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On the Structure of Commutative Rings with ${p_1}^{k_1} \cdots {p_n}^{k_n}$ $(1 \leq k_i \leq 7)$ Zero-Divisors II

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Abstract. In this paper, we determine the structure of nonlocal commutative rings with p^6 zero-divisors and characterize the structure of nonlocal commutative rings with p^7 zero-divisors. Also, the structure and classification up to isomorphism all commutative rings with ${p_1}^{k_1} \ldots {p_n}^{k_n}$ zero-divisors, where n is a positive integer, p_i's are distinct prime number and $1 \leq k_i \leq 4$, are determined.

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1. Introduction

The present paper is a sequel to [2] and so the notations introduced in Introduction of [2] will remain in force. In particular, all rings are associative rings with identity elements, $J(R)$ denotes the jacobson radical of R, $Z(R)$ denotes the set of all zero-divisors of R and for any finite subset Y of R, we denote $|Y|$ for the cardinality of Y. Also, F_q is the finite field of order q, ${F_q}^*$ is the group of nonzero elements of F_q and for a prime number p, Σ_m is a set of coset representation of $({F_p}^*)^m$ in ${F_p}^*$, $\Sigma_m^0 = \Sigma_m \cup \{0\}$ and $GR(p^{nr}, p^r)$ is the Galois ring of order p^{nr} and characteristic p^r.

In [2] the structure and classification up to isomorphism all rings with ${p_1}^{k_1} \ldots {p_s}^{k_s}$ zero-divisors, where s is a positive integer, p_i's are distinct prime number and $1 \leq k_i \leq 3$ were determined. Also we determined the structure of nonlocal rings with p^k zero-divisors where $k = 4$ or 5. In the paper we develop these results. In fact the structure and classification up to isomorphism all rings with ${p_1}^{k_1}{p_2}^{k_2} \ldots {p_s}^{k_s}$ zero-divisors, where s is a positive integer, p_i's are distinct prime number and $1 \leq k_i \leq 4$ are determined. Also we determine the structure of nonlocal rings with p^6 zero-divisors and characterize the structure of nonlocal rings with p^7 zero-divisors.

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2. On Rings with p^k Zero-divisors

We recall the following facts that we will use them in the paper:

(i) An Artinian commutative ring R is called *completely primary* if $R/J(R)$ is a field. One can easily see that an Artinian commutative ring R is completely primary if and only if $Z(R)$ is an ideal of R, if and only if R is a local ring.

(ii) Let R_i $(1 \leq i \leq t)$ be a nonzero finite commutative ring with m_i elements and n_i zero-divisors. Then by [6, Theorem 2], the ring $R_1 \times \ldots \times R_t$ has $m_1 m_2 \ldots m_t - (m_1 - n_1)(m_2 - n_2)\ldots(m_t - n_t)$ zero-divisors.

(iii) Every finite commutative ring is uniquely expressible as a direct sum of completely primary (local) rings (see for example [7, p.95]).

We need the following two lemmas which are crucial in our investigation.

Lemma 1. *[8, Theorem 2] Let R be a finite completely primary ring. Then*

1. $Z(R) = J(R)$;
2. $|Z(R)| = p^{(n-1)r}$ *and* $|R| = p^{nr}$ *for some prime number p, and some positive integers n, r;*
3. $Z(R)^n = 0$;
4. $char(R) = p^k$ *for some integer k with* $1 \leq k \leq n$;
5. $R/J(R) \cong F_q$, *where* $q = p^r$.

Lemma 2. *[2, Theorem 2] Let R be a commutative ring such that* $|Z(R)| = p^k$ *for some prime number p and a positive number k. Then either*

(i) *R is local,*

(ii) *R is reduced or*

(iii) $k \geq 3$ *and* $R \cong R_1 \times \ldots \times R_s \times F_{q_1} \times \ldots \times F_{q_t}$ *where s and t are positive integers, each* F_{q_i} *is a field, and where each* R_i *is a commutative finite local ring with* $|Z(R_i)| = p^{t_i}$, $|R_i| = p^{k_i}$ *for some positive integers* k_i *and* t_i *with* $1 \leq \sum_{i=1}^{s} t_i \leq \sum_{i=1}^{s} k_i - s \leq k - s - 1$ *such that*

$$p^{k-\Sigma_{i=1}^{s} t_i} = q_1 \ldots q_t p^{\Sigma_{i=1}^{s}(k_i - t_i)} - (q_1 - 1)\ldots(q_t - 1)\Pi_{i=1}^{s}(p^{k_i - t_i} - 1). \quad (1)$$

Consequently, in the latter case, $q_i \equiv 1\ (p)$ *and for each* $i = 1, \ldots, s,$ $t_i \leq k - 2$. *Moreover, if* $t_j = k - 2$ *for some* $j \in \{1, \ldots, s\}$, *then* $s = t = 1$, *i.e.,* $R \cong R_1 \times F_q$ *where* $|Z(R_1)| = p^{k-2}$ *and so* $p^2 = p + q - 1$.

Also we need the following construction [3, p.5071].

Construction A. Let R_0 be the Galois ring $GR(p^{2r},p^2)$ or $GR(p^{3r},p^3)$. Let s, d, t, λ be integers with either $1 \le t \le s^2$, $1 \le 1+t \le s^2$ or $1 \le d+t \le s^2$ if $\text{char}(R_0)=p^2$ or $1 \le 1+d+t \le 1+s^2$ if $\text{char}(R_0)=p^3$, and $\lambda \ge 0$. Let V, W be R_0/pR_0-spaces which when considered as R_0-modules have generating sets $\{v_1,\dots,v_\lambda\}$ and $\{w_1,\dots,w_t\}$ respectively. Let U be an R_0-module with an R_0-modules generating set $\{u_1,\dots,u_s\}$; and suppose that $d \ge 0$ of the u_i are such that $pu_i \ne 0$. Since R_0 is commutative, we can think of them as left and right R_0-module.
Let (a^l_{ij}), for $l=0,1,\dots,t,t+1$ or $d+t$, be $s\times s$ matrices with entries in R_0/pR_0 if $\text{char}(R_0)=p^2$ or $l=0,1,\dots,d+t$ be $(1+s)\times(1+s)$ matrices with entries in R_0/pR_0 if $\text{char}(R_0)=p^3$.
Consider the additive group direct sum $R=R_0\oplus U\oplus V\oplus W$ and define a multiplication on R by
$(\alpha_0,\ \sum_{i=1}^s \alpha_i u_i,\ \sum_{j=1}^\lambda \beta_j v_j,\ \sum_{k=1}^t \gamma_k w_k).(\alpha'_0,\ \sum_{i=1}^s \alpha'_i u_i,\ \sum_{j=1}^\lambda \beta'_j v_j,\ \sum_{k=1}^t \gamma'_k w_k)=$
$(\alpha_0\alpha'_0+p^f\ \sum_{i,j=1}^s\ a^0_{ij}[\alpha_i\alpha'_j+pR_0],\ \sum_{i=1}^s[\alpha_0\alpha'_i+\alpha_i\alpha'_0+p\sum_{i,j=1}^s a^i_{ij}[\alpha_i\alpha'_j+pR_0]]u_i,\ \sum_{j=1}^\lambda[(\alpha_0+pR_0)\beta'_j+\beta_j(\alpha'_0+pR_0)]v_j,\ \sum_{k=1}^t[(\alpha_0+pR_0)\gamma'_k+\gamma_k(\alpha'_0+pR_0)+\sum_{i,j=1}^s\ a^{d+k}_{ij}[\alpha_i\alpha'_j+pR_0]]w_k)$
where $f=1$ or 2, depending on whether $char(R)=p^2$ or p^3. Then by [3, Theorem 6.1], this multiplication turns R into a ring and any local ring with $Z(R)^3=0$, $Z(R)^2\ne 0$ of characteristic p^2 or p^3, is isomorphic to one given by construction A.

Proposition 1. *Let R be a commutative ring with $|Z(R)|=p^4$ and $|R|=p^6$ where p is a prime number. Then R is isomorphic to one of the rings $GR(p^6,p^3)$, $F_{p^2}\oplus F_{p^2}\oplus F_{p^2}$ with multiplication $(r_0,r_1,r_2)(s_0,s_1,s_2)=(r_0s_0,r_0s_1+r_1s_0,r_0s_2+r_2s_0)$, $S\oplus F$ with multiplication $(r_0,r_1)(s_0,s_1)=(r_0s_0,r_0s_1+r_1s_0)$, where $S=GR(p^4,p^2)$ and $F=S/pS$, $F_{p^2}\oplus F_{p^2}\oplus F_{p^2}$ with multiplication $(\alpha_0,\alpha,\gamma)(\alpha'_0,\alpha',\gamma')=(\alpha_0\alpha'_0,\alpha_0\alpha'+\alpha\alpha'_0,\alpha_0\gamma'+\gamma\alpha'_0+\alpha\alpha')$ or $R_0\oplus R_0/pR_0$ with multiplication $(\alpha_0,\alpha+pR_0)(\alpha'_0,\alpha'+pR_0)=(\alpha_0\alpha'_0+\alpha\alpha' p,\alpha_0\alpha'+\alpha\alpha'_0+pR_0)$ where $R_0=GR(p^4,p^2)$.*

Proof. Since R is a ring with $|Z(R)|=p^4$ and $|R|=p^6$, by Lemma 1, $Z(R)^3=0$. Thus we consider the following cases.

Case 1: $Z(R)^2=0$ i.e., R is a ring in which the multiplication of any two zero-divisors is zero. Then by [1, Theorem 1], R is isomorphic to one of the rings $S\oplus F^k$, where S is either the field of p^r elements or the Galois ring $GR(p^{2r},p^2)$ and $F=S/pS$ with the multiplication

$$(r_0,r_1,\dots,r_k)(s_0,s_1,\dots,s_k)=(r_0s_0,r_0s_1+r_1s_0,\dots,r_0s_k+r_ks_0)$$

for some positive integers r and k.
Now since $|Z(R)|=p^4$ and $|R|=p^6$, we can conclude that $S=F_{p^2}$ and $k=2$ or $S=GR(p^4,p^2)$ and $k=1$. Thus R is isomorphic to one of the rings $F_{p^2}\oplus F_{p^2}\oplus F_{p^2}$ with $(r_0,r_1,r_2)(s_0,s_1,s_2)=(r_0s_0,r_0s_1+r_1s_0,r_0s_2+r_2s_0)$ or $S\oplus F$ with $(r_0,r_1)(s_0,s_1)=(r_0s_0,r_0s_1+r_1s_0)$, where $S=GR(p^4,p^2)$ and $F=S/pS$.

Case 2: $Z(R)^2\ne 0$. If $\text{char}(R)=p$, then by [3, Theorem 4.1], any commutative local ring of characteristic p in which the multiplication of any two zero-divisors is zero, is isomorphic to one of the rings $F \oplus U \oplus V \oplus W$ with multiplication

$$(\alpha_0, \sum_{i=1}^{s} \alpha_i u_i, \sum_{j=1}^{\lambda} \beta_j v_j, \sum_{k=1}^{t} \gamma_k w_k)(\alpha_0', \sum_{i=1}^{s} \alpha_i' u_i, \sum_{j=1}^{\lambda} \beta_j' v_j, \sum_{k=1}^{t} \gamma_k' w_k)$$

$$= (\alpha_0\alpha_0', \sum_{i=1}^{s}[\alpha_0\alpha_i' + \alpha_i\alpha_0']u_i, \sum_{j=1}^{\lambda}[\alpha_0\beta_j' + \beta_j\alpha_0']v_j, \sum_{k=1}^{t}[\alpha_0\gamma_k' + \gamma_k\alpha_0' + \sum_{i,j=1}^{s} a_{i,j}^k \alpha_i\alpha_j']w_k)$$

where F is the field of order r and U, V, W are s, λ, t -dimensional F-spaces respectively, for some integers s, λ, t with $\lambda \geq 0$ and $1 \leq t \leq s^2$, where $\{u_i\}$, $\{v_i\}$ and $\{w_i\}$ are bases for U, V and W respectively, and $(a_{i,j}^k)$ $1 \leq k \leq t$ are t matrices of size $s \times s$ with entries in F. Since $|R| = p^6$, we can conclude that $t = s = 1$ and $\lambda = 0$. On the other hand by [3, Corollary 5.2], we can put $a_{11}^1 = 1$. Thus $R \cong F_{p^2} \oplus F_{p^2} \oplus F_{p^2}$ with multiplication $(\alpha_0, \alpha, \gamma)(\alpha_0', \alpha', \gamma') = (\alpha_0\alpha_0', \alpha_0\alpha' + \alpha\alpha_0', \alpha_0\gamma' + \gamma\alpha_0' + \alpha\alpha')$.
Now suppose that $\text{char}(R) = p^2$ or p^3. Since $|R| = p^6$ and $|R/J(R)| = p^2$, by construction A we conclude that $s = 1$ and $t = \lambda = 0$ if $\text{char}(R) = p^2$ and $s = t = \lambda = 0$ if $\text{char}(R) = p^3$. Also by [3, Lemma 7.1], we can put $a_{11}^0 = 1$ and so the following rings is obtained.
If $\text{char}(R) = p^2$, then $R \cong R_0 \oplus R_0/pR_0$ with multiplication $(\alpha_0, \alpha + pR_0).(\alpha_0', \alpha' + pR_0) = (\alpha_0\alpha_0' + \alpha\alpha' p, \alpha_0\alpha' + \alpha\alpha_0' + pR_0)$, where $R_0 = GR(p^4, p^2)$ and if $\text{char}(R) = p^3$, then $R \cong GR(p^6, p^3)$.

Proposition 2. *Let R be a commutative ring with $|Z(R)| = p^4$ and $|R| = p^5$ where p is a prime number. Then R is isomorphic to one of the rings $\mathbb{Z}_{p^5}$, $F_p[x]/(x^5)$, $F_p[x,y]/(x^4, xy, y^2)$, $F_p[x,y]/(x^4, xy, y^2 - x^3)$, $\mathbb{Z}_p[x,y,z,t]/(x,y,z,t)^2$, $\mathbb{Z}_{p^2}[x]/(px, x^4 - ap)$ where $a \in \Sigma_4^0$, $\mathbb{Z}_{p^2}[x]/(px^2, x^3 - bp)$ where $b \in \Sigma_3$ and $p \neq 3$, $\mathbb{Z}_9[x]/(3x^2, x^3 - 3 - 3bx)$ where $b \in \{-1, 0, 1\}$, $\mathbb{Z}_{p^2}[x]/(px^2, x^3 - apx)$ where $a \in \Sigma_2^0$, $\mathbb{Z}_{p^2}[x,y,z]/(p,x,y,z)^2$, $\mathbb{Z}_{p^3}[x]/(p^2x, x^2 - ap)$ where $a \in \Sigma_2$ and $p \neq 2$, $\mathbb{Z}_8[x]/(4x, x^2 - 2a - 2bx)$ where $(a,b) \in \{(1,0),(1,1),(-1,1)\}$, $\mathbb{Z}_{p^3}[x]/(px, x^3 - ap^2)$ where $a \in \Sigma_3^0$, $\mathbb{Z}_{p^3}[x]/(p^2x, x^2 - ap^2)$ where $a \in \Sigma_2^0$ and $p \neq 2$, $\mathbb{Z}_8[x]/(4x, x^2 - 4a - 2bx)$ where $(a,b) \in \{(0,0),(0,1),(1,1)\}$, $\mathbb{Z}_{p^4}[x]/(px, x^2 - ap^3)$ where $a \in \Sigma_2^0$,*
$\langle 1, x_1, x_2, y_1, y_2;\ p1 = 0,\ {x_1}^2 = y_1,\ {x_2}^2 = 0,\ x_1x_2 = y_2,\ x_iy_i = y_iy_j = 0\rangle$,
$\langle 1, x_1, x_2, y_1, y_2;\ p1 = 0,\ {x_1}^2 = {x_2}^2 = y_1,\ x_1x_2 = y_2,\ x_iy_i = y_iy_j = 0\rangle$ where $p \neq 2$,
$\langle 1, x_1, x_2, y_1, y_2;\ p1 = 0,\ {x_1}^2 = y_1,\ {x_2}^2 = \xi y_1,\ x_1x_2 = y_2,\ x_iy_i = y_iy_j = 0\rangle$ where $p \neq 2$ and ξ is a non-square in F_p,
$\langle 1, x_1, x_2, y_1, y_2;\ 2.1 = 0,\ {x_1}^2 = y_1,\ {x_2}^2 = y_2,\ x_1x_2 = y_2,\ x_iy_i = y_iy_j = 0\rangle$,
$\langle 1, x_1, x_2, y_1, y_2;\ 2.1 = 0,\ {x_1}^2 = y_1,\ {x_2}^2 = y_1 + y_2,\ x_1x_2 = y_2,\ x_iy_i = y_iy_j = 0\rangle$,
$\langle 1, x_1, x_2, x_3, y;\ p1 = 0,\ {x_1}^2 = y,\ {x_2}^2 = {x_3}^2 = x_ix_j = x_iy = y^2 = 0, \text{for } i \neq j\rangle$,
$\langle 1, x_1, x_2, x_3, y;\ p1 = 0,\ {x_1}^2 = {x_2}^2 = y,\ {x_3}^2 = x_ix_j = x_iy = y^2 = 0, \text{for } i \neq j\rangle$,
$\langle 1, x_1, x_2, x_3, y;\ p1 = 0,\ {x_i}^2 = y,\ x_ix_j = x_iy = y^2 = 0, \text{for } i \neq j\rangle$,
$\langle 1, x_1, x_2, x_3, y;\ p1 = 0,\ {x_1}^2 = y,\ {x_2}^2 = \epsilon y,\ {x_3}^2 = x_ix_j = x_iy = y^2 = 0, \text{for } i \neq j\rangle$ where $p \neq 2$ and ϵ is a non-square in F_p,

$\langle 1, x_1, x_2, x_3, y;\ 2.1 = 0,\ {x_i}^2 = x_1x_2 = x_1x_3 = x_iy = y^2 = 0,\ x_2x_3 = y\rangle$,
$\langle 1, x, y, z, p;\ p^2 1 = 0,\ x^2 = \alpha z,\ xz = p,\ y^2 = \delta p,\ z^2 = xy = yz = 0\rangle$ *where* $\alpha \in \Sigma_3$ *and* $\delta \in \Sigma_2^0$,
$\langle 1, x, y;\ p^2 1 = p^2 x = py = 0,\ x^2 = \alpha p,\ y^2 = \delta px,\ xy = 0\rangle$ *where* $p \neq 2$, $\alpha \in \Sigma_2$ *and* $\delta = 0$ *or* $\alpha \in \Sigma_4$ *and* $\delta = 1$,
$\langle 1, x, y;\ 4.1 = 4x = 2y = 0,\ x^2 = 2,\ y^2 = xy = 0\rangle$,
$\langle 1, x, y;\ 4.1 = 4x = 2y = 0,\ x^2 = 2 + 2x,\ y^2 = xy = 0\rangle$,
$\langle 1, x, y;\ 4.1 = 4x = 2y = 0,\ x^2 = 2,\ y^2 = 2x,\ xy = 0\rangle$,
$\langle 1, x_1, x_2, x_3, p;\ p^2 1 = px_i = 0,\ {x_1}^2 = \nu p,\ {x_2}^2 = {x_3}^2 = 0,\ x_ix_j = 0$ *for* $i \neq j\rangle$ *where* $p \neq 2$, ϵ *is a non-square in* F_p *and* $\nu \in \{1, \epsilon\}$,
$\langle 1, x_1, x_2, x_3, p;\ p^2 1 = px_i = 0,\ {x_1}^2 = 1,\ {x_2}^2 = \nu p,\ {x_3}^2 = 0,\ x_ix_j = 0$ *for* $i \neq j\rangle$ *where* $p \neq 2$, ϵ *is a non-square in* F_p *and* $\nu \in \{1, \epsilon\}$,
$\langle 1, x_1, x_2, x_3, p;\ p^2 1 = px_i = 0,\ {x_1}^2 = {x_2}^2 = 1,\ {x_3}^2 = \nu p,\ x_ix_j = 0$ *for* $i \neq j\rangle$ *where* $p \neq 2$, ϵ *is a non-square in* F_p *and* $\nu \in \{1, \epsilon\}$,
$\langle 1, x_1, x_2, x_3;\ 4.1 = 2x_i = 0,\ {x_1}^2 = 2,\ {x_2}^2 = {x_3}^2 = 0,\ x_ix_j = 0$ *for* $i \neq j\rangle$,
$\langle 1, x_1, x_2, x_3;\ 4.1 = 2x_i = 0,\ {x_1}^2 = 1,\ {x_2}^2 = 2,\ {x_3}^2 = 0, x_ix_j = 0$ *for* $i \neq j\rangle$,
$\langle 1, x_1, x_2, x_3;\ 4.1 = 2x_i = 0,\ {x_1}^2 = {x_2}^2 = 1,\ {x_3}^2 = 2,\ x_ix_j = 0$ *for* $i \neq j\rangle$,
$\langle 1, x_1, x_2, y;\ p^2 1 = px_i = py = 0,\ {x_1}^2 = 1,\ {x_2}^2 = 0,\ x_1x_2 = x_1y = x_2y = 0\rangle$,
$\langle 1, x_1, x_2, y;\ p^2 1 = px_i = py = 0,\ {x_1}^2 = 1,\ {x_2}^2 = y,\ x_1x_2 = x_1y = x_2y = 0\rangle$,
$\langle 1, x_1, x_2, y;\ p^2 1 = px_i = py = 0,\ {x_1}^2 = 1,\ {x_2}^2 = \xi y,\ x_1x_2 = x_1y = x_2y = 0\rangle$ *where* $p \neq 2$, ξ *is a non-square in* F_p,
$\langle 1, x_1, x_2, y;\ 4.1 = 2x_i = 2y = 0,\ {x_1}^2 = {x_2}^2 = x_1y = x_2y = 0,\ x_1x_2 = y\rangle$,
$\langle 1, x, y, p;\ p^2 1 = p^2 x = py = 0,\ x^2 = 0,\ y^2 = \delta px,\ xy = 0\rangle$ *where* $\delta \in \{0, 1\}$ *and* $p \neq 2$,
$\langle 1, x, y;\ 4.1 = 4x = 2y = 0,\ x^2 = \alpha 2x,\ y^2 = \delta 2x,\ xy = 0\rangle$ *where* $(\alpha, \delta) \in \{(0, 0), (1, 0), (1, 1)\}$,
$\langle 1, x_1, x_2;\ p^3 1 = px_i = 0,\ {x_1}^2 = {x_2}^2 = 0,\ x_1x_2 = 0\rangle$,
$\langle 1, x_1, x_2;\ p^3 1 = px_i = 0,\ {x_1}^2 = p^2,\ {x_2}^2 = x_1x_2 = 0\rangle$,
$\langle 1, x_1, x_2;\ p^3 1 = px_i = 0,\ {x_1}^2 = \epsilon p^2,\ {x_2}^2 = x_1x_2 = 0\rangle$ *where* $p \neq 2$,
$\langle 1, x_1, x_2;\ p^3 1 = px_i = 0,\ {x_1}^2 = {x_2}^2 = p^2,\ x_1x_2 = 0\rangle$,
$\langle 1, x_1, x_2;\ p^3 1 = px_i = 0,\ {x_1}^2 = p^2,\ {x_2}^2 = \epsilon p^2,\ x_1x_2 = 0\rangle$ *where* $p \neq 2$ *and* ϵ *is a non-square in* F_p,
$\langle 1, x_1, x_2;\ 8.1 = 2x_i = 0,\ {x_1}^2 = {x_2}^2 = 0,\ x_1x_2 = 4\rangle$ *or*
one of the rings given in full in [9]. The number of these rings is 10 *or* 6 *according to whether* $p \neq 2$ *or* $p = 2$.

Proof. By using [5] and [9] one can check that R is isomorphic to one of the above rings.

Theorem 1. *Let* R *be a ring with* $|Z(R)| = p^4$, *where* p *is a prime number. Then* R *is isomorphic to one of the rings described in Proposition 1, Proposition 2, the Galois ring* $GR(p^8, p^2)$, $F_{p^4}[x]/(x^2)$, $\mathbb{Z}_{p^2} \times F_{q_1} \times \ldots \times F_{q_t}$, $\mathbb{Z}_p[x]/(x^2) \times F_{q_1} \times \ldots \times F_{q_t}$ *where* $p^3 = p^2 q_1 \ldots q_t - (p^2 - p)(q_1 - 1) \ldots (q_t - 1)$, $F_{q_1} \times \ldots \times F_{q_t}$ *where* $p^4 = q_1 q_2 \ldots q_t - (q_1 - 1)(q_2 - 1) \ldots (q_t - 1)$ *or* $R_1 \times F_q$ *with* $p^2 = p + q - 1$ *where* R_1 *is isomorphic to one of the rings* $\mathbb{Z}_{p^3}$, $F_p[x, y]/(x, y)^2$, $F_p[x]/(x^3)$ *or* $\mathbb{Z}_{p^2}[x]/(px, x^2 - \varepsilon p)$ *where* $\varepsilon \in \Sigma_2^0$.

Proof. Suppose that R is a local ring. Then by Lemma 1, $|R| = p^5$, p^6 or p^8. If $|R| = p^5$ or p^6, then R is isomorphic to one of the rings described in Proposition 1 or Proposition 2. If $|R| = p^8$, then by [8, Theorem 12], R is isomorphic to the Galois ring $GR(p^8, p^2)$ or $F_{p^4}[x]/(x^2)$. Now suppose R is a nonlocal ring. If R is reduced, then we are down. Otherwise by Lemma 2, $1 \leq \sum_{i=1}^{s} t_i \leq 2$ and hence $1 \leq s \leq 2$. Thus we proceed by cases.

Case 1: $s = 1$. Then $t_1 = 1$ or 2. If $t_1 = 1$, then $R \cong R_1 \times F_{q_1} \times \ldots \times F_{q_t}$, where R_1 is a local ring of order p^2 with p zero-divisors. By [4, p.687], R_1 is isomorphic to $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p[x]/(x^2)$. If $t_1 = 2$, then by Lemma 2, $R \cong R_1 \times F_q$, where R_1 is a local ring of order p^3 with p^2 zero-divisors and $p^2 = p + q - 1$. Moreover by [4, p.687], R_1 is isomorphic to $\mathbb{Z}_{p^3}$, $F_p[x, y]/(x, y)^2$, $F_p[x]/(x^3)$ or $\mathbb{Z}_{p^2}[x]/(px, x^2 - \varepsilon p)$ where $\varepsilon \in \Sigma_2^0$.

Case 2: $s = 2$, i.e., $t_1 = t_2 = 1$. Then $R \cong R_1 \times R_2 \times F_{q_1} \times \ldots \times F_{q_t}$, where each R_i is a local ring with $|Z(R_i)| = p$. Now by Lemma 1, $|R_1| = |R_2| = p^2$. If $t > 1$, then clearly $|Z(R)| > p^4$, a contradiction. Therefore $t = 1$ and hence by relation (1) in Lemma 2, p^2 is a divisor of $q_1 - 1$. Thus $q_1 > p^2$ and so $|Z(R)| > |Z(R_1)||R_2||F_{q_1}| \geq p^5$, a contradiction.

Corollary 1. *Let R be a ring with $|Z(R)| = p_1^{k_1} p_2^{k_2} \ldots p_n^{k_n}$, where $n \geq 1$, $1 \leq k_i \leq 4$ and p_i's are distinct prime numbers. Then there exist $0 \leq s \leq \Sigma_{i=1}^{n} k_i$ and $t \geq 0$ such that*

$$R \cong R_1 \times \ldots \times R_s \times F_{q_1} \times \ldots \times F_{q_t}$$

where F_{q_i}'s are finite fields and each R_i is local ring with $|Z(R_i)| = p_j^{t_j}$ for some p_j $(1 \leq j \leq n)$ and $1 \leq t_j \leq k_j$. Consequently, each R_i is isomorphic to one of the local rings described in [2, Theorem 5] or Theorem 1.

Proof. We put

$$R \cong R_1 \times \ldots \times R_s \times F_{q_1} \times \ldots \times F_{q_t},$$

where $F_{q_1}, \ldots, F_{q_t}$ are finite fields and each R_i is a commutative finite local ring with identity that is not a field. By Lemma 2, for each i, $|Z(R_i)| = p^k$ for some prime number p and $k \geq 1$ such that p^k is a divisor of $|Z(R)|$ and also $0 \leq s \leq \Sigma_{i=1}^{n} k_i$. Thus $|Z(R_i)| = p_j^{t_j}$ where $1 \leq t_j \leq k_j$, $1 \leq j \leq n$ and $1 \leq i \leq s$. Hence for each $1 \leq i \leq s$, $t_i = 1, 2, 3$ or 4 and so each R_i is isomorphic to one of the local rings described in [2, Theorem 5] or Theorem 1.

Theorem 2. *Let R be a commutative nonlocal ring with $|Z(R)| = p^6$ where p is a prime number. Then R is isomorphic to one of the rings $F_{q_1} \times \ldots \times F_{q_t}$ with $p^6 = q_1 q_2 \ldots q_t - (q_1 - 1)(q_2 - 1) \ldots (q_t - 1)$, $\mathbb{Z}_4 \times \mathbb{Z}_4 \times F_5, \mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_4 \times F_5$, $\mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2[x]/(x^2) \times F_5$, $R_1 \times F_{q_1} \times \ldots \times F_{q_t}$, where R_1 is isomorphic to $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p[x]/(x^2)$ and $p^5 = pq_1 q_2 \ldots q_t - (p - 1)(q_1 - 1)(q_2 - 1) \ldots (q_t - 1)$, $R_1 \times F_{q_1} \times \ldots \times F_{q_t}$ where R_1 is isomorphic to one the rings $\mathbb{Z}_{p^3}$, $F_p[x, y]/(x, y)^2$, $F_p[x]/(x^3)$ or $\mathbb{Z}_{p^2}[x]/(px, x^2 - \varepsilon p)$ where $\varepsilon \in \Sigma_2^0$ and $p^4 = pq_1 q_2 \ldots q_t - (p - 1)(q_1 - 1)(q_2 - 1) \ldots (q_t - 1)$, $R_1 \times F_{q_1} \times \ldots \times F_{q_t}$, where R_1 is isomorphic to $F_{p^2}[x]/(x^2)$ or $GR(p^4, p^2)$ and $p^4 = p^2 q_1 q_2 \ldots q_t - (p^2 - 1)(q_1 - 1)(q_2 - 1) \ldots (q_t - 1)$, $R_1 \times F_{q_1} \times \ldots \times F_{q_t}$,* *where R_1 is isomorphic to one of the local rings of order p^4 described in [2, corollary 3] and $p^3 = pq_1q_2\dots q_t - (p-1)(q_1-1)(q_2-1)\dots(q_t-1)$ or $R_1 \times F_q$ where R_1 is isomorphic to one of the rings described in Proposition 2, and $p^2 = p+q-1$.*

Proof. If R is reduced, then we are down. Now suppose that R is not reduced. Then by [2, Theorem 4], either R is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4 \times F_5$, $\mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_4 \times F_5$, $\mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2[x]/(x^2) \times F_5$ or $R \cong R_1 \times F_{q_1} \times \dots \times F_{q_t}$, where R_1 is a local ring with $|Z(R_1)| = p^k$ $(1 \le k \le 4)$ and $t \ge 1$. Thus we proceed by cases.

Case 1: $|Z(R_1)| = p$. Then by [4, p.687], R_1 is isomorphic to $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p[x]/(x^2)$ and $p^5 = pq_1q_2\dots q_t - (p-1)(q_1-1)(q_2-1)\dots(q_t-1)$.

Case 2: $|Z(R_1)| = p^2$. Then by Lemma 1, $|R_1| = p^3$ or p^4. If $|R_1| = p^3$, then by [4, p.687], R_1 is isomorphic to one the rings $\mathbb{Z}_{p^3}$, $F_p[x,y]/(x,y)^2$, $F_p[x]/(x^3)$ or $\mathbb{Z}_{p^2}[x]/(px, x^2-\varepsilon p)$ where $\varepsilon \in \Sigma_2^0$ and $p^4 = pq_1q_2\dots q_t - (p-1)(q_1-1)(q_2-1)\dots(q_t-1)$. If $|R_1| = p^4$, then by [8, Theorem 12], R_1 is isomorphic to $F_{p^2}[x]/(x^2)$ or $GR(p^4, p^2)$ and $p^4 = p^2q_1q_2\dots q_t - (p^2-1)(q_1-1)(q_2-1)\dots(q_t-1)$.

Case 3: $|Z(R_1)| = p^3$. Then by Lemma 1, $|R_1| = p^4$ or p^6. If $|R_1| = p^6$, then $|Z(R)| > |R_1|$ which is impossible. Thus $|R_1| = p^4$ and so R_1 is isomorphic to one of the local rings of order p^4 described in [2, corollary 3] and $p^3 = pq_1q_2\dots q_t - (p-1)(q_1-1)(q_2-1)\dots(q_t-1)$.

Case 4: $|Z(R_1)| = p^4$. Then by Lemma 1, $|R_1| = p^5, p^6$ or p^8. If $|R_1| = p^6$ or p^8, then $|Z(R)| > |R_1|$ which is impossible. Thus $|R_1| = p^5$ and so R_1 is isomorphic to one of the rings described in Proposition 2. Also by Lemma 2, $R \cong R_1 \times F_q$ and $p^2 = p+q-1$.

Theorem 3. *Let R be a commutative nonlocal ring with $|Z(R)| = p^7$ where p is a prime number. Then R is isomorphic to one of the rings $F_{q_1} \times \dots \times F_{q_t}$ with $p^7 = q_1q_2\dots q_t - (q_1-1)(q_2-1)\dots(q_t-1)$, $\mathbb{Z}_4 \times \mathbb{Z}_4 \times F_3 \times F_3$, $\mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_4 \times F_3 \times F_3$, $\mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2[x]/(x^2) \times F_3 \times F_3$, $R_1 \times R_2 \times F_5$ where R_1 is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$ and R_2 is isomorphic to one of the rings $\mathbb{Z}_8$, $\mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_2[x]/(x^3)$ or $\mathbb{Z}_4[x]/(2x, x^2-2\varepsilon)$ where $\varepsilon \in \Sigma_2^0$, $R_1 \times R_2 \times F_{q_1} \times \dots \times F_{q_t}$, where p is an odd prime number, each R_i is isomorphic to $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p[x]/(x^2)$ and $p^5 = p^2q_1q_2\dots q_t - (p-1)^2(q_1-1)(q_2-1)\dots(q_t-1)$, $R_1 \times F_{q_1} \times \dots \times F_{q_t}$ where R_1 is isomorphic to $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p[x]/(x^2)$ with $p^6 = pq_1q_2\dots q_t - (p-1)(q_1-1)(q_2-1)\dots(q_t-1)$, $R_1 \times F_{q_1} \times \dots \times F_{q_t}$ where R_1 is isomorphic to one the rings $\mathbb{Z}_{p^3}$, $F_p[x,y]/(x,y)^2$, $F_p[x]/(x^3)$ or $\mathbb{Z}_{p^2}[x]/(px, x^2-\varepsilon p)$ where $\varepsilon \in \Sigma_2^0$ with $p^5 = pq_1q_2\dots q_t - (p-1)(q_1-1)(q_2-1)\dots(q_t-1)$, $R_1 \times F_{q_1} \times \dots \times F_{q_t}$ where $R_1 \cong F_{p^2}[x]/(x^2)$ or $GR(p^4, p^2)$ with $p^5 = p^2q_1q_2\dots q_t - (p^2-1)(q_1-1)(q_2-1)\dots(q_t-1)$, $R_1 \times F_{q_1} \times \dots \times F_{q_t}$ where R_1 is isomorphic to one of the local rings of order p^4 described in [2, Corollary 3] and $p^4 = pq_1q_2\dots q_t - (p-1)(q_1-1)(q_2-1)\dots(q_t-1)$, $R_1 \times F_q$ where $R_1 \cong F_{p^3}[x]/(x^2)$ or $GR(p^6, p^2)$ with $p^4 = p^3+p-1$, $R_1 \times F_{q_1} \times \dots \times F_{q_t}$ where R_1 is isomorphic to one of the rings described in Proposition 2, with $p^3 = pq_1q_2\dots q_t - (p-1)(q_1-1)(q_2-1)\dots(q_t-1)$, $R_1 \times F_q$ where R_1 is isomorphic to one of the rings described in Proposition 1, with $p^3 = p^2+q-1$ or $R_1 \times F_q$, where R_1 is a local ring of order p^6 with p^5 zero-divisors and $p^2 = p+q-1$.*

Proof. If R is reduced or $R \cong R_1 \times R_2 \times F_5$ where R_1 is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$ and R_2 is isomorphic to one of the rings $\mathbb{Z}_8$, $\mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_2[x]/(x^3)$ or $\mathbb{Z}_4[x]/(2x, x^2-2\varepsilon)$ where $\varepsilon \in \Sigma_2^0$, then we are done. Otherwise by [2, Theorem 4], we have the following two cases.

Case 1: $R \cong R_1 \times F_{q_1} \times \ldots \times F_{q_t}$, where each F_{q_i} $(1 \le i \le t)$ is a finite field and R_1 is a local ring with $|Z(R_1)| = p^m$, $|R_1| = p^n$ such that $0 < m < n \le 6$ and

$$p^7 = p^n q_1 q_2 \ldots q_t - (p^n - p^m)(q_1-1)(q_2-1)\ldots(q_t-1).$$

Case 2: $R \cong R_1 \times R_2 \times F_{q_1} \times \ldots \times F_{q_t}$, where each F_{q_i} $(1 \le i \le t)$ is a finite field, each R_i is isomorphic to $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p[x]/(x^2)$ and

$$p^5 = p^2 q_1 q_2 \ldots q_t - (p-1)^2(q_1-1)(q_2-1)\ldots(q_t-1). \tag{2}$$

In case 1, as in the proof of Theorem 2, R is isomorphic to one of the rings $R_1 \times F_{q_1} \times \ldots \times F_{q_t}$ where R_1 is isomorphic to $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p[x]/(x^2)$ with $p^6 = pq_1q_2\ldots q_t - (p-1)(q_1-1)(q_2-1)\ldots(q_t-1)$, $R_1 \times F_{q_1} \times \ldots \times F_{q_t}$, where R_1 is isomorphic to one the rings $\mathbb{Z}_{p^3}$, $F_p[x,y]/(x,y)^2$, $F_p[x]/(x^3)$ or $\mathbb{Z}_{p^2}[x]/(px, x^2-\varepsilon p)$ where $\varepsilon \in \Sigma_2^0$ with $p^5 = pq_1q_2\ldots q_t - (p-1)(q_1-1)(q_2-1)\ldots(q_t-1)$, $R_1 \times F_{q_1} \times \ldots \times F_{q_t}$, where R_1 is isomorphic to $F_{p^2}[x]/(x^2)$ or $GR(p^4, p^2)$ with $p^5 = p^2q_1q_2\ldots q_t - (p^2-1)(q_1-1)(q_2-1)\ldots(q_t-1)$, $R_1 \times F_{q_1} \times \ldots \times F_{q_t}$, where R_1 is isomorphic to one of the local rings of order p^4 described in [2, Corollary 3] with $p^4 = pq_1q_2\ldots q_t - (p-1)(q_1-1)(q_2-1)\ldots(q_t-1)$, $R_1 \times F_q$ where R_1 is isomorphic to $F_{p^3}[x]/(x^2)$ or $GR(p^6, p^2)$ with $p^4 = p^3 + p - 1$, $R_1 \times F_{q_1} \times \ldots \times F_{q_t}$ where R_1 is isomorphic to one of the rings described in Proposition 2, with $p^3 = pq_1q_2\ldots q_t - (p-1)(q_1-1)(q_2-1)\ldots(q_t-1)$, $R_1 \times F_q$ where R_1 is isomorphic to one of the rings described in Proposition 1, with $p^3 = p^2 + q - 1$ or $R_1 \times F_q$, where R_1 is a local ring of order p^6 with p^5 zero-divisors.

In case 2, If $p = 2$, then Since $|Z(R)| > |R_1||R_2|q_1\ldots q_{t-1}$, $t \le 3$. We claim that $t = 2$. If $t = 1$, then the relation (2) implies that $q_1 = 31/3$, a contradiction. If $t = 3$, then the relation (2) implies that 4 is a divisor of $(q_1-1)(q_2-1)(q_3-1)$. Without loss of generality we can assume that either 4 is a divisor of (q_1-1) or 2 is a divisor of both (q_1-1) and (q_2-1). Therefore either $q_1 \ge 5$ or $q_1 \ge 3$ and $q_2 \ge 3$. This implies that either $2^7 = |Z(R)| > 5|R_1||R_2|q_2 = 80q_2$ or $2^7 = |Z(R)| > 9|R_1||R_2| = 144$. But it is impossible in any case. Thus $t = 2$ and by the relation (2) we have

$$2^5 = 2^2 q_1 q_2 - (q_1-1)(q_2-1). \tag{3}$$

If 4 is a divisor of (q_i-1) for some i, then $q_i \ge 5$ and hence $2^5 = 3q_1q_2 + q_1 + q_2 - 1 \ge 30 + 7 - 1 = 36$, a contradiction. Thus 2 is a divisor of both (q_1-1) and (q_2-1). Then $q_1 - 1 = 2k_1$, $q_2 - 1 = 2k_2$ for some positive integers k_1, k_2 and put them into (3). Then we obtain $8 = 3k_1k_2 + 2k_1 + 2k_2 + 1$. It yields $k_1 = k_2 = 1$ and hence $q_1 = q_2 = 3$. Thus $R \cong R_1 \times R_2 \times F_3 \times F_3$ where R_1 and R_2 are isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$.

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