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Inversion of the Generalized Dunkl Intertwining Operator on $\mathbb{R}$ and its Dual using Generalized Wavelets

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Abstract. We establish an inversion formula for a continuous wavelet transform associated with a class of singular differential-difference operators on $\mathbb{R}$. We apply this result to derive new expressions for the inverse generalized Dunkl intertwining operator and its dual on $\mathbb{R}$.

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1. Introduction

Consider the second-order singular differential operator on the real line

$$\Delta = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)}\frac{d}{dx} \tag{1}$$

where

$$A(x) = |x|^{2\alpha+1}B(x), \quad \alpha > -\frac{1}{2},$$

B being a positive C^∞ even function on $\mathbb{R}$. In addition we suppose that

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(i) A is increasing on $[0,\infty[$ and $\lim_{x\to\infty} A(x)=\infty$;

(ii) A'/A is decreasing on $]0,\infty[$ and $\lim_{x\to\infty} A'(x)/A(x)=0$;

(iii) There exists a constant $\delta>0$ such that the function $e^{\delta x}B'(x)/B(x)$ is bounded for large $x\in]0,\infty[$ together with its derivatives.

Lions [5] has constructed an automorphism $\mathscr{X}$ of the space $\mathscr{E}_e(\mathbb{R})$ of C^∞ even functions on $\mathbb{R}$, which intertwines Δ and the second derivative operator d^2/dx^2; that is, satisfying the intertwining relation

$$\mathscr{X}\,\frac{d^2}{dx^2}f=\Delta\,\mathscr{X} f,\qquad f\in\mathscr{E}_e(\mathbb{R}).$$

It is known [14] that the Lions operator $\mathscr{X}$ admits the integral representation

$$\mathscr{X} f(x)=\int_0^{|x|}G(x,y)f(y)dy,\quad x\neq 0,$$

where $G(x,\cdot)$ is an even positive function on $\mathbb{R}$, continuous on $]-|x|,|x|[$ and supported in $[-|x|,|x|]$. Furthermore, the dual Lions operator

$${}^t\mathscr{X} f(y)=\int_{|y|}^{\infty}G(x,y)f(x)A(x)dx,\quad y\in\mathbb{R},$$

is an automorphism of the space $\mathscr{S}_e(\mathbb{R})$ of even Schwartz functions on $\mathbb{R}$, satisfying the intertwining relation

$$\frac{d^2}{dx^2}\,{}^t\mathscr{X} f={}^t\mathscr{X}\Delta f,\qquad f\in\mathscr{S}_e(\mathbb{R}).$$

In [8] the second author has introduced on the space $\mathscr{E}(\mathbb{R})$ of C^∞ functions on $\mathbb{R}$, the following operator

$$Vf=\mathscr{X}(f_e)+\frac{d}{dx}\mathscr{X} I(f_o),\tag{2}$$

where

$$f_e(x)=\frac{f(x)+f(-x)}{2},\quad f_o(x)=\frac{f(x)-f(-x)}{2},\tag{3}$$

and I is the map defined by $Ih(x)=\int_0^x h(t)dt$.

Mainly, he showed that V is an automorphism of $\mathscr{E}(\mathbb{R})$ satisfying for all $f\in\mathscr{E}(\mathbb{R})$,

$$V\frac{d}{dx}f=\Lambda V f,\tag{4}$$

where Λ is a first-order differential-difference operator on $\mathbb{R}$ given by

$$\Lambda f(x)=\frac{df}{dx}+\frac{A'(x)}{A(x)}\left(\frac{f(x)-f(-x)}{2}\right)\tag{5}$$

For $A(x) = |x|^{2\alpha+1}$, $\alpha > -1/2$, the intertwining operator V reads

$$V(f)(x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)} \int_{-1}^{1} f(tx)(1-t^2)^{\alpha-1/2}(1+t)\,dt,$$

and referred to as the Dunkl intertwining operator of index $\alpha + 1/2$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. The differential-difference operator Λ reduces to the one-dimensional Dunkl operator

$$D_\alpha f = \frac{df}{dx} + (\alpha + \frac{1}{2})\,\frac{f(x) - f(-x)}{x}.$$

Such operators have been introduced by Dunkl in connection with a generalization of the classical theory of spherical harmonics (see [1, 11] and the references therein). During the last years, the theory of Dunkl operators has found a wide area of applications in mathematics and mathematical physics. In fact, Dunkl operators have been used in the study of multivariable orthogonality structures with certain reflection symmetries [12, 16]. Moreover, they have been successfully involved in the description and solution of Calogero-Moser-Sutherland type quantum many body systems [4] .

Define the dual operator ${}^tV$ of V on the space $\mathscr{S}(\mathbb{R})$ of Schwartz functions on $\mathbb{R}$, by the relation

$${}^tVf = {}^t\mathscr{X}(f_e) + \frac{d}{dx}\,{}^t\mathscr{X}J(f_o), \tag{6}$$

where J is the map defined by

$$Jh(x) = \int_{-\infty}^{x} h(y)dy, \quad x \in \mathbb{R}. \tag{7}$$

In this paper, it is shown that the dual operator ${}^tV$ is an automorphism of $\mathscr{S}(\mathbb{R})$ which satisfies the intertwining relation

$$\frac{d}{dx}\,{}^tVf = {}^tV\Lambda f, \qquad f \in \mathscr{S}(\mathbb{R}).$$

Moreover, the following inversion formulas for V and ${}^tV$ on certain specific subspaces of $\mathscr{S}(\mathbb{R})$ are provided

$$\begin{aligned} f &= V\,\mathscr{K}\,{}^tVf; \\ f &= \mathscr{M}\,V\,{}^tVf; \\ f &= {}^tV\,\mathscr{M}\,Vf; \\ f &= \mathscr{K}\,{}^tV\,Vf; \end{aligned}$$

$\mathscr{K}$ and $\mathscr{M}$ being pseudo-differential operators. But the main contribution of this work is the determination of the inverse operators V^{-1} and ${}^tV^{-1}$ through a continuous wavelet transform on $\mathbb{R}$ associated with the differential-difference operator Λ. For examples of use of wavelet type transforms in inverse problems the reader is referred to [2, 6, 7, 10, 15] and the references therein. The content of this paper is as follows. In Section 2 we provide some harmonic analysis results related to the differential-difference operator Λ. Next we list some basic properties of the generalized Dunkl intertwining operator V and its dual ${}^tV$. In section 3 we introduce the generalized continuous wavelet transform associated with Λ, and we prove for this transform Plancherel and reconstruction formulas. Using generalized wavelets, we obtain in Section 4 formulas which give the inverse operators V^{-1} and ${}^tV^{-1}$ on Schwartz type spaces.

2. Preliminaries

In this section we provide some facts about harmonic analysis related to the differential-difference operator Λ. We cite here, as briefly as possible, only those properties actually required for the discussion. For more details we refer to [8].

Notation. We denote by

- $\mathscr{S}(\mathbb{R})$ the space of C^∞ functions f on $\mathbb{R}$, which are rapidly decreasing together with their derivatives, i.e., such that for all $m, n = 0, 1, \ldots,$

$$P_{m,n}(f) = \sup_{x\in\mathbb{R}} (1+x^2)^m \left| \frac{d^n}{dx^n} f(x) \right| < \infty.$$

  The topology of $\mathscr{S}(\mathbb{R})$ is defined by the semi-norms $P_{m,n}$, $m, n = 0, 1, \ldots$.

- $\mathscr{S}_e(\mathbb{R})$ (resp. $\mathscr{S}_o(\mathbb{R})$) the subspace of $\mathscr{S}(\mathbb{R})$ consisting of even (rep. odd) functions.

- $\mathscr{B}(\mathbb{R})$ the subspace of $\mathscr{S}(\mathbb{R})$ consisting of functions f such that for all $n = 0, 1, \ldots,$

$$\int_{\mathbb{R}} f(x) b_n(x) A(x) dx = 0,$$

  with $b_n(x) = V\left(\frac{y^n}{n!}\right)(x)$, V being the generalized Dunkl intertwining operator given by (2).

- $\mathscr{W}(\mathbb{R})$ the subspace of $\mathscr{S}(\mathbb{R})$ consisting of functions f such that for all $n = 0, 1 \ldots,$

$$\int_{\mathbb{R}} f(x) x^n dx = 0.$$

- $\mathscr{H}(\mathbb{R})$ the subspace of $\mathscr{S}(\mathbb{R})$ consisting of functions f such that for all $n = 0, 1 \ldots,$

$$\frac{d^n}{dx^n} f(0) = 0.$$

  Put

$$\mathscr{B}_e(\mathbb{R}) = \mathscr{S}_e(\mathbb{R}) \cap \mathscr{B}(\mathbb{R}), \quad \mathscr{B}_o(\mathbb{R}) = \mathscr{S}_o(\mathbb{R}) \cap \mathscr{B}(\mathbb{R}),$$
$$\mathscr{W}_e(\mathbb{R}) = \mathscr{S}_e(\mathbb{R}) \cap \mathscr{W}(\mathbb{R}), \quad \mathscr{W}_o(\mathbb{R}) = \mathscr{S}_o(\mathbb{R}) \cap \mathscr{W}(\mathbb{R}),$$
$$\mathscr{H}_e(\mathbb{R}) = \mathscr{S}_e(\mathbb{R}) \cap \mathscr{H}(\mathbb{R}), \quad \mathscr{H}_o(\mathbb{R}) = \mathscr{S}_o(\mathbb{R}) \cap \mathscr{H}(\mathbb{R}).$$

Remark 1.

(i) Due to our assumptions on the function A there is a positive constant k such that

$$A(x) \sim k\,|x|^{2\alpha+1}, \quad \text{as } |x| \to \infty.$$

(ii) It follows from (4) that

$$\Lambda b_{n+1} = b_n \tag{8}$$

for all $n \in \mathbb{N}$. Further, by [9] we have for any $n \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$|b_n(x)| \quad \leq k\,|x|^n,$$

k being a positive constant depending only on n.

(iii) It is easily checked that the space $\mathscr{S}(\mathbb{R})$ is invariant under the differential-difference operator Λ.

For each $\lambda \in \mathbb{C}$ the differential-difference equation

$$\Lambda u = i\lambda u, \quad u(0) = 1, \tag{9}$$

admits a unique C^∞ solution on $\mathbb{R}$, denoted Ψ_λ given by

$$\Psi_\lambda(x) = \begin{cases} \varphi_\lambda(x) + \frac{1}{i\lambda}\frac{d}{dx}\varphi_\lambda(x) & \text{if } \lambda \neq 0, \\ 1 & \text{if } \lambda = 0, \end{cases} \tag{10}$$

where φ_λ designates the solution of the differential equation

$$\Delta u = -\lambda^2 u, \quad u(0) = 1, \quad u'(0) = 0, \tag{11}$$

Δ being the differential operator defined by (1).

Remark 2.

(i) If $A(x) = |x|^{2\alpha+1}$, $\alpha > -1/2$, then

$$\Psi_\lambda(x) = j_\alpha(\lambda x) + \frac{i\lambda x}{2(\alpha+1)} j_{\alpha+1}(\lambda x),$$

where j_γ $(\gamma > -1/2)$ stands for the normalized spherical Bessel function of index γ given by

$$j_\gamma(z) = \Gamma(\gamma+1) \sum_{n=0}^{\infty} \frac{(-1)^n\,(z/2)^{2n}}{n!\,\Gamma(n+\gamma+1)} \quad (z \in \mathbb{C}).$$

(ii) It follows by (4) and (9) that

$$\Psi_\lambda(x) = V\left(e^{i\lambda\cdot}\right)(x) \tag{12}$$

for all $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$.

The next statement provides a new estimate for the eigenfunction $\Psi_\lambda(x)$.

Lemma 1. *For all λ, $x \in \mathbb{R}$, we have*

$$|\Psi_\lambda(x)| \leq 1.$$

Proof. For $\lambda = 0$, the result is obvious. For $\lambda \neq 0$, set

$$u_\lambda(x) = |\Psi_\lambda(x)|^2 = \left|\varphi_\lambda(x) + \frac{1}{i\lambda}\frac{d}{dx}\varphi_\lambda(x)\right|^2 = (\varphi_\lambda(x))^2 + \frac{1}{\lambda^2}\left(\frac{d}{dx}\varphi_\lambda(x)\right)^2.$$

Notice that $u_\lambda(x)$ is even in x. By (11),

$$\frac{d}{dx}u_\lambda(x) = 2\varphi_\lambda(x)\frac{d}{dx}\varphi_\lambda(x) + \frac{2}{\lambda^2}\frac{d}{dx}\varphi_\lambda(x)\frac{d^2}{dx^2}\varphi_\lambda(x) = -\frac{2}{\lambda^2}\frac{A'(x)}{A(x)}\left(\frac{d}{dx}\varphi_\lambda(x)\right)^2.$$

As the function A is increasing on $[0,\infty[$, it follows that u_λ is decreasing on $]0,\infty[$. As $u_\lambda(0) = 1$, we deduce that $u_\lambda(x) \leq 1$ for all $x \geq 0$. This ends the proof.

Notation. For a positive Borel measure μ on $\mathbb{R}$, and $p = 1$ or 2, we write $L^p(\mathbb{R}, d\mu)$ for the class of measurable functions f on $\mathbb{R}$ for which

$$\|f\|_{p,\mu} = \left(\int_{\mathbb{R}} |f(x)|^p d\mu(x)\right)^{1/p} < \infty.$$

Definition 1. *The generalized Fourier transform of a function f in $L^1(\mathbb{R}, A(x)dx)$ is defined by*

$$\mathscr{F}_\Lambda(f)(\lambda) = \int_{\mathbb{R}} f(x)\Psi_{-\lambda}(x)A(x)dx. \tag{13}$$

Remark 3. *Let $f \in L^1(\mathbb{R}, A(x)dx)$. By Lemma 1, it follows that $\mathscr{F}_\Lambda(f)$ is continuous on $\mathbb{R}$ and $||\mathscr{F}_\Lambda(f)||_\infty \leq \|f\|_{1,A}$.*

An outstanding result about the generalized Fourier transform $\mathscr{F}$ is as follows.

Theorem 1. *[8]*

(i) For every $f \in L^1 \cap L^2(\mathbb{R}, A(x)dx)$ we have the Plancherel formula

$$\int_{\mathbb{R}} |f(x)|^2 A(x)dx = \int_{\mathbb{R}} |\mathscr{F}_\Lambda(f)(\lambda)|^2 d\sigma(\lambda).$$

where

$$d\sigma(\lambda) = \frac{d\lambda}{|c(|\lambda|)|^2},$$

$c(z)$ being a continuous functions on $]0,\infty[$ such that

$$\begin{array}{rcl} c(z)^{-1} & \sim & k_1 z^{\alpha+\frac{1}{2}}, \quad as \quad z \to \infty, \\ c(z)^{-1} & \sim & k_2 z^{\alpha+\frac{1}{2}}, \quad as \quad z \to 0, \end{array}$$

for some $k_1, k_2 \in \mathbb{C}$.

(ii) The generalized Fourier transform $\mathscr{F}_\Lambda$ extends uniquely to a unitary isomorphism from $L^2(\mathbb{R}, A(x)dx)$ onto $L^2(\mathbb{R}, d\sigma)$. The inverse transform is given by

$$\mathscr{F}_\Lambda^{-1} g(x) = \int_{\mathbb{R}} g(\lambda)\Psi_\lambda(x) d\sigma(\lambda)$$

where the integral converges in $L^2(\mathbb{R}, A(x)dx)$.

Remark 4.

(i) The tempered measure σ is called the spectral measure associated with the differential-difference operator Λ.

(ii) For $A(x) = |x|^{2\alpha+1}$, $\alpha > -1/2$, we have

$$c(s) = \frac{2^{\alpha+1}\,\Gamma(\alpha+1)}{s^{\alpha+1/2}}.$$

The following lemma will play a key role in the remainder of this section.

Lemma 2. *The map J, given by (7), is a topological isomorphism*

- *from $\mathscr{S}_o(\mathbb{R})$ onto $\mathscr{S}_e(\mathbb{R})$;*
- *from $\mathscr{B}_o(\mathbb{R})$ onto $\mathscr{B}_e(\mathbb{R})$.*

Proof.

(i) It is sufficient to show that J maps continuously $\mathscr{S}_o(\mathbb{R})$ into $\mathscr{S}_e(\mathbb{R})$. Let $f \in \mathscr{S}_o(\mathbb{R})$. Clearly Jf is a C^∞ even function on $\mathbb{R}$. For $n = 1, 2, \ldots$, $P_{m,n}(Jf) = P_{m,n-1}(f)$. Moreover,

$$(1+x^2)^m\,|Jf(x)| \quad \le \quad (1+x^2)^m \int_{|x|}^{\infty} |f(t)| dt$$

$$\begin{aligned} &\leq \int_{|x|}^{\infty}(1+t^2)^m\,|f(t)|dt \\ &\leq P_{m+1,0}(f)\int_{|x|}^{\infty}\frac{dt}{(1+t^2)} \end{aligned}$$

Hence $P_{m,0}(Jf) \leq \dfrac{\pi}{2} P_{m+1,0}(f)$.

(ii) Let $f \in \mathscr{B}_o(\mathbb{R})$. By using (8) and by integrating by parts we have for any $n = 0, 1, \ldots,$

$$\begin{aligned} \int_{\mathbb{R}} Jf(x)\, b_n(x) A(x) dx &= \int_{\mathbb{R}} Jf(x)\, \Lambda b_{n+1}(x) A(x) dx \\ &= -\int_{\mathbb{R}} \Lambda Jf(x)\, b_{n+1}(x) A(x) dx \\ &= -\int_{\mathbb{R}} f(x) b_{n+1}(x) A(x) dx = 0, \end{aligned}$$

which shows that $Jf \in \mathscr{B}_e(\mathbb{R})$. Conversely, let $f \in \mathscr{B}_e(\mathbb{R})$. Identity (8) together with an integration by parts yields for any $n = 1, 2, \ldots,$

$$\begin{aligned} \int_{\mathbb{R}} f'(x) b_n(x) A(x) dx &= \int_{\mathbb{R}} \Lambda f(x) b_n(x) A(x) dx \\ &= -\int_{\mathbb{R}} f(x) \Lambda b_n(x) A(x) dx \\ &= -\int_{\mathbb{R}} f(x) b_{n-1}(x) A(x) dx = 0, \end{aligned}$$

which shows that $f' \in \mathscr{B}_o(\mathbb{R})$.

Proposition 1.

(i) For all f in $\mathscr{S}(\mathbb{R})$, we have

$$\mathscr{F}_\Lambda(\Lambda f)(\lambda) = i\lambda \mathscr{F}_\Lambda(f)(\lambda). \tag{14}$$

(ii) For all f in $\mathscr{S}(\mathbb{R})$, we have

$$\mathscr{F}_\Lambda(f)(\lambda) = \mathscr{F}_\Delta(f_e)(\lambda) + i\lambda \mathscr{F}_\Delta(Jf_o)(\lambda), \tag{15}$$

where $\mathscr{F}_\Delta$ stands for the Fourier transform related to the differential operator Δ, defined on $\mathscr{S}_e(\mathbb{R})$ by

$$\mathscr{F}_\Delta(h)(\lambda) = \int_{\mathbb{R}} h(x) \varphi_\lambda(x) A(x) dx, \quad \lambda \in \mathbb{R},$$

f_e and f_o being respectively the even and odd parts of f given by (3).

Proof.

(i) Let $f \in \mathscr{S}(\mathbb{R})$. By (5), (10) and (13),

$$\begin{aligned}\mathscr{F}_\Lambda(\Lambda f)(\lambda) &= \int_{\mathbb{R}} \left(f_o'(x) + \frac{A'(x)}{A(x)} f_o(x) \right) \varphi_\lambda(x) A(x) dx \\ &- \frac{1}{i\lambda} \int_{\mathbb{R}} f_e'(x) \varphi_\lambda'(x) A(x) dx \\ &= \kappa_1 - \frac{\kappa_2}{i\lambda}.\end{aligned}$$

By integrating by parts we get

$$\kappa_1 = \int_{\mathbb{R}} (A(x) f_o(x))' \varphi_\lambda(x) dx = -\int_{\mathbb{R}} f_o(x) \varphi_\lambda'(x) A(x) dx$$

and

$$\begin{aligned}\kappa_2 &= \int_{\mathbb{R}} f_e'(x) \varphi_\lambda'(x) A(x) dx \\ &= -\int_{\mathbb{R}} f_e(x) (A(x) \varphi_\lambda'(x))' dx \\ &= -\int_{\mathbb{R}} f_e(x) \Delta \varphi_\lambda(x) A(x) dx \\ &= \lambda^2 \int_{\mathbb{R}} f_e(x) \varphi_\lambda(x) A(x) dx\end{aligned}$$

by virtue of (11). Hence

$$\begin{aligned}\kappa_1 - \frac{\kappa_2}{i\lambda} &= i\lambda \int_{\mathbb{R}} \left(f_e(x) \varphi_\lambda(x) - f_o(x) \frac{\varphi_\lambda'(x)}{i\lambda} \right) A(x) dx \\ &= i\lambda \int_{\mathbb{R}} f(x) \Phi_{-\lambda}(x) A(x) dx.\end{aligned}$$

This clearly yields (14).

(ii) If $f \in \mathscr{S}_e(\mathbb{R})$, identity (15) is obvious. Assume $f \in \mathscr{S}_o(\mathbb{R})$. By using (10), (11), (13) and by integrating by parts we obtain

$$\begin{aligned}\mathscr{F}_\Lambda(f)(\lambda) &= -\frac{1}{i\lambda} \int_{\mathbb{R}} f(x) \varphi_\lambda'(x) A(x) dx \\ &= \frac{1}{i\lambda} \int_{\mathbb{R}} J f(x) (A(x) \varphi_\lambda'(x))' dx\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{i\lambda}\int_{\mathbb{R}} Jf(x)\Delta\varphi_\lambda(x)A(x)dx \\
&= i\lambda\int_{\mathbb{R}} Jf(x)\varphi_\lambda(x)A(x)dx \\
&= i\lambda\mathscr{F}_\Delta Jf(\lambda),
\end{aligned}
$$

which completes the proof.

Theorem 2. *The generalized Fourier transform $\mathscr{F}_\Lambda$ is a topological isomorphism*

- *from $\mathscr{S}(\mathbb{R})$ onto itself;*
- *from $\mathscr{B}(\mathbb{R})$ onto $\mathscr{H}(\mathbb{R})$.*

Proof. By [13] we know that the transform $\mathscr{F}_\Delta$ is a topological isomorphism

- from $\mathscr{S}_e(\mathbb{R})$ onto itself;
- from $\mathscr{B}_e(\mathbb{R})$ onto $\mathscr{H}_e(\mathbb{R})$.
  The result follows then from (15), Lemma 2 and the fact that the operator $\lambda \mapsto \lambda f$ is a topological isomorphism
- from $\mathscr{S}_e(\mathbb{R})$ onto $\mathscr{S}_o(\mathbb{R})$;
- from $\mathscr{H}_e(\mathbb{R})$ onto $\mathscr{H}_o(\mathbb{R})$.

Proposition 2.

(i) For all $f \in \mathscr{S}(\mathbb{R})$,

$$\mathscr{F}_\Lambda(f) = \mathscr{F}_u \circ {}^tV(f), \tag{16}$$

where $\mathscr{F}_u$ denotes the usual Fourier transform on $\mathbb{R}$ given by

$$\mathscr{F}_u(f)(\lambda) = \int_{\mathbb{R}} f(x)e^{-i\lambda x}dx.$$

(ii) For all $f \in \mathscr{S}(\mathbb{R})$,

$$\frac{d}{dx}\,{}^tVf = {}^tV\Lambda f. \tag{17}$$

Proof. Assertion (i) follows by applying the usual Fourier transform $\mathscr{F}_u$ to both sides of (6) and by using the identity

$$\mathscr{F}_\Delta h(\lambda) = \mathscr{F}_u\left({}^t\mathscr{X}h\right)(\lambda), \qquad h \in \mathscr{S}_e(\mathbb{R}),$$

(see [13]). The intertwining relation (17) follows by applying the usual Fourier transform $\mathscr{F}_u$ to both its sides and by using (14) and (16).

Theorem 3. *The intertwining operator* ${}^tV$ *is a topological isomorphism*

- *from* $\mathscr{S}(\mathbb{R})$ *onto itself;*
- *from* $\mathscr{B}(\mathbb{R})$ *onto* $\mathscr{W}(\mathbb{R})$.

Proof. We deduce the result from (16), Theorem 2 and the fact that the usual Fourier transform $\mathscr{F}_u$ is a topological isomorphism

- from $\mathscr{S}(\mathbb{R})$ onto itself;
- from $\mathscr{W}(\mathbb{R})$ onto $\mathscr{H}(\mathbb{R})$.

Definition 2.

(i) *The generalized translation operators* T^x, $x \in \mathbb{R}$, *are defined on* $L^2(\mathbb{R}, A(x)dx)$ *by the relation*

$$\mathscr{F}_\Lambda(T^x f)(\lambda) = \Psi_\lambda(x)\mathscr{F}_\Lambda(f)(\lambda). \tag{18}$$

(ii) *The generalized convolution product of two functions* f *and* g *in* $L^2(\mathbb{R}, A(x)dx)$ *is defined by*

$$f \# g(x) = \int_{\mathbb{R}} T^x f(-y)g(y)A(y)dy. \tag{19}$$

Remark 5. *Let* f *and* g *be in* $L^2(\mathbb{R}, A(x)dx)$. *Then*

(i) *By (18), Lemma 1 and Theorem 1, we deduce that*

$$\left\|T^x f\right\|_{2,A} \leq \left\|f\right\|_{2,A} \tag{20}$$

for any $x \in \mathbb{R}$.

(ii) *It follows from (19), (20) and Schwarz inequality that* $f \# g \in L^\infty(\mathbb{R})$ *and*

$$\left\|f \# g\right\|_\infty \leq \left\|f\right\|_{2,A}\left\|g\right\|_{2,A}. \tag{21}$$

(iii) *By virtue of (18), (19) and Theorem 1,* $f \# g$ *may be rewritten as*

$$f \# g(x) = \int_{\mathbb{R}} \mathscr{F}_\Lambda(f)(\lambda)\mathscr{F}_\Lambda(g)(\lambda)\Psi_\lambda(x)d\sigma(\lambda). \tag{22}$$

Proposition 3. *Let* $f \in L^2(\mathbb{R}, A(x)dx)$ *and* $g \in L^1 \cap L^2(\mathbb{R}, A(x)dx)$. *Then* $f \# g \in L^2(\mathbb{R}, A(x)dx)$,

$$\left\|f \# g\right\|_{2,A} \leq \left\|f\right\|_{2,A}\left\|g\right\|_{1,A}, \tag{23}$$

and

$$\mathscr{F}_\Lambda(f \# g) = \mathscr{F}_\Lambda(f)\mathscr{F}_\Lambda(g). \tag{24}$$

Proof. By Schwarz inequality, $\mathscr{F}_\Lambda(f)\mathscr{F}_\Lambda(g) \in L^1(\mathbb{R}, d\sigma)$. Moreover, by Remark 3, $\mathscr{F}_\Lambda(f)\mathscr{F}_\Lambda(g) \in L^2(\mathbb{R}, d\sigma)$ and $\left\|\mathscr{F}_\Lambda(f)\mathscr{F}_\Lambda(g)\right\|_{2,\sigma} \leq \left\|\mathscr{F}_\Lambda(f)\right\|_{2,\sigma} \left\|g\right\|_{1,A}$. The result follows then by combining (22) and Theorem 1.

Proposition 4. *If $f, g \in \mathscr{S}(\mathbb{R})$, then $f\#g \in \mathscr{S}(\mathbb{R})$ and*

$$^tV(f\#g) = {}^tVf * {}^tVg, \tag{25}$$

where $*$ denotes the usual convolution on $\mathbb{R}$.

Proof. The fact that $f\#g \in \mathscr{S}(\mathbb{R})$ follows from (24) and Theorem 2. Identity (25) follows by applying the usual Fourier transform to both its sides and by using (16) and (24).

Remark 6. *Notice by (24) and Theorem 2 that $\mathscr{B}(\mathbb{R})\#\mathscr{S}(\mathbb{R}) \subset \mathscr{B}(\mathbb{R})$.*

3. Generalized Wavelets

Definition 3. *We say that a function $g \in L^2(\mathbb{R}, A(x)dx)$ is a generalized wavelet if it satisfies the admissibility condition :*

$$0 < C_g = \int_0^\infty |\mathscr{F}_\Lambda g(a\lambda)|^2 \frac{da}{a} < \infty, \tag{26}$$

for almost all $\lambda \in \mathbb{R}$.

Remark 7.

(i) The admissibility condition (26) can also be written as

$$0 < C_g = \int_0^\infty |\mathscr{F}_\Lambda(g)(\lambda)|^2 \frac{d\lambda}{\lambda} = \int_0^\infty |\mathscr{F}_\Lambda(g)(-\lambda)|^2 \frac{d\lambda}{\lambda} < \infty.$$

(ii) If g is real-valued we have $\mathscr{F}_\Lambda(g)(-\lambda) = \overline{\mathscr{F}_\Lambda(g)(\lambda)}$, so (26) reduces to

$$0 < C_g = \int_0^\infty |\mathscr{F}_\Lambda(g)(\lambda)|^2 \frac{d\lambda}{\lambda} < \infty.$$

(iii) If $0 \neq g \in L^2(\mathbb{R}, A(x)dx)$ is real-valued and satisfies

$$\exists\, \eta > 0 \quad \textit{such that} \quad \mathscr{F}_\Lambda(g)(\lambda) - \mathscr{F}_\Lambda(g)(0) = \mathscr{O}(\lambda^\eta), \quad \text{as } \lambda \to 0^+,$$

then (26) is equivalent to $\mathscr{F}_\Lambda(g)(0) = 0$.

(iv) According to (iii) and Theorem 2, each real-valued function g in $\mathscr{B}(\mathbb{R})$ is a generalized wavelet.

Proposition 5.

(i) Let $h \in L^2(\mathbb{R}, d\sigma)$ and $a > 0$. Then the function $\lambda \mapsto h(a\lambda)$ belongs to $L^2(\mathbb{R}, d\sigma)$ and we have

$$\|h(a\cdot)\|_{2,\sigma} \leq \frac{k(a)}{\sqrt{a}} \|h\|_{2,\sigma},$$

where

$$k(a) = \sup_{\lambda>0} \frac{|c(\lambda)|}{|c(\lambda/a)|}.$$

(ii) For every $a > 0$, the dilatation operator

$$H_a(f)(x) = \frac{1}{\sqrt{a}} f\left(\frac{x}{a}\right), \quad x \in \mathbb{R},$$

is a topological automorphism of $L^2(\mathbb{R}, d\sigma)$.

Proof.

(i) Notice first that according to the properties of the function $c(\lambda)$ given in Theorem 1, there exist two positive constants m_1 and m_2 such that

$$\frac{m_1}{a^{\alpha+1/2}} \leq k(a) \leq \frac{m_2}{a^{\alpha+1/2}} \quad \text{for all } a > 0.$$

We have

$$\begin{aligned} \|h(a\cdot)\|^2_{2,\sigma} &= \int_{\mathbb{R}} |h(a\lambda)|^2 \frac{d\lambda}{|c(|\lambda|)|^2} \\ &= \frac{1}{a} \int_{\mathbb{R}} |h(s)|^2 \frac{|c(|s|)|^2}{|c(|s|/a)|^2} \frac{ds}{|c(|s|)|^2} \\ &\leq \frac{k^2(a)}{a} \|h\|^2_{2,\sigma} \end{aligned}$$

(ii) We deduce the result from (i).

Proposition 6. *Let $g \in L^2(\mathbb{R}, A(x)dx)$ and $a > 0$. Then there exists a function $g_a \in L^2(\mathbb{R}, A(x)dx)$ (and only one) such that*

$$\mathscr{F}_\Lambda(g_a)(\lambda) = \mathscr{F}_\Lambda(g)(a\lambda) \tag{27}$$

for almost every $\lambda \in \mathbb{R}$. This function is given by the relation

$$g_a = \frac{1}{\sqrt{a}} \mathscr{F}_\Lambda^{-1} \circ H_{a^{-1}} \circ \mathscr{F}_\Lambda(g) \tag{28}$$

and satisfies

$$\left\|g_a\right\|_{2,A} \leq \frac{k(a)}{\sqrt{a}} \left\|g\right\|_{2,A}.$$

Proof. The result follows by combining Theorem 1 and Proposition 5.

Remark 8. *For $A(x) = |x|^{2\alpha+1}$, $\alpha > -1/2$, the function g_a, $a > 0$, is given by*

$$g_a(x) = \frac{1}{a^{2\alpha+2}}\, g\left(\frac{x}{a}\right), \quad x \in \mathbb{R}.$$

Proposition 7. *Let g be in $\mathscr{S}(\mathbb{R})$. Then for all $a > 0$, the function g_a belongs to $\mathscr{S}(\mathbb{R})$ and we have the relation*

$$g_a = \frac{1}{\sqrt{a}}\, {}^tV^{-1} \circ H_a \circ {}^tV(g). \tag{29}$$

Proof. The result follows from (16), (28), Theorem 2, and the fact that $\mathscr{F}_u \circ H_a = H_{a^{-1}} \circ \mathscr{F}_u$.

Notation. For a function g in $L^2(\mathbb{R}, A(x)dx)$ and for $(a, b) \in]0, \infty[\times \mathbb{R}$ we write

$$g_{a,b}(x) := \sqrt{a}\, T^{-b} g_a(x), \tag{30}$$

where T^{-b} are the generalized translation operators given by (18).

Definition 4. *Let $g \in L^2(\mathbb{R}, A(x)dx)$ be a generalized wavelet. The generalized continuous wavelet transform Φ_g is defined for regular functions f on $\mathbb{R}$ by :*

$$\Phi_g(f)(a, b) = \int_{\mathbb{R}} f(x)\overline{g_{a,b}(x)} A(x)dx.$$

This transform can also be written in the form

$$\Phi_g(f)(a, b) = \sqrt{a}\, f \# \widetilde{g_a}(b), \tag{31}$$

where # is the generalized convolution product given by (19), and $\widetilde{g}_a(x) = \overline{g_a(-x)}$, $x \in \mathbb{R}$.

Lemma 3. *For all $f, g \in L^2(\mathbb{R}, A(x)dx)$ and all $h \in \mathscr{S}(\mathbb{R})$ we have the identity*

$$\int_{\mathbb{R}} f \# g(x) \mathscr{F}_\Lambda^{-1}(h)(x) A(x)dx = \int_{\mathbb{R}} \mathscr{F}_\Lambda(f)(\lambda) \mathscr{F}_\Lambda(g)(\lambda) h^-(\lambda)\, d\sigma(\lambda)$$

where $h^-(\lambda) = h(-\lambda)$, $\lambda \in \mathbb{R}$.

Proof. Fix $g \in L^2(\mathbb{R}, A(x)dx)$ and $h \in \mathscr{S}(\mathbb{R})$. For $f \in L^2(\mathbb{R}, A(x)dx)$ put

$$S_1(f) = \int_{\mathbb{R}} f \# g(x) \mathscr{F}_\Lambda^{-1}(h)(x) A(x)dx$$

and

$$S_2(f) = \int_{\mathbb{R}} \mathscr{F}_\Lambda(f)(\lambda) \mathscr{F}_\Lambda(g)(\lambda) h^-(\lambda) d\sigma(\lambda).$$

In view of Proposition 3 and Theorem 1, we see that $S_1(f) = S_2(f)$ for each $f \in L^1 \cap L^2(\mathbb{R}, A(x)dx)$. Moreover, by using (21), Schwarz inequality and Theorem 1 we get

$$|S_1(f)| \le \left\|f \# g\right\|_\infty \left\|\mathscr{F}_\Lambda^{-1}(h)\right\|_{1,A} \le \left\|f\right\|_{2,A} \left\|g\right\|_{2,A} \left\|\mathscr{F}_\Lambda^{-1}(h)\right\|_{1,A}$$

and

$$\begin{aligned} |S_2(f)| &\le \left\|\mathscr{F}_\Lambda(f)\mathscr{F}_\Lambda(g)\right\|_{1,\sigma} \|h\|_\infty \\ &\le \left\|\mathscr{F}_\Lambda(f)\right\|_{2,\sigma} \left\|\mathscr{F}_\Lambda(g)\right\|_{2,\sigma} \|h\|_\infty \\ &\le \left\|f\right\|_{2,A} \left\|g\right\|_{2,A} \|h\|_\infty, \end{aligned}$$

which shows that the linear functionals S_1 and S_2 are bounded on $L^2(\mathbb{R}, A(x)dx)$. Therefore $S_1 \equiv S_2$, and the lemma is proved.

Lemma 4. *Let $f_1, f_2 \in L^2(\mathbb{R}, A(x)dx)$. Then $f_1 \# f_2 \in L^2(\mathbb{R}, A(x)dx)$ if and only if $\mathscr{F}_\Lambda(f_1)\mathscr{F}_\Lambda(f_2) \in L^2(\mathbb{R}, d\sigma)$ and we have*

$$\mathscr{F}_\Lambda(f_1 \# f_2) = \mathscr{F}_\Lambda(f_1)\, \mathscr{F}_\Lambda(f_2)$$

in the L^2-case.

Proof. Suppose $f_1 \# f_2 \in L^2(\mathbb{R}, A(x)dx)$. By Lemma 3 and Theorem 1, we have for any $h \in \mathscr{S}(\mathbb{R})$,

$$\begin{aligned} \int_{\mathbb{R}} \mathscr{F}_\Lambda(f_1)(\lambda)\, \mathscr{F}_\Lambda(f_2)(\lambda)\, h(\lambda)\, d\sigma(\lambda) &= \int_{\mathbb{R}} f_1 \# f_2(x)\, \mathscr{F}_\Lambda^{-1}(h^-)(x)\, A(x)dx \\ &= \int_{\mathbb{R}} f_1 \# f_2(x)\, \overline{\mathscr{F}_\Lambda^{-1}\left(\overline{h}\right)(x)} A(x)dx \\ &= \int_{\mathbb{R}} \mathscr{F}_\Lambda(f_1 \# f_2)(\lambda)\, h(\lambda)\, d\sigma(\lambda), \end{aligned}$$

which shows that $\mathscr{F}_\Lambda(f_1)\mathscr{F}_\Lambda(f_2) = \mathscr{F}_\Lambda(f_1 \# f_2)$. Conversely, if $\mathscr{F}_\Lambda(f_1)\mathscr{F}_\Lambda(f_2) \in L^2(\mathbb{R}, d\sigma)$, then by Lemma 3 and Theorem 1, we have for any $h \in \mathscr{S}(\mathbb{R})$,

$$\begin{aligned} \int_{\mathbb{R}} f_1 \# f_2(x)\, \mathscr{F}_\Lambda^{-1}(h)(x) A(x)dx &= \int_{\mathbb{R}} \mathscr{F}_\Lambda(f_1)(\lambda)\, \mathscr{F}_\Lambda(f_2)(\lambda) \overline{\widetilde{h}(\lambda)}\, d\sigma(\lambda) \\ &= \int_{\mathbb{R}} \mathscr{F}_\Lambda^{-1}[\mathscr{F}_\Lambda(f_1)\, \mathscr{F}_\Lambda(f_2)](x)\, \mathscr{F}_\Lambda^{-1}(h)(x) A(x)dx, \end{aligned}$$

which shows, in view of Theorem 2, that $f_1 \# f_2 = \mathscr{F}_\Lambda^{-1}[\mathscr{F}_\Lambda(f_1)\, \mathscr{F}_\Lambda(f_2)]$. This achieves the proof.

A combination of Lemma 4 and Theorem 1 gives us the following.

Lemma 5. *Let $f_1, f_2 \in L^2(\mathbb{R}, A(x)dx)$. Then*

$$\int_{\mathbb{R}} |f_1 \# f_2(x)|^2 A(x)dx = \int_{\mathbb{R}} |\mathscr{F}_\Lambda(f_1)(\lambda)|^2 |\mathscr{F}_\Lambda(f_2)(\lambda)|^2 d\sigma(\lambda)$$

where both sides are finite or infinite.

Theorem 4. *Let $g \in L^2(\mathbb{R}, A(x)dx)$ be a generalized wavelet. Then for all $f \in L^2(\mathbb{R}, A(x)dx)$, we have the Plancherel formula*

$$\int_{\mathbb{R}} |f(x)|^2 A(x)dx = \frac{1}{C_g}\int_0^\infty \int_{\mathbb{R}} |\Phi_g(f)(a,b)|^2 A(b)db\frac{da}{a^2}.$$

Proof. Using (26), (27), (31), Fubini's Theorem and Lemma 5, we have

$$\begin{aligned}
\frac{1}{C_g}\int_0^\infty \int_{\mathbb{R}} |\Phi_g(f)(a,b)|^2 A(b)db\frac{da}{a^2} &= \\
&= \frac{1}{C_g}\int_0^\infty \left(\int_{\mathbb{R}} |f \# \widetilde{g}_a(b)|^2 A(b)db\right)\frac{da}{a} \\
&= \frac{1}{C_g}\int_0^\infty \left(\int_{\mathbb{R}} |\mathscr{F}_\Lambda(f)(\lambda)|^2 |\mathscr{F}_\Lambda(g)(a\lambda)|^2 d\sigma(\lambda)\right)\frac{da}{a} \\
&= \int_{\mathbb{R}} |\mathscr{F}_\Lambda(f)(\lambda)|^2 \left(\frac{1}{C_g}\int_0^\infty |\mathscr{F}_\Lambda(g)(a\lambda)|^2 \frac{da}{a}\right) d\sigma(\lambda) \\
&= \int_{\mathbb{R}} |\mathscr{F}_\Lambda(f)(\lambda)|^2 d\sigma(\lambda)
\end{aligned}$$

The result is now a direct consequence of Theorem 1.

Theorem 5. *Let $g \in L^2(\mathbb{R}, A(x)dx)$ be a generalized wavelet. Then for $f \in L^1 \cap L^2(\mathbb{R}, A(x)dx)$ such that $\mathscr{F}_\Lambda(f) \in L^1(\mathbb{R}, d\sigma)$, we have*

$$f(x) = \frac{1}{C_g}\int_0^\infty \left(\int_{\mathbb{R}} \Phi_g(f)(a,b) g_{a,b}(x) A(b)db\right)\frac{da}{a^2}, \quad \text{a.e.,}$$

where, for each $x \in \mathbb{R}$, both the inner integral and the outer integral are absolutely convergent, but possibly not the double integral.

Proof. Put

$$\mathscr{I}(a,x) = \int_{\mathbb{R}} \Phi_g(f)(a,b) g_{a,b}(x) A(b)db$$

and

$$\mathscr{J}(x) = \frac{1}{C_g}\int_0^\infty \mathscr{I}(a,x)\frac{da}{a^2}.$$

By (30) and (31) we have

$$\mathscr{I}(a,x) = a\int_{\mathbb{R}} f\#\tilde{g}_a(b)\,\overline{T^{-x}\,\tilde{g}_a(b)}A(b)db.$$

From (20), (23) and Schwarz inequality we deduce that the integral $\mathscr{I}(a,x)$ is absolutely convergent. On the other hand, by (18), (24) and (27),

$$\mathscr{F}_\Lambda(f\#\tilde{g}_a)(\lambda) = \mathscr{F}_\Lambda(f)(\lambda)\,\overline{\mathscr{F}_\Lambda(g)(a\lambda)}$$

and

$$\mathscr{F}_\Lambda(T^{-x}\tilde{g}_a)(\lambda) = \Psi_\lambda(-x)\,\overline{\mathscr{F}_\Lambda(g)(a\lambda)}.$$

So using Theorem 1 we obtain

$$\mathscr{I}(a,x) = a\int_{\mathbb{R}} \mathscr{F}_\Lambda(f)(\lambda)\Psi_\lambda(x)|\mathscr{F}_\Lambda(g)(a\lambda)|^2 d\sigma(\lambda).$$

In particular, this implies that

$$\begin{aligned}\frac{1}{C_g}\int_0^\infty |\mathscr{I}(a,x)|\frac{da}{a^2} &\leq \int_{\mathbb{R}} |\mathscr{F}_\Lambda(f)(\lambda)|\left(\frac{1}{C_g}\int_0^\infty |\mathscr{F}_\Lambda(g)(a\lambda)|^2\frac{da}{a}\right)d\sigma(\lambda)\\ &= \left\|\mathscr{F}_\Lambda(f)\right\|_{1,\sigma} < \infty,\end{aligned}$$

that is, the integral $\mathscr{J}(x)$ is absolutely convergent. Finally, using Fubini's theorem we get

$$\begin{aligned}\mathscr{J}(x) &= \frac{1}{C_g}\int_0^\infty \left(\int_{\mathbb{R}} \mathscr{F}_\Lambda(f)(\lambda)|\mathscr{F}_\Lambda(g)(a\lambda)|^2\Psi_\lambda(x)d\sigma(\lambda)\right)\frac{da}{a}\\ &= \int_{\mathbb{R}} \mathscr{F}_\Lambda(f)(\lambda)\left(\frac{1}{C_g}\int_0^\infty |\mathscr{F}_\Lambda(g)(a\lambda)|^2\frac{da}{a}\right)\Psi_\lambda(x)d\sigma(\lambda)\\ &= \int_{\mathbb{R}} \mathscr{F}_\Lambda(f)(\lambda)\Psi_\lambda(x)d\sigma(\lambda),\end{aligned}$$

which ends the proof in view of Theorem 1.

4. Inversion of the Intertwining Operators Using Generalized Wavelets

In this section we suppose that the function $|c(\lambda)|^{-2}$ is C^∞ on $]0,\infty[$, and for all $n \in \mathbb{N}$:

(i) $d^n/d\lambda^n|c(\lambda)|^{-2} \neq 0$ on $]0,\infty[$;

(ii) $\exists\, p_n \in \mathbb{N}$ and $k_n > 0$ such that $d^n/d\lambda^n|c(\lambda)|^{-2} \leq k_n\lambda^{p_n}$ for $\lambda \geq 1$;

(iii) $d^n/d\lambda^n|c(\lambda)|^{-2} \sim_{0^+} a_n\lambda^{q_n}$, where $a_n \in \mathbb{R}$ and $q_n \in \mathbb{Z}$.

Remark 9. *These conditions are satisfied in the Dunkl operator case.*

Proposition 8. *The operator $\mathscr{K}$ (resp. $\mathscr{M}$) defined by*

$$\mathscr{K}(f) = \mathscr{F}_u^{-1}\left[2\pi\,|c(|\lambda|)|^{-2}\mathscr{F}_u(f)\right] \tag{32}$$

$$\left(\text{resp. } \mathscr{M}(f) = \mathscr{F}_\Lambda^{-1}\left[2\pi\,|c(|\lambda|)|^{-2}\mathscr{F}_\Lambda(f)\right]\right) \tag{33}$$

is a topological automorphism of $\mathscr{W}(\mathbb{R})$ (resp. $\mathscr{B}(\mathbb{R})$).

Proof. Clearly, the mapping $f \mapsto 2\pi\,|c(\lambda)|^{-2} f$ is a topological automorphism of $\mathscr{H}(\mathbb{R})$, and its inverse is given by $f \longmapsto \frac{1}{2\pi}|c(|\lambda|)|^2 f$. We deduce the result from Theorem 2 and the fact that the usual Fourier transform $\mathscr{F}_u$ is a topological isomorphism from $\mathscr{W}(\mathbb{R})$ onto $\mathscr{H}(\mathbb{R})$.

Proposition 9. *For f in $\mathscr{B}(\mathbb{R})$, we have*

$$\mathscr{M}(f) = {}^tV^{-1} \circ \mathscr{K} \circ {}^tV(f). \tag{34}$$

Proof. By (16), (32) and (33),

$$\begin{aligned}
\mathscr{M}(f) &= \mathscr{F}_\Lambda^{-1}\left[2\pi\,|c(|\lambda|)|^{-2}\mathscr{F}_\Lambda(f)\right] \\
&= {}^tV^{-1} \circ \mathscr{F}_u^{-1}\left[2\pi\,|c(|\lambda|)|^{-2}\,\mathscr{F}_u \circ {}^tV(f)\right] \\
&= {}^tV^{-1} \circ \mathscr{K} \circ {}^tV(f).
\end{aligned}$$

Proposition 10.

(i) *For all f in $\mathscr{W}(\mathbb{R})$ and g in $\mathscr{S}(\mathbb{R})$, we have*

$$\mathscr{K}(f * g) = \mathscr{K}(f) * g\,.$$

(ii) *For all f in $\mathscr{B}(\mathbb{R})$ and g in $\mathscr{S}(\mathbb{R})$, we have*

$$\mathscr{M}(f \# g) = \mathscr{M}(f) \# g\,.$$

Proof. We have

$$\begin{aligned}
\mathscr{K}(f * g) &= \mathscr{F}_u^{-1}\left[2\pi\,|c(|\lambda|)|^{-2}\,\mathscr{F}_u(f * g)\right] \\
&= \mathscr{F}_u^{-1}\left[2\pi\,|c(|\lambda|)|^{-2}\,\mathscr{F}_u(f)\mathscr{F}_u(g)\right] \\
&= \left\{\mathscr{F}_u^{-1}\left[2\pi\,|c(|\lambda|)|^{-2}\,\mathscr{F}_u(f)\right]\right\} * g \\
&= \mathscr{K}(f) * g
\end{aligned}$$

and

$$\mathscr{M}(f \# g) = \mathscr{F}_\Lambda^{-1}\left[2\pi\,|c(|\lambda|)|^{-2}\mathscr{F}_\Lambda(f \# g)\right]$$

$$\begin{aligned} &= \mathscr{F}_\Lambda^{-1}\left[2\pi\,|c(|\lambda|)|^{-2}\mathscr{F}_\Lambda(f)\mathscr{F}_\Lambda(g)\right] \\ &= \left\{\mathscr{F}_\Lambda^{-1}\left[2\pi\,|c(|\lambda|)|^{-2}\mathscr{F}_\Lambda(f)\right]\right\}\#g \\ &= \mathscr{M}(f)\#g, \end{aligned}$$

which ends the proof.

Theorem 6. *1. The intertwining operator V is a topological isomorphism from $\mathscr{W}(\mathbb{R})$ onto $\mathscr{B}(\mathbb{R})$.*

2. We have the following inverse formulas for V and ${}^tV$:

(a) For $f \in \mathscr{B}(\mathbb{R})$,

$$f = V\,\mathscr{K}\,{}^tV(f); \tag{35}$$

$$f = \mathscr{M}\,V\,{}^tV(f). \tag{36}$$

(b) For $f \in \mathscr{W}(\mathbb{R})$,

$$f = \mathscr{K}\,{}^tV\,V(f); \tag{37}$$

$$f = {}^tV\,\mathscr{M}\,V(f). \tag{38}$$

Proof. Let $f \in \mathscr{B}(\mathbb{R})$. From (12), (16) and Theorem 1 we have for all $x \in \mathbb{R}$,

$$\begin{aligned} f(x) &= \int_{\mathbb{R}} \mathscr{F}_\Lambda(f)(\lambda)\,\Psi_\lambda(x)\frac{d\lambda}{|c(|\lambda|)|^2} \\ &= V\left(\int_{\mathbb{R}} \mathscr{F}_\Lambda(f)(\lambda)\,e^{i\lambda\cdot}\frac{d\lambda}{|c(|\lambda|)|^2}\right)(x) \\ &= V\left(\frac{1}{2\pi}\int_{\mathbb{R}}\left[2\pi\,|c(|\lambda|)|^{-2}\mathscr{F}_u\circ{}^tV(f)\right]e^{i\lambda\cdot}d\lambda\right)(x) \\ &= V\,\mathscr{K}\,{}^tV(f)(x). \end{aligned}$$

This when combined with (34) yields formula (36). By replacing f respectively by Vf and ${}^tV^{-1}f$ into formulas (35) and (36), we regain identities (37) and (38). From (35), Proposition 8 and Theorem 3, we deduce that V is a topological isomorphism from $\mathscr{W}(\mathbb{R})$ onto $\mathscr{B}(\mathbb{R})$.

In order to invert the intertwining operators V and ${}^tV$ we shall need some technical lemmas.

Lemma 6. *For all f in $\mathscr{W}(\mathbb{R})$ and g in $\mathscr{S}(\mathbb{R})$, we have*

$$V(f * g) = V(f)\#\,{}^tV^{-1}(g). \tag{39}$$

Proof. By using relations (25), (35), (37) and Proposition 10(i) we have

$$\begin{aligned} V^{-1}\left[V(f)\#\,{}^tV^{-1}(g)\right] &= \mathscr{K}\,{}^tV\left[V(f)\#\,{}^tV^{-1}(g)\right] \\ &= \mathscr{K}\left[{}^tV\,V(f) * g\right] \\ &= \left[\mathscr{K}\,{}^tV\,V(f)\right] * g \\ &= f * g. \end{aligned}$$

Definition 5. *The classical continuous wavelet transform on* $\mathbb{R}$ *is defined for regular functions by*

$$S_g(f)(a,b) = \int_{\mathbb{R}} f(x)\,\overline{g^0_{a,b}(x)}\,dx, \quad a>0,\ b\in\mathbb{R},$$

where

$$g^0_{a,b}(x) := \frac{1}{\sqrt{a}} g\left(\frac{x-b}{a}\right)$$

The function g *is a classical wavelet on* $\mathbb{R}$*, i.e., a function in* $L^2(\mathbb{R},dx)$ *satisfying the admissibility condition:*

$$0 < C^0_g = \int_0^\infty |\mathscr{F}_u(g)(a\lambda)|^2 \frac{da}{a} < \infty,$$

for almost all $\lambda\in\mathbb{R}$.

A more complete and detailed discussion of the properties of the classical wavelet transform on $\mathbb{R}$ can be found in [3], from which we have the following inversion formula.

Theorem 7. *Let* $g \in L^2(\mathbb{R},dx)$ *be a classical wavelet. If both* f *and* $\mathscr{F}_u(f)$ *are in* $L^1(\mathbb{R},dx)$ *then we have*

$$f(x) = \frac{1}{C^0_g}\int_0^\infty \left(\int_{\mathbb{R}} S_g(f)(a,b) g^0_{a,b}(x) db\right)\frac{da}{a^2}$$

for almost every $x\in\mathbb{R}$.

Remark 10. *According to (16) and Definitions 3, 5,* $g\in\mathscr{S}(\mathbb{R})$ *is a generalized wavelet, if and only if,* ${}^tV(g)$ *is a classical wavelet and we have:*

$$C^0_{{}^tV(g)} = C_g. \tag{40}$$

Lemma 7. *Let* $g\in\mathscr{W}(\mathbb{R})$ *be real-valued. Then for all* $f\in\mathscr{S}(\mathbb{R})$ *we have*

$$\Phi_{V\mathscr{K}g}(f)(a,b) = \mathscr{M}V\left[S_g\left({}^tVf\right)(a,\cdot)\right](b).$$

Proof. Notice that $V\mathscr{K}g = {}^tV^{-1}g$ by virtue of (35). Further, g is a classical wavelet according to [3]. So it follows from Remark 10 that $V\mathscr{K}g \in \mathscr{B}(\mathbb{R})$ is a generalized wavelet and

$$C_{V\mathscr{K}g} = C^0_g. \tag{41}$$

Due to (25), (29), (31), (35), (38) and Definition 5 we have

$$\begin{aligned}
\Phi_{V\mathscr{K}g}(f)(a,b) &= \sqrt{a}\, f \# (V\mathscr{K}g\breve{)}_a(b)\\
&= \sqrt{a}\,{}^tV^{-1}\left[{}^tVf * {}^tV(V\mathscr{K}g\breve{)}_a\right](b)\\
&= {}^tV^{-1}\left[{}^tVf * H_a\left({}^tVV\mathscr{K}\widetilde{g}\right)\right](b)\\
&= \mathscr{M}V\left[{}^tVf * H_a(\widetilde{g})\right](b)\\
&= \mathscr{M}V\left[S_g\left({}^tVf\right)(a,\cdot)\right](b).
\end{aligned}$$

Lemma 8. *Let $g \in \mathscr{B}(\mathbb{R})$ be real-valued. Then for all $f \in \mathscr{W}(\mathbb{R})$, we have*

$$S_{{}^tVg}(f)(a,b) = \mathscr{K}\,{}^tV\left[\Phi_g(Vf)(a,\cdot)\right](b).$$

Proof. Observe that by Remarks 7(iv) and 10, ${}^tVg$ is a classical wavelet. Using (29), (31), (39) and Definition 5 we have

$$\begin{aligned}
V\left(S_{{}^tVg}(f)(a,\cdot)\right)(b) &= V\left(f * H_a\left({}^tV\widetilde{g}\right)\right)(b)\\
&= \sqrt{a}\,V\left(f * {}^tV(\widetilde{g}_a)\right)(b)\\
&= \sqrt{a}\,V(f) \# \widetilde{g}_a(b)\\
&= \Phi_g(Vf)(a,b).
\end{aligned}$$

Thus

$$\begin{aligned}
S_{{}^tVg}(f)(a,b) &= V^{-1}\left[\Phi_g(Vf)(a,\cdot)\right](b)\\
&= \mathscr{K}\,{}^tV[\Phi_g(Vf)(a,\cdot)](b)
\end{aligned}$$

by virtue of (35).

We can now state our main result.

Theorem 8.

(i) Let $g \in \mathscr{W}(\mathbb{R})$ be real-valued. Then for all $f \in \mathscr{S}(\mathbb{R})$ we have

$${}^tV^{-1}f(x) = \frac{1}{C_g^0}\int_0^\infty \left(\int_{\mathbb{R}} \mathscr{M}V[S_g(f)(a,\cdot)](b)\,(V\mathscr{K}g)_{a,b}(x)A(b)db\right)\frac{da}{a^2}.$$

(ii) Let $g \in \mathscr{B}(\mathbb{R})$ be real-valued. Then for all $f \in \mathscr{B}(\mathbb{R})$ we have

$$V^{-1}f(x) = \frac{1}{C_g}\int_0^\infty \left(\int_{\mathbb{R}} \mathscr{K}\,{}^tV[\Phi_g(f)(a,\cdot)](b)\left({}^tVg\right)^0_{a,b}(x)db\right)\frac{da}{a^2}.$$

Proof. The result follows by combining Theorems 5, 7, Lemmas 7, 8 and identities (40), (41).

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