On $\mathcal{I}$-Convergence in the Topology Induced by Probabilistic Norms

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Abstract. The concepts of $\mathcal{I}$-convergence is a natural generalization of statistical convergence and it is dependent on the notion of the ideal of subsets of $\mathbb{N}$ of positive integer set. In this paper we study the $\mathcal{I}$-convergence of sequences, $\mathcal{I}$-convergence of sequences of functions and $\mathcal{I}$-Cauchy sequences in probabilistic normed spaces and prove some important results.

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1. Introduction

The concepts of statistical convergence was introduced (independently) by Fast [7] and Steinhouse [25]. In their studies, the concept of ordinary convergence of sequence of real numbers was extended to statistical convergence in the following way: a sequence \( \{x_n\} \subset \mathbb{R} \) is said to be statistically convergent to the real number \( x_0 \in \mathbb{R} \) provided that each \( \varepsilon \) neighborhood \( N_\varepsilon(x_0) \) of \( x_0 \), the set consisting of all elements not contained by \( N_\varepsilon(x_0) \) has natural density zero for any \( \varepsilon > 0 \). The notion of natural density here can be described as a function \( \delta : 2^{\mathbb{N}} \to [0,1] \) and given by \( \delta(K) := \lim_{n \to \infty} n^{-1}|\{k \in K : k \leq n\}| \) where \( K \subset \mathbb{N} \), and \( |A| \) denotes the cardinality of the set \( A \). The concept of statistical convergence was further discussed and developed by many authors including [1,4,8–11,19,21]. Statistical convergence has also been discussed in more general abstract spaces such as the fuzzy number spaces [22], locally convex spaces [18], Banach spaces [15] and characterization of Banach spaces [5]. Recently, Karakus [13] has extended the concept of statistical convergence for sequences in probabilistic normed spaces (PN space) and proved several interesting results. In another paper, Karakus and Demirci [14] studied the concept of statistical convergence of double sequences on PN spaces. The idea of \( \mathcal{I} \)-convergence for sequences, was inspired by the concept of statistical convergence introduced in [7], see Kostyrko et al. [16] for a comprehensive bibliography. It is a natural generalization of the concept of statistical convergence. The \( \mathcal{I} \)-convergence is based on the notion of the ideal \( \mathcal{I} \) of subsets of \( \mathbb{N} \), the set of positive integers. Here, a sequence \( \{x_n\} \subset \mathbb{R} \) is said to be \( \mathcal{I} \)-convergent to the real number \( x_0 \in \mathbb{R} \) provided that each \( \varepsilon \) neighborhood \( N_\varepsilon(x_0) \) of \( x_0 \), the set consisting of all elements not contained by \( N_\varepsilon(x_0) \) belongs to \( \mathcal{I} \) for any \( \varepsilon > 0 \). Further works on ideal convergence can be found in [2,3,6,12,17,20] The work of Karakus [13] inspired us to study the \( \mathcal{I} \)-convergence and other related properties in PN spaces. In this context, we obtain
some results that parallel to the one given in [3, 12, 20, 24].

Now we recall some notation and definitions used in this paper (see [26]).

**Definition 1.1.** A function \( f : \mathbb{R} \rightarrow \mathbb{R}_0^+ \) is called a distribution function if it is non-decreasing and left-continuous with \( \inf_{t \in \mathbb{R}} f(t) = 0 \) and \( \sup_{t \in \mathbb{R}} f(t) = 1 \). We denote the set of all distribution function by \( \Delta^+ \).

**Definition 1.2.** A \( t \)-norm \( T \) is a continuous mapping \( T : [0, 1] \times [0, 1] \rightarrow [0, 1] \) such that for all \( a, b, c, d \in [0, 1] \)

(i) \( T(a, b) = T(b, a) \);
(ii) \( T(a, T(b, c)) = T(T(a, b), c) \);
(iv) \( T(a, b) \leq T(c, d) \) whenever \( a \leq c \) and \( b \leq d \);
(v) \( T(a, 1) = a \).

**Example 1.1.** The operation \( T(a, b) = ab \), \( T(a, b) = \max(a + b - 1, 0) \) and \( T(a, b) = \min(a, b) \) on \([0, 1]\) are \( t \)-norms.

The following definition is due to A. N. Šerstnev [23].

**Definition 1.3.** A probabilistic normed space (briefly, a PN space) is a triplet \((X, F, T)\), where \( X \) is a real linear space, \( T \) is a continuous \( t \)-norm, and \( F \) (called probabilistic norm) is a mapping from \( X \) into \( \Delta^+ \) (writing \( F(x) \) as \( F_x \)), the following conditions hold for every \( x, y \in X \) and every \( s, t > 0 \):

(N1) \( F_x(t) = 1 \) if and only if \( x = \theta \) (the null vector of \( X \));
(N2) \( F_{\alpha x}(t) = F_x(\frac{t}{|\alpha|}) \) for \( \alpha \neq 0 \);
(N3) \( F_{x+y}(s + t) \geq T(F_x(s), F_y(t)) \);

**Example 1.2.** Let \((X, \| \cdot \|)\) is a normed space and \( T(a, b) = ab \) (or \( T(a, b) = \min(a, b) \)).
Define

\[
F_x(t) = \frac{t}{t + \|x\|}
\]
where $x \in X$ and $t > 0$. Then $(X, F, T)$ is a PN space.

Let $(X, F, T)$ be a PN space. Since $T$ is a continuous $t$-norm, the system of $(\varepsilon, \lambda)$-neighborhoods of $\theta$ (the null vector in $X$)

$$\mathcal{N}_\theta(\varepsilon, \lambda) = \{x \in X : F_x(\varepsilon) > 1 - \lambda\}$$

(1.1)

determines a first countable Hausdorff topology on $X$, called the $F$-topology. Thus, the $F$-topology can be completely specified by means of $F$-convergence of sequences.

It is clear that $x - y \in \mathcal{N}_\theta$ means $y \in \mathcal{N}_x$ and vice-versa.

A sequence $(x_n)$ is said to be $F$-convergent to $\xi \in X$ if for every $\varepsilon > 0$, and for every $\lambda \in (0, 1)$ there exists a number $N \in \mathbb{N}$ such that

$$x_n - \xi \in \mathcal{N}_\theta(\varepsilon, \lambda) \quad \text{for all} \quad n \geq N.$$

or equivalently,

$$x_n \in \mathcal{N}_\xi(\varepsilon, \lambda) \quad \text{for all} \quad n \geq N.$$

In this case we write $F - \lim x_n = \xi$.

**Lemma 1.1.** Let $(X, \| \cdot \|)$ be a real normed space and $(X, F, T)$ be a PN space induced by the probabilistic norm $F_x(t) = \frac{t}{t + \|x\|}$, where $x \in X$ and $t > 0$. Then for every $\{x_n\}$ in $X$

$$\lim x_n = \xi \Rightarrow 3 - \lim x_n = \xi.$$

**Proof.** Let suppose that $\lim x_n = \xi$. Then for every $t > 0$ there exists a positive integer $N = N(t)$ such that

$$\|x_n - \xi\| < t \quad \text{for all} \quad n \geq N.$$

We observe that for any given $\varepsilon > 0$,
\[
\frac{\epsilon + \|x_n - \xi\|}{\epsilon} < \frac{\epsilon + t}{\epsilon}
\]

which is equivalent to

\[
\frac{\epsilon}{\epsilon + \|x_n - \xi\|} > \frac{\epsilon}{\epsilon + t} = 1 - \frac{t}{\epsilon + t}.
\]

Therefore, by letting \(\lambda = \frac{t}{\epsilon + t} \in (0, 1)\) we have

\[
F_{x_n-\xi}(\epsilon) > 1 - \lambda \quad \text{for all} \quad n \geq N.
\]

This implies that \(x_n \in \mathcal{M}_\xi(\epsilon, \lambda)\) for all \(n \geq N\) as required.

We recall the definition and notations of ideal.

**Definition 1.4.** A non-empty subset \(\mathcal{I}\) of \(2^\mathbb{N}\) is called an ideal on \(\mathbb{N}\) if

(i) \(B \in \mathcal{I}\) whenever \(B \subseteq A\) for some \(A \in \mathcal{I}\) (closed under subsets),

(ii) \(A \cup B \in \mathcal{I}\) whenever \(A, B \in \mathcal{I}\) (closed under unions).

An ideal called proper if \(\mathbb{N} \notin \mathcal{I}\). An ideal called admissible if its proper and contains all finite subsets.

Filter \(\mathcal{F}\) is a dual notion to ideal \(\mathcal{I}\)-it is closed under supersets and intersections. It holds that \(\{\mathbb{N}\setminus A : A \in \mathcal{I}\}\) is a filter if and only if \(\mathcal{I}\) is ideal. The filter \(\mathcal{F}(\mathcal{I})\) is called the filter associated with the ideal \(\mathcal{I}\). Thus, one can write

\[
A \in \mathcal{I} \iff A^c \in \mathcal{F}(\mathcal{I}).
\]

where \(A^c\) denotes the complement of \(A\).

Ideal can be viewed as a way to describe which sets will be considered "small", i.e., finite. Filter is collection of all "large" sets.
2. $\mathcal{I}$-Convergence for Sequences in PN Spaces

In this section we define the ideal convergence of a sequence in $(X, F, T)$ and prove some important results.

**Definition 2.1.** Let $\mathcal{I} \subseteq 2^\mathbb{N}$ be a proper ideal in $\mathbb{N}$ and $(X, F, T)$ be a PN space. The sequence $(x_n)$ in $X$ is said to be $\mathcal{I}^F$-convergent to $x \in X$ ($\mathcal{I}$-convergent to $x \in X$ with respect to $F$-topology) if for each $\epsilon > 0$, and $\lambda \in (0, 1)$

$$\{n \in \mathbb{N}: x_n /\not\in \mathcal{N}_\epsilon(x, \lambda)\} \in \mathcal{I}.$$  

The vector $x$ is called the $\mathcal{I}^F$-limit of the sequence $\{x_n\}$ and we write $\mathcal{I}^F$-lim $x_n = x$.

**Definition 2.2.** Let $(X, F, T)$ be a PN space and $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. The sequence $(x_n)$ in $X$ is said to be $\mathcal{I}^{F*}$-convergent to $\xi \in X$ (i.e., $\mathcal{I}^{F*}$-lim $x_n = \xi$) if and only if there exists a set $M = \{m_1 < m_2 < \cdots\} \in \mathcal{F}(\mathcal{I})$ such that $\mathcal{I}$-lim $x_{m_k} = \xi$.

**Lemma 2.1.** Let $(X, F, T)$ be a PN space. $\mathcal{I}^F$-limit of any sequence if exists is unique.

*Proof.* Let $\{x_n\}$ be any sequence and suppose that $\mathcal{I}^F$-lim $x_n = \xi$, $\mathcal{I}^F$-lim $x_n = \eta$ where $\xi \neq \eta$. Since $\xi \neq \eta$, select $\epsilon > 0$ and $\lambda \in (0, 1)$ such that $\mathcal{N}_\xi(\epsilon, \lambda)$ and $\mathcal{N}_\eta(\epsilon, \lambda)$ are disjoint neighborhoods of $\xi$ and $\eta$. Since $\xi$ and $\eta$ both are $\mathcal{I}^F$-limit of the sequence $(x_n)$, we have $A = \{n \in \mathbb{N}: x_n /\not\in \mathcal{N}_\xi(\epsilon, \lambda)\}$ and $B = \{n \in \mathbb{N}: x_n /\not\in \mathcal{N}_\eta(\epsilon, \lambda)\}$ are both belongs to $\mathcal{I}$. This implies that the sets $A^c = \{n \in \mathbb{N}: x_n \in \mathcal{N}_\xi(\epsilon, \lambda)\}$ and $B^c = \{n \in \mathbb{N}: x_n \in \mathcal{N}_\eta(\epsilon, \lambda)\}$ belongs to $\mathcal{F}(\mathcal{I})$. Since $\mathcal{F}(\mathcal{I})$ is a filter in $\mathbb{N}$, we have $A^c \cap B^c$ is a nonempty in $\mathcal{F}(\mathcal{I})$. In this way we obtain a contradiction to the fact that the neighborhoods $\mathcal{N}_\xi(\epsilon, \lambda)$ and $\mathcal{N}_\eta(\epsilon, \lambda)$ of $\xi$ and $\eta$ are disjoints. Hence we have $\xi = \eta$. This completes the proof.
Lemma 2.2. Let \((X, F, T)\) be a PN space and \(\mathcal{J}_{\text{fin}}\) be Fréchet ideal (finite subsets on \(\mathbb{N}\)). Then \(F\) convergence implies \(\mathcal{J}_{\text{fin}}^F\) convergence.

Proof. Let \(\epsilon > 0\) and \(\lambda \in (0, 1)\). Suppose that \(\{x_n\}\) is \(F\) convergent to \(\xi\). Then, there exists a number \(N \in \mathbb{N}\) such that \(x_n \in \mathcal{N}_\xi(\epsilon, \lambda)\) for every \(n \geq N\). This implies that the set \(A = \{n \in \mathbb{N}: x_n \notin \mathcal{N}_\xi(\epsilon, \lambda)\} \subseteq \{1, 2, \cdots, N - 1\}\). Since the right hand side belongs to \(\mathcal{J}_f\), we have \(A \in \mathcal{J}_{\text{fin}}\). This shows that \(\{x_n\}\) is \(\mathcal{J}_{\text{fin}}^F\) convergent to \(\xi\).

The following example shows that the converse of above theorem is not valid.

Example 2.1. By letting \(X = \mathbb{R}\) in Example 1, we have \((\mathbb{R}, F, T)\) is a PN space induced by the probabilistic norm \(F_x(\epsilon) = \frac{\epsilon}{\epsilon + \|x\|}\). Let us suppose that \(A \in \mathcal{J}_{\text{fin}}\). Define a sequence \(\{x_n\}\) in \(\mathbb{R}\) via

\[
x_n = \begin{cases} 
1, & \text{if } n \in A \\
0, & \text{otherwise.}
\end{cases}
\]

Then, for every \(\epsilon > 0\) and \(\lambda \in (0, 1)\), let \(K = \{n \in \mathbb{N}: x_n \notin \mathcal{N}_\theta(\epsilon, \lambda)\}\). We observe that

\[
x_n \notin \mathcal{N}_\theta(\epsilon, \lambda) \Rightarrow F_x(\epsilon) \leq 1 - \lambda \Rightarrow \frac{\epsilon}{\epsilon + \|x_n\|} \leq 1 - \lambda \Rightarrow \|x_n\| \geq \frac{\epsilon \lambda}{1 - \lambda} > 0.
\]

Hence, we have

\[
K = \{n \in \mathbb{N}: \|x_n\| > 0\} = \{n \in \mathbb{N}: x_n = 1\} = A \in \mathcal{J}_f.
\]

Therefore \(\mathcal{J}_{\text{fin}}^F - \lim x_n = \theta\). But the sequence \(\{x_n\}\) is not convergent to \(\theta\) in \((\mathbb{R}, \|\cdot\|)\). By Lemma 1, this implies that \(F - \lim x_n \neq \theta\).
Lemma 2.3. Let \((X, F, T)\) be a PN space and \(\mathcal{I}\) is an admissible ideal on \(X\). Then \(\mathcal{I}_{\text{fin}}\)-convergence implies \(\mathcal{I}\)-convergence.

Proof. For \(\mathcal{I}\) be an admissible ideal, we have \(\bigcup \mathcal{I} = \mathbb{N}\). This implies that \(\mathcal{I}_{\text{fin}} \subset \mathcal{I}\). So, \(\mathcal{I}_{\text{fin}}\)-convergence implies \(\mathcal{I}\)-convergence.

The following lemma is an immediate consequence of definition of statistical convergence sequence.

Lemma 2.4. Let \((X, F, T)\) be a PN space. If \(\mathcal{I}_\delta = \{ A \subseteq \mathbb{N} : \delta(A) = 0 \}\) where \(\delta(A)\) be the density of \(A\), then \(\mathcal{I}_\delta\)-convergence coincide with statistical convergence.

Lemma 2.5. If \(\{x_n\}\) and \(\{y_n\}\) are two sequences in \((X, F, T)\) with \(T(a, a) > a\) for every \(a \in (0, 1)\), then

(i) If \(\mathcal{I}\)-lim \(x_n = \xi\) and \(\mathcal{I}\)-lim \(y_n = \eta\), then \(\mathcal{I}\)-lim \((x_n + y_n) = \xi + \eta\).

(ii) If \(\mathcal{I}\)-lim \(x_n = \xi\) and \(a \in \mathbb{R}\), then \(\mathcal{I}\)-lim \(ax_n = a\xi\).

(iii) If \(\mathcal{I}\)-lim \(x_n = \xi\) and \(\mathcal{I}\)-lim \(y_n = \eta\), then \(\mathcal{I}\)-lim \((x_n - y_n) = \xi - \eta\).

Proof. (i) Let \(\epsilon > 0\) and \(\lambda \in (0, 1)\). Since \(\mathcal{I}\)-lim \(x_n = \xi\) and \(\mathcal{I}\)-lim \(y_n = \eta\), the sets \(A = \{ n \in \mathbb{N} : x_n \notin \mathcal{N}_\xi(\frac{\epsilon}{2}, \lambda) \}\) and \(B = \{ n \in \mathbb{N} : x_n \notin \mathcal{N}_\eta(\frac{\epsilon}{2}, \lambda) \}\) are belongs to \(\mathcal{I}\). Let \(C = \{ n \in \mathbb{N} : x_n + y_n \notin \mathcal{N}_{\xi + \eta}(\epsilon, \lambda) \}\). Since \(\mathcal{I}\) is an ideal it is sufficient to show that \(C \subset A \cup B\). This is equivalent to show that \(C^c \supset A^c \cap B^c\) where \(A^c\) and \(B^c\) are belongs to \(\mathcal{F}(\mathcal{I})\). Let \(n \in A^c \cap B^c\), i.e., \(n \in A^c\) and \(n \in B^c\) then by (N4) we have

\[
F_{(x_n+y_n)-(\xi+\eta)}(\epsilon) \geq \tau_T(F_{x_n-\xi}, F_{y_n-\eta})(\epsilon) \\
\geq T(F_{x_n-\xi}(\frac{\epsilon}{2}), F_{y_n-\eta}(\frac{\epsilon}{2})) \\
> T(1 - \lambda, 1 - \lambda) \\
> 1 - \lambda.
\]

Hence, \(n \in C^c \supset A^c \cap B^c \in \mathcal{F}(\mathcal{I})\) which implies \(C \subset A \cup B \in \mathcal{I}\) and the result follows.
(ii) Let $\varepsilon > 0$ and $\lambda \in (0, 1)$. Since $\mathcal{F}^F - \lim x_n = \xi$, we have $A = \{n \in \mathbb{N}: x_n \notin \mathcal{N}_\xi(\varepsilon, \lambda)\} \in \mathcal{I}$. This implies that $A^c = \{n \in \mathbb{N}: x_n \in \mathcal{N}_\xi(\varepsilon, \lambda)\} \in \mathcal{F}(\mathcal{I})$. Let $n \in A^c$.

For the case $\alpha = 0$, we have

$$F_{\alpha x_n - \xi}(\varepsilon) = F_0 \varepsilon = 1 > 1 - \lambda$$

and for the case $\alpha \neq 0$, we have

$$F_{\alpha x_n - \alpha \xi}(\varepsilon) = F_{\alpha x_n - \xi}(\varepsilon) \geq T \left( F_{x_n - \xi}(\varepsilon), F_0 \left( \frac{\varepsilon}{|\alpha|} - \varepsilon \right) \right) > T(1 - \lambda, 1) = 1 - \lambda.$$

This shows that $\{n \in \mathbb{N}: \alpha x_n \notin \mathcal{N}_{\alpha \xi}(\varepsilon, \lambda)\} \in \mathcal{I}$ and consequently we have $\mathcal{F}^3 - \lim \alpha x_n = \alpha \xi$.

(iii) The proof is obvious from (i) and (ii).

**Definition 2.3.** Let $(X, F, T)$ be a PN space. A subset $A = \{x_n\}$ of $X$ is said to be $\mathcal{F}^F$-bounded on PN spaces if for every $\lambda \in (0, 1)$, there exists $\varepsilon > 0$ such that

$$\{n \in \mathbb{N}: \alpha x_n \notin \mathcal{N}_{\alpha \xi}(\varepsilon, \lambda)\} \in \mathcal{I}.$$

Let $(X, F, T)$. We denote $\mathcal{F}^F_b(X)$ the set of all $\mathcal{F}^F$-bounded $I^F -$ convergent sequences on $X$ and $l^F_\infty(X)$ the set of all $\mathcal{F}^F$-bounded sequences on $X$.

**Theorem 2.1.** Let $(X, F, T)$ be a PN space such that $T(a, a) > a$ for every $a \in (0, 1)$. Let $\mathcal{I} \subset 2^\mathbb{N}$ be an admissible ideal in $\mathbb{N}$. Then $\mathcal{F}^F_b(X)$ is a closed linear subspace of the set $l^F_\infty(X)$.

**Proof.** In view of Lemma (i), it is clear that the set $\mathcal{F}^F_b(X)$ is a linear subspace of the set $l^F_\infty(X)$. So to prove the result it is sufficient to prove that $\mathcal{F}^F_b(X) = \mathcal{F}^F_\infty(X)$. It
is clear that $\mathcal{I}_b^F(X) \subset \mathcal{I}_b^{F_n}(X)$. Now we show that $\mathcal{I}_b^{F_m}(X) \subset \mathcal{I}_b^F(X)$. Let $y \in \mathcal{I}_b^F(X)$.

We notice that since $\mathcal{N}_\epsilon(e, \lambda) \cap \mathcal{I}_b^F(X) \neq \emptyset$ for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists an $x \in \mathcal{N}_\epsilon(e, \lambda) \cap \mathcal{I}_b^F(X)$ such that the set $K = \{n \in \mathbb{N}: x \notin \mathcal{N}_\epsilon(e, \lambda)\}$ belongs to $\mathcal{I}$. This implies that $K^c = \{n \in \mathbb{N}: x \in \mathcal{N}_\epsilon(e, \lambda)\} \in \mathcal{F}(\mathcal{I})$. Now let $n \in K^c$, then by (N4), we have

$$F_{y_n}(\epsilon) = F_{y_n-x_n+x_n}(\epsilon) \geq T \left( F_{y_n-x_n}(\frac{\epsilon}{2}), F_{x_n}(\frac{\epsilon}{2}) \right) > T(1-\lambda, 1-\lambda) > 1-\lambda.$$  

Thus, we have $\{n \in K^c: y_n \notin \mathcal{N}_\epsilon(e, \lambda) > 1-\lambda\} \in \mathcal{F}(\mathcal{I})$ which implies that $\{n \in \mathbb{N}: y_n \notin \mathcal{N}_\epsilon(e, \lambda) > 1-\lambda\} \in \mathcal{I}$. Thus $y \in \mathcal{I}_b^F(X)$ and this completes the proof.

**Lemma 2.6.** If a sequence in a PN space $(X, F, T)$ is $\mathcal{F}^{F_n}$-convergent, then it is $\mathcal{I}_b^F$-convergent to the same limit.

*Proof.* Let $\mathcal{F}^{F_n}$-lim $x_n = \xi$, then by definition, there exists $M = \{m_1 < m_2 < \cdots \} \in \mathcal{F}(\mathcal{I})$ such that $F - \text{lim} x_{m_k} = \xi$. Let $\epsilon > 0$ and $\lambda \in (0, 1)$ be given. Since $F - \text{lim} x_{m_k} = \xi$, there exists $N \in \mathbb{N}$ such that $x_{m_k} \in \mathcal{N}_{\epsilon}(\xi, \lambda)$ for every $k \geq N$. Let $A = \{k \in \mathbb{N}: x_{m_k} \notin \mathcal{N}_{\epsilon}(\xi, \lambda)\}$. Then it is clear that $A \subset \{1, 2, \cdots, N-1\} \in \mathcal{I}$. Therefore, the sequence $\{x_n\}$ is $\mathcal{F}^{F_n}$-lim $x_n = \xi$.

### 3. $\mathcal{I}_i$-Convergence for Continuous Functions in PN Spaces

In this short section, we extend the study of ideal convergence to a sequence of function $f_n$ in $(X, F, T)$ and prove a theorem about ideal convergence. We begin with the following definition.
**Definition 3.1.** Let \((X, F, T)\) be a PN spaces and \(\mathcal{I}\) be an arbitrary admissible ideal in \(\mathbb{N}\). We say that a sequence of functions \(f_n : X \rightarrow X\) is \(\mathcal{I}^F\)-convergent to a function \(f : X \rightarrow X\) denoted \(\mathcal{I}^F - \lim f_n = f\), if for every \(x \in X\), \(\epsilon > 0\) and \(\lambda \in (0, 1)\) the set
\[
\{n \in \mathbb{N} : f_n(x) - f(x) \notin \mathcal{N}_\theta(\epsilon, \lambda)\}
\]
belongs to \(\mathcal{I}\).

**Theorem 3.1.** Let \((X, F, T)\) be a PN spaces such that \(\sup_{a<1} T(a, a) = 1\) and let \(\mathcal{I}\) be an arbitrary admissible ideal in \(\mathbb{N}\). Let \(\mathcal{I}^F - \lim f_n = f\) (on \(X\)) where \(f_n : X \rightarrow X\), \(n \in \mathbb{N}\), are equi-continuous (on \(X\)) and \(f : X \rightarrow X\). Then \(f\) is \(F\)-continuous (on \(X\)).

**Proof.** Let \(x_0 \in X\) and \(x - x_0 \in \mathcal{N}_\theta(\epsilon, \lambda)\) be fixed. By equi-continuity of \(f_n\)'s, for every \(\epsilon > 0\), there exists a \(\gamma \in (0, 1)\) with \(\gamma < \lambda\) such that
\[
f_n(x) - f_n(x_0) \in \mathcal{N}_\theta(\frac{\epsilon}{3}, \gamma)
\]
for every \(n \in \mathbb{N}\). Since \(\mathcal{I}^F - \lim f_n = f\), the set
\[
K = \{n \in \mathbb{N} : f_n(x_0) - f(x_0) \notin \mathcal{N}_\theta(\frac{\epsilon}{3}, \gamma)\} \bigcup \{n \in \mathbb{N} : f_n(x) - f(x) \notin \mathcal{N}_\theta(\frac{\epsilon}{3}, \gamma)\}
\]
is in \(\mathcal{I}\) and different from \(\mathbb{N}\). Hence, there exists \(n \in \mathcal{I}(K)\) such that
\[
f_n(x_0) - f(x_0) \in \mathcal{N}_\theta(\frac{\epsilon}{3}, \gamma)) \quad \text{and} \quad f_n(x) - f(x) \in \mathcal{N}_\theta(\frac{\epsilon}{3}, \gamma).
\]
It follows that
\[
F_{f(x_0) - f(x)}(\epsilon) \geq T \left( F_{f_n(x_0) - f_n(x)}(\frac{\epsilon}{3}), T(F_{f_n(x_0) - f_n(x)}(\frac{\epsilon}{3}), F_{f_n(x) - f(x)}(\frac{\epsilon}{3})) \right)
\]
\[
> T(1 - \gamma, T(1 - \gamma, 1 - \gamma))
\]
\[
> T(1 - \gamma, 1 - \gamma)
\]
\[
> 1 - \gamma
\]
\[
> 1 - \lambda.
\]

This implies that \(f\) is \(F\)-continuous (on \(X\)).
4. $\mathcal{I}$-Continuity of a Function in PN Spaces

We begin with the definition of continuity an important type of sequential continuity in PN space.

**Definition 4.1.** Let $\mathcal{I}$ be an ideal and $(X,F,T)$ be a PN space. A map $f : X \to X$ is called $F$-continuous at a point $\xi \in X$, if

$$F - \lim x_n = \xi \implies F - \lim f(x_n) = f(\xi).$$

This means for every $\epsilon > 0$ and $\lambda \in (0,1)$, there exists a number $N \in \mathbb{N}$ such that for $n \geq N$, we have $x_n - \xi \in \mathcal{N}_0(\epsilon,\lambda)$ implies $f(x_n) - f(\xi) \in \mathcal{N}_0(\epsilon,\lambda)$.

**Definition 4.2.** Let $\mathcal{I}$ be an ideal and $(X,F,T)$ be a PN space. A map $f : X \to X$ is called $\mathcal{I}^F$-continuous at a point $\xi \in X$, if

$$\mathcal{I}^F - \lim x_n = \xi \implies \mathcal{I}^F - \lim f(x_n) = f(\xi).$$

**Theorem 4.1.** Let $(X,F,T)$ be a PN space and $\mathcal{I}$ be an arbitrary ideal in $\mathbb{N}$. If $f : X \to X$ is $F$-continuous then it is $\mathcal{I}^F$-continuous.

*Proof.* Let $\{x_n\} \in X$ and $\mathcal{I}^F - \lim x_n = \xi$. Then by $F$-continuity of $f$ at $\xi \in X$ we means for every $\epsilon > 0$ and $\lambda \in (0,1)$, we have $x_n - \xi \in \mathcal{N}_0(\epsilon,\lambda)$ implies $f(x_n) - f(\xi) \in \mathcal{N}_0(\epsilon,\lambda)$. Thus $\{n \in \mathbb{N} : f(x_n) - f(\xi) \notin \mathcal{N}_0(\epsilon,\lambda)\} \subseteq \{n \in \mathbb{N} : x_n - \xi \notin \mathcal{N}_0(\epsilon,\lambda)\}$. Since $\mathcal{I}^F - \lim x_n = \xi$, we have $\{n \in \mathbb{N} : x_n - \xi \notin \mathcal{N}_0(\epsilon,\lambda)\} \in \mathcal{I}$. This implies that $\{n \in \mathbb{N} : f(x_n) - f(\xi) \notin \mathcal{N}_0(\epsilon,\lambda)\} \in \mathcal{I}$ which means $\mathcal{I}^F - \lim f(x_n) = f(\xi)$. Hence, $f$ is an $\mathcal{I}^F$-continuous.

**Theorem 4.2.** Let $(X,F,T)$ be a PN space and $\mathcal{I}$ be an arbitrary admissible ideal in $\mathbb{N}$. If $f : X \to X$ is $\mathcal{I}^F$-continuous then $f$ is $\mathcal{I}^F_{\text{fin}}$-continuous.
Proof. Let $f$ is $F^\mathcal{I}$ -continuous at $\xi \in X$. Suppose that $f$ is not $F^\mathcal{I}$ -continuous, then the set $A = \{ n \in \mathbb{N} : f(x_n) - f(\xi) \notin \mathcal{N}_\theta(\epsilon, \lambda) \} \notin \mathcal{I}_f$, i.e., $A$ is infinite set whenever $\{ n \in \mathbb{N} : x_n - \xi \notin \mathcal{N}_\theta(\epsilon, \lambda) \} \notin \mathcal{I}_f$. Let $\{ y_n \}$ be the subsequence of $\{ x_n \}$ given by the subset $A$ of $\mathbb{N}$. Then $\{ n \in \mathbb{N} : f(y_n) - f(\xi) \notin \mathcal{N}_\theta(\epsilon, \lambda) \} = \mathbb{N}$. Also, the subsequence $\{ y_n \}$ holds $F^\mathcal{I} - \lim y_n = \xi$. By Lemma 4, this implies $F^\mathcal{I} - \lim y_n = \xi$. Thus, by $F^\mathcal{I}$ continuity of $f$, we have $F^\mathcal{I} - \lim f(y_n) = f(\xi)$. Hence $\{ n \in \mathbb{N} : f(y_n) - f(\xi) \notin \mathcal{N}_\theta(\epsilon, \lambda) \} = \mathbb{N} \in \mathcal{I}$, a contradiction. Therefore $f$ is $F^\mathcal{I}$ -continuous.

From theorem 4.3 and 4.4, we can easily prove the following lemma.

**Lemma 4.1.** Let $(X, F, T)$ be a PN space and $\mathcal{I}$ be an arbitrary admissible ideal in $\mathbb{N}$. If $f : X \to X$ is a map, then the following implication hold:

$$F - \text{continuous} \Rightarrow F^\mathcal{I} - \text{continuous} \Rightarrow F^\mathcal{I}_\text{fin} - \text{continuous}$$

### 5. $\mathcal{I}$-Cauchy Sequences in PN Spaces

**Definition 5.1.** Let $(X, F, T)$ be a PN space. A sequence $\{ x_n \}$ in $X$ is said to be $F$ -Cauchy, if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists a number $N = N(\epsilon, \lambda) \in \mathbb{N}$ such that

$$x_n - x_m \in \mathcal{N}_\theta(\epsilon, \lambda) \quad \text{for every} \quad n, m \geq N.$$

**Definition 5.2.** Let $(X, F, T)$ be a PN space and $\mathcal{I}$ be an admissible ideal. Then a sequence $(x_n)$ in $X$ is called $F^\mathcal{I}$ -Cauchy sequence in $X$ if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists $M = M(\epsilon, \lambda) \in \mathbb{N}$ such that

$$\{ n \in \mathbb{N} : x_n - x_M \notin \mathcal{N}_\theta(\epsilon, \lambda) \} \notin \mathcal{I}.$$

**Definition 5.3.** Let $(X, F, T)$ be a PN space and $\mathcal{I}$ be an admissible ideal. Then a sequence $(x_n)$ in $X$ is called $F^\mathcal{I}_\text{fin}$ -Cauchy sequence in $X$ if for every $\epsilon > 0$ and $\lambda \in (0, 1)$,
there exists a set \( M = \{m_1 < m_2 < \cdots < m_k, \cdots \} \in \mathcal{F}(\mathcal{I}) \) such that the subsequence \( x_M = (x_{m_k}) \) is \( F \)-Cauchy in \( X \), i.e., there exists a number \( k_0 \in \mathbb{N} \) such that

\[
x_{m_k} - x_{m_p} \in \mathcal{N}_\theta(\epsilon, \lambda) \quad \text{for every} \quad k, p \geq k_0.
\]

**Theorem 5.1.** Let \( (X, F, T) \) be a PN space and \( \mathcal{I} \) in \( \mathbb{N} \) is an admissible ideal. If \( \{x_n\} \) in \( X \) is \( \mathcal{I}^F \)-Cauchy then it is \( \mathcal{I}^F \)-Cauchy.

**Proof.** Let \( \{x_n\} \) be a \( \mathcal{I}^F \)-Cauchy sequence. Then for every \( \epsilon > 0 \) and \( \lambda \in (0, 1) \) there exists a set \( M = \{m_1 < m_2 < \cdots < m_k, \cdots \} \in \mathcal{F}(\mathcal{I}) \) and a number \( k_0 \in \mathbb{N} \) such that \( x_{m_k} - x_{m_p} \in \mathcal{N}_\theta(\epsilon, \lambda) \) for every \( k, p \geq k_0 \). Now, fix \( N = m_{k_0+1} \). Then for every \( \epsilon > 0 \) and \( \lambda \in (0, 1) \), we have \( x_{m_k} - x_N \in \mathcal{N}_\theta(\epsilon, \lambda) \) for every \( k \geq k_0 \). Let \( H = \mathbb{N} \setminus M \). It is obvious that \( H \in \mathcal{I} \) and \( A(\epsilon, \lambda) = \{n \in \mathbb{N} : x_n - x_N \notin \mathcal{N}_\theta(\epsilon, \lambda)\} \subset H \cup \{m_1 < m_2 < \cdots < m_{k_0}\} \). Clearly, the right hand side of the last argument is belongs to \( \mathcal{I} \). Therefore, for every \( \epsilon > 0 \) and \( \lambda \in (0, 1) \) we can find \( N = N(\epsilon, \lambda) \in \mathbb{N} \) such that \( A(\epsilon, \lambda) \in \mathcal{I} \), i.e., \( \{x_n\} \) is \( \mathcal{I}^F \)-Cauchy sequence in \( X \).

**Theorem 5.2.** Let \( (X, F, T) \) be a PN space such that \( T(a, a) > a \) for every \( a \in (0, 1) \) and \( \mathcal{I} \) be an admissible ideal. A sequence \( \{x_n\} \) in \( X \) is \( \mathcal{I}^F \)-convergent if and only if it is \( \mathcal{I}^F \)-Cauchy.

**Proof:** Necessity: Suppose that \( \{x_n\} \) is \( \mathcal{I}^F \)-convergent to \( \xi \in X \). Let \( \epsilon > 0 \) and \( \lambda \in (0, 1) \) be given. Since \( \mathcal{I}^F \)-lim \( x_n = \xi \), we have \( A = \{n \in \mathbb{N} : x_n \notin \mathcal{N}_\xi(\xi, \lambda)\} \in \mathcal{I} \). This implies that \( A' = \{n \in \mathbb{N} : x_n \in \mathcal{N}_\xi(\xi, \lambda)\} \in \mathcal{F}(\mathcal{I}) \). Now, by (N4), for every \( n, m \in A' \),

\[
v_{x_n - x_m}(\epsilon) \geq T \left( v_{x_n - \xi}(\frac{\epsilon}{2}), v_{x_m - \xi}(\frac{\epsilon}{2}) \right) > T(1 - \lambda, 1 - \lambda) > 1 - \lambda.
\]

Hence, \( \{n \in \mathbb{N} : x_n - x_m \in \mathcal{N}_\theta(\epsilon, \lambda)\} \in \mathcal{F}(\mathcal{I}) \). This implies that \( \{n \in \mathbb{N} : x_n - x_m \notin \mathcal{N}_\theta(\epsilon, \lambda)\} \in \mathcal{I} \), i.e., \( \{x_n\} \) is a \( \mathcal{I}^F \)-Cauchy sequence.
Proof. Sufficiency: Assume that \( \{x_n\} \) is a \( \mathcal{E} \)-Cauchy sequence. We shall prove that \( \{x_n\} \) is \( \mathcal{E} \)-convergent sequence. For this, let \( \{\varepsilon_p\} \) be a strictly decreasing sequence of positive real numbers such that \( \varepsilon_p \to 0 \) as \( p \to \infty \). Since \( \{x_n\} \) is a \( \mathcal{E} \)-Cauchy sequence, there exists a strictly increasing sequence \( \{m_p\} \) of positive integers such that

\[
A_p = \{ n \in \mathbb{N} : x_n - x_{m_p} \notin \mathcal{N}_0(\varepsilon_p, \lambda) \} \in \mathcal{E} \quad p = 1, 2, 3, \ldots .
\]

This implies that

\[
0 \neq \{ n \in \mathbb{N} : x_n - x_{m_p} \notin \mathcal{N}_0(\varepsilon_p, \lambda) \} \in \mathcal{F}(\mathcal{E}) \quad p = 1, 2, 3, \ldots .
(5.1)
\]

Let \( p \) and \( q \) be two positive integers such that \( p \neq q \). Then by (3), both the sets \( \{ n \in \mathbb{N} : x_n - x_{m_p} \notin \mathcal{N}_0(\varepsilon_p, \lambda) \} \) and \( \{ n \in \mathbb{N} : x_n - x_{m_q} \notin \mathcal{N}_0(\varepsilon_q, \lambda) \} \) are nonempty elements of \( \mathcal{F}(\mathcal{E}) \). Since \( \mathcal{F}(\mathcal{E}) \) is a filter on \( \mathbb{N} \), therefore

\[
0 \neq \{ n \in \mathbb{N} : x_n - x_{m_p} \notin \mathcal{N}_0(\varepsilon_p, \lambda) \} \cap \{ n \in \mathbb{N} : x_n - x_{m_q} \notin \mathcal{N}_0(\varepsilon_q, \lambda) \} \in \mathcal{F}(\mathcal{E}).
\]

Thus, for each \( p \) and \( q \) with \( p \neq q \), we can select \( n_p, n_q \in \mathbb{N} \) such that \( x_{n_p} - x_{m_p} \in \mathcal{N}_0(\varepsilon_p, \lambda) \) and \( x_{n_q} - x_{m_q} \in \mathcal{N}_0(\varepsilon_q, \lambda) \). Let \( \varepsilon = \varepsilon_p + \varepsilon_q \). Then by (N4), we have

\[
\nu_{x_{m_p} - x_{m_q}}(\varepsilon) \geq T(\nu_{x_{n_p} - x_{m_p}}(\varepsilon_p), \nu_{x_{n_q} - x_{m_q}}(\varepsilon_q)) > T(1 - \lambda, 1 - \lambda) > 1 - \lambda.
\]

This implies that \( \{x_{m_p}\} \) is a \( F \)-Cauchy sequence and satisfies the Cauchy criterion. Say \( \lim x_{m_p} = \xi \). Also we have \( \varepsilon \to 0 \) as \( p \to \infty \), so for each \( \varepsilon > 0 \) we can choose \( p_0 \in \mathbb{N} \) such that \( \varepsilon_{p_0} < \frac{\varepsilon}{2} \) and

\[
x_{m_p} \in \mathcal{N}_\xi(\frac{\varepsilon}{2}, \lambda) \quad \text{for} \quad p \geq p_0.
\]

Next we prove that \( A = \{ n \in \mathbb{N} : x_n \notin \mathcal{N}_\xi(\varepsilon, \lambda) \} \subset A_{p_0} = \{ n \in \mathbb{N} : x_n - x_{m_{p_0}} \notin \mathcal{N}_0(\varepsilon_{p_0}, \lambda) \} \). Since \( A \) and \( A_{p_0} \) are both in \( \mathcal{E} \), it is sufficient to show that \( A^c \supset A_{p_0}^c \).
Let \( n \in A_{p_0}^c \), then we have

\[
\nu_{x_n-\xi}(\epsilon) \geq T \left( \nu_{x_n-x_{n+p_0}} \left( \frac{\epsilon}{2} \right), \nu_{x_{n+p_0}-\xi} \left( \frac{\epsilon}{2} \right) \right) = T(1 - \lambda, 1 - \lambda) > 1 - \lambda.
\]

This implies that \( n \in A^c \). Therefore \( A \subset A_{p_0} \). Since \( A_{p_0} \subset I \), we conclude that \( A \subset I \).

This proves that the sequence \((x_n)\) is \( I^F \)-convergent to \( \xi \).

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**References**


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