(\mu>1)$. And let $\mathscr{N}(\mu)$ be the subclass of the class $\mathscr{A}$ consisting of functions $f(z)$ which satisfy the inequality

$$\Re\{\frac{zf''(z)}{f'(z)}\}<\mu,\quad (z\in U)$$

for some $\mu(\mu>1)$. Then $f\in\mathscr{N}(\mu)$ if and only if $zf'\in\mathscr{M}(\mu)$. In this paper we define and study the following subclasses involving the generalized differential operator (2). Let $\mathscr{M}^k_{\alpha,\lambda}(\mu)$ be the subclass of the class $\mathscr{A}$ consisting of functions $f(z)$ which satisfy the inequality

$$\Re\{\frac{z[D^k_{\alpha,\lambda}f(z)]'}{D^k_{\alpha,\lambda}f(z)}\}<\mu,\quad (z\in U)$$

for some $\mu(\mu > 1)$. And let $\mathscr{N}^k_{\alpha,\lambda}(\mu)$ be the subclass of the class $\mathscr{A}$ consisting of functions $f(z)$ which satisfy the inequality

$$\Re\{\frac{z[D^k_{\alpha,\lambda}f(z)]''}{[D^k_{\alpha,\lambda}f(z)]'}\} < \mu, \quad (z \in U)$$

for some $\mu(\mu > 1)$. Then $f \in \mathscr{N}^k_{\alpha,\lambda}(\mu)$ if and only if $zf' \in \mathscr{M}^k_{\alpha,\lambda}(\mu)(\mu)$.

Remark 1. *When $k = 0$, then the classes*

$$\mathscr{M}^0_{\alpha,\lambda}(\mu) \equiv \mathscr{M}(\mu) \text{ and } \mathscr{N}^0_{\alpha,\lambda}(\mu) \equiv \mathscr{N}(\mu)$$

were introduced by

- *(i) For $1 < \mu \leq \frac{4}{3}$, $k = 0$, Uralegaddi et al. [13, 12].*
- *(ii) For $\mu > 1, k = 0$, Owa and Srivastava [9] and Owa and Nishiwaki [10].*
- *(iii) For $\mu > 1$, $\alpha = 1$ Bulut [2].*

2. Coefficient estimates.

In this section we derive sufficient conditions for $f(z)$ to belongs to the classes $\mathscr{M}^k_{\alpha,\lambda}(\mu)$ and $\mathscr{N}^k_{\alpha,\lambda}(\mu)$, which are obtained by using coefficient inequalities.

Theorem 1. *If $f(z) \in \mathscr{A}$ satisfies the inequality*

$$\sum_{n=2}^{\infty} |[(n-1)(\lambda-\alpha)+n]^k| \Big\{(n-\kappa) + |n+\kappa-2\mu|\Big\} |a_n| \leq 2(\mu-1) \tag{3}$$

for some $0 \leq \kappa \leq 1$ and $\mu > 1$, then $f \in \mathscr{M}^k_{\alpha,\lambda}(\mu)$.

Proof. Assume that the inequality (3) holds. It suffices to show that

$$\left| \frac{\frac{z[D^k_{\alpha,\lambda}f(z)]'}{D^k_{\alpha,\lambda}f(z)} - \kappa}{\frac{z[D^k_{\alpha,\lambda}f(z)]'}{D^k_{\alpha,\lambda}f(z)} - (2\mu-\kappa)} \right| < 1, \quad (z \in U).$$

We observe

$$\begin{aligned}
\left| \frac{\frac{z[D^k_{\alpha,\lambda}f(z)]'}{D^k_{\alpha,\lambda}f(z)} - \kappa}{\frac{z[D^k_{\alpha,\lambda}f(z)]'}{D^k_{\alpha,\lambda}f(z)} - (2\mu-\kappa)} \right| &= \left| \frac{1-\kappa+\sum_{n=2}^{\infty}(n-\kappa)[(n-1)(\lambda-\alpha)+n]^k a_n z^{n-1}}{1+\kappa-2\mu+\sum_{n=2}^{\infty}(n+\kappa-2\mu)[(n-1)(\lambda-\alpha)+n]^k a_n z^{n-1}} \right| \\
&\leq \frac{1-\kappa+\sum_{n=2}^{\infty}(n-\kappa)|[(n-1)(\lambda-\alpha)+n]^k||a_n||z|^{n-1}}{2\mu-1-\kappa-\sum_{n=2}^{\infty}|(n+\kappa-2\mu)||[(n-1)(\lambda-\alpha)+n]^k||a_n||z|^{n-1}} \\
&< \frac{1-\kappa+\sum_{n=2}^{\infty}(n-\kappa)|[(n-1)(\lambda-\alpha)+n]^k||a_n|}{2\mu-1-\kappa-\sum_{n=2}^{\infty}|(n+\kappa-2\mu)||[(n-1)(\lambda-\alpha)+n]^k||a_n|}.
\end{aligned}$$

The last expression is bounded above by 1 if

$$1-\kappa+\sum_{n=2}^{\infty}(n-\kappa)|[(n-1)(\lambda-\alpha)+n]^k||a_n| < 2\mu-1-\kappa-\sum_{n=2}^{\infty}|(n+\kappa-2\mu)||[(n-1)(\lambda-\alpha)+n]^k||a_n|$$

which is equivalent to assertion (3), hence the proof.

When $k=0$, the next result can found in [10].

Corollary 1. *If $f(z) \in \mathscr{A}$ satisfies the inequality*

$$\sum_{n=2}^{\infty}\Big\{(n-\kappa)+|n+\kappa-2\mu|\Big\}|a_n| \le 2(\mu-1) \tag{4}$$

for some $0 \le \kappa \le 1$ and $\mu > 1$, then $f \in \mathscr{M}^0_{\alpha,\lambda}(\mu) \equiv \mathscr{M}(\mu)$.

When $\kappa = 1$, we obtain the next result

Corollary 2. *If $f(z) \in \mathscr{A}$ satisfies the inequality*

$$\sum_{n=2}^{\infty}(n-\mu)|[(n-1)(\lambda-\alpha)+n]^k||a_n| \le \mu-1 \tag{5}$$

for $1 < \mu \le \frac{3}{2}$, then $f \in \mathscr{M}^k_{\alpha,\lambda}(\mu)$.

When $k=0, \kappa=1$ the next result can found in [10].

Corollary 3. *If $f(z) \in \mathscr{A}$ satisfies the inequality*

$$\sum_{n=2}^{\infty}(n-\mu)|a_n| \le \mu-1 \tag{6}$$

for $1 < \mu \le \frac{3}{2}$, then $f \in \mathscr{M}(\mu)$.

Theorem 2. *If $f(z) \in \mathscr{A}$ satisfies the inequality*

$$\sum_{n=2}^{\infty}n|[(n-1)(\lambda-\alpha)+n]^k|\Big\{n-\kappa+1+|n+\kappa-2\mu|\Big\}|a_n| \le 2(\mu-1) \tag{7}$$

for some $0 \le \kappa \le 1$ and $\mu > 1$, then $f \in \mathscr{N}^k_{\alpha,\lambda}(\mu)$.

When $k=0$, the next result can be found in [10].

Corollary 4. *If $f(z) \in \mathscr{A}$ satisfies the inequality*

$$\sum_{n=2}^{\infty}n\Big\{n-\kappa+1+|n+\kappa-2\mu|\Big\}|a_n| \le 2(\mu-1) \tag{8}$$

for some $0 \le \kappa \le 1$ and $\mu > 1$, then $f \in \mathscr{N}^0_{\alpha,\lambda}(\mu) \equiv \mathscr{N}(\mu)$.

3. Integral operator.

Analogous to the generalized differential operator (2), we define and study a new integral operator $I^k_{\alpha,\lambda} : \mathscr{A} \to \mathscr{A}$ as follows. Let

$$\phi(z) := \frac{(\lambda-\alpha)z}{(1-z)^2} - \frac{(\lambda-\alpha)z}{1-z} + \frac{z}{(1-z)^2}$$

and

$$F(z) = \underbrace{\phi(z) * \ldots * \phi(z)}_{k-times}$$
$$= z + \sum_{n=2}^{\infty} [(n-1)(\lambda-\alpha)+n]^k z^n$$

Now we define the integral operator $I^k_{\alpha,\lambda}$ such that

$$I^k_{\alpha,\lambda} := [F(z)]^{-1} * f(z), \quad (z \in U)$$

where $f \in \mathscr{A}$ and

$$F(z) * [F(z)]^{-1} = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n, \quad (z \in U).$$

Implies

$$[F(z)]^{-1} = z + \sum_{n=2}^{\infty} \frac{1}{[(n-1)(\lambda-\alpha)+n]^k} z^n, \quad (z \in U)$$

thus we have

$$I^k_{\alpha,\lambda} f(z) = z + \sum_{n=2}^{\infty} \frac{a_n}{[(n-1)(\lambda-\alpha)+n]^k} z^n, \quad (z \in U). \tag{9}$$

Remark 2. *Note that when $\alpha = \lambda$, the integral operator (9) reduces to the integral operator*

$$I^k_{\alpha,\alpha} f(z) = z + \sum_{n=2}^{\infty} \frac{a_n}{n^k} z^n, \quad (z \in U),$$

which defined and studied by Sălăgean [see 11].

Lemma 1. *Let $f \in \mathscr{A}$. Then*

(i) $I^0_{\alpha,\lambda} f(z) = f(z)$,

(ii) $I^1_{\alpha,\alpha} f(z) = \int_0^z \frac{f(t)}{t} dt.$

Proof.

(i)

$$I^0_{\alpha,\lambda}f(z) = z + \sum_{n=2}^{\infty} a_n z^n = f(z),$$

(ii)

$$\begin{aligned}\int_0^z \frac{f(t)}{t}dt &= \int_0^z [1+\sum_{n=2}^{\infty} a_n t^{n-1}]dt \\ &= z + \sum_{n=2}^{\infty} \frac{a_n}{n} z^n \\ &= I^1_{\alpha,\alpha} f(z).\end{aligned}$$

Define the subclasses involving the generalized integral operator (9). Let $\mathscr{S}^k_{\alpha,\beta,\lambda}(\mu)$ be the subclass of the class $\mathscr{A}$ consisting of functions $f(z)$ which satisfy the inequality

$$\Re\{\frac{z[I^k_{\alpha,\lambda}f(z)]'}{I^k_{\alpha,\lambda}f(z)}\} < \mu, \quad (z \in U)$$

for some $\mu(\mu > 1)$. It is clear that

$$\mathscr{S}^0_{\alpha,\lambda}(\mu) \equiv \mathscr{M}(\mu).$$

And let $\mathscr{K}^k_{\alpha,\beta,\lambda}(\mu)$ be the subclass of the class $\mathscr{A}$ consisting of functions $f(z)$ which satisfy the inequality

$$\Re\{\frac{z[I^k_{\alpha,\lambda}f(z)]''}{[I^k_{\alpha,\lambda}f(z)]'}\} < \mu, \quad (z \in U)$$

for some $\mu(\mu > 1)$. Then $f \in \mathscr{K}^k_{\alpha,\lambda}(\mu)$ if and only if $zf' \in \mathscr{S}^k_{\alpha,\lambda}(\mu)(\mu)$. Also we have

$$\mathscr{K}^0_{\alpha,\lambda}(\mu) \equiv \mathscr{N}(\mu).$$

In the same manner of Theorem 1 and Theorem 2, we have the following results.

Theorem 3. *If $f(z) \in \mathscr{A}$ satisfies the inequality*

$$\sum_{n=2}^{\infty} \frac{\Big\{(n-\kappa) + |n+\kappa-2\mu|\Big\}}{|[(n-1)(\lambda-\alpha)+n]^k|}|a_n| \leq 2(\mu-1) \tag{10}$$

for some $0 \leq \kappa \leq 1$ and $\mu > 1$, then $f \in \mathscr{S}^k_{\alpha,\lambda}(\mu)$.

Theorem 4. *If $f(z) \in \mathscr{A}$ satisfies the inequality*

$$\sum_{n=2}^{\infty} \frac{n\Big\{n-\kappa+1+|n+\kappa-2\mu|\Big\}}{|[(n-1)(\lambda-\alpha)+n]^k|}|a_n| \leq 2(\mu-1) \tag{11}$$

for some $0 \leq \kappa \leq 1$ and $\mu > 1$, then $f \in \mathscr{K}^k_{\alpha,\lambda}(\mu)$.

4. Conclusion.

This work is a generalization for well known differential and integral operators of univalent functions. Moreover, the classes which are studied here also generalized the ones studied by different authors

$$\mathscr{M}^0_{\alpha,\lambda}(\mu) \equiv \mathscr{S}^0_{\alpha,\lambda}(\mu) \equiv \mathscr{M}(\mu)$$

and

$$\mathscr{N}^0_{\alpha,\lambda}(\mu) \equiv \mathscr{K}^0_{\alpha,\lambda}(\mu) \equiv \mathscr{N}(\mu).$$

In fact, many other operators can be seen in [1,3-8,14] for different problems.

ACKNOWLEDGEMENTS The work presented here was supported by UKM-ST-06-FRGS0107-2009.

References

[1] M.H. Al-Abbadi and M. Darus, Differential Subordination Defined by New Generalised Derivative Operator for Analytic Functions, International Journal of Mathematics and Mathematical Sciences, Article ID 369078, 15 pages. 2010.

[2] S. Bulut, Some properties for an integral operator defined by AL-Oboudi differential operator, J. Ineq. Pure Appl. Math., Vol 9, Art. 115, pp5. 2008.

[3] M.Darus and K.Al-Shaqsi, On Subordinations for certain analytic functions associated with generalized integral operator. Lobachevskii Journal of Mathematics, 29 (2), 90-97. 2008.

[4] M. Darus, R. W. Ibrahim, Coefficient Inequalities for a New Class of Univalent Functions, Lobachevskii Journal of Mathematics, 29(4), 221-229. 2008.

[5] R. W. Ibrahim, M. Darus , Subordination and superordination for univalent solutions for fractional differential equations, J. Math. Anal. Appl., 345, 871-879. 2008.

[6] R. W. Ibrahim, M. Darus, On Subordination Theorems for New Classes of Normalize Analytic Functions, Applied Mathematical Sciences, 56(2), 2785 - 2794. 2008.

[7] R. W. Ibrahim, M. Darus, Diferential subordination results for new classes of the family $\mathscr{E}(\Phi,\Psi)$, J. Ineq. Pure and Appl. Math., Volume 9, Issue 4, Article draft, 9 pp. 2009.

[8] R. W. Ibrahim and M. Darus, Subordination and superordination for analytic functions involving fractional integral operator, Complex Variables and Elliptic Equations, 53(11), 1021-1031. 2008.

[9] S. Owa, H. M. Srivastava, Some generalized convolution properties associated with certain subclasses of analytic functions, J. Ineq. Pure Appl. Math., 3(3), Article 42. 2002.

[10] S. Owa, J. Nishiwaki, Coefficent estimate for certain classes of analytic functions, J. Ineq. Pure Appl. Math., 3(5), Article 72. 2002.

[11] G. S. Sălăgean, Subclasses of univalent functions, Lecture Notes in Math., 1013, Springer-Verlag, Berlin, 362-372. 1983.

[12] B. A. Uralegaddi, A. R. Desai, Convolutions of univalent functions with positive coefficients, Tamkang J. Math., 29, 279-285, 1998.

[13] B. A. Uralegaddi, M. D. Ganigi and S. M. Sarangi, Univalent functions with positive coefficients, Tamkang J. Math., 25, 225-230. 1994.

[14] Ri-G. Xiang, Z.-Gang Wang and M. Darus, A family of integral operators preserving subordination and superordination, Bull. Malays. Math. Sci., (2) 33(1), 121-131. 2010.