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A Generalization of the Concept of Cointegration to Harmonizable and Class (KF) Processes

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Abstract. In this paper we consider the generalization of the concept of cointegration to non-stationary processes which are not necessarily $I(d)$. Two cases are of special interest. First the case of non-stationary processes which adjust to an equilibrium not necessarily according to a linear adjustment process. Second the case of non-stationary (possibly $I(1)$) series which co-move according to a non linear or heteroscedastic adjustment process . The non-stationary processes considered here belong to the Kampé de Fériet (KF) class.

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1. Introduction

The theory of cointegration as introduced by Engle and Granger [7] refers to the situation where multiple I(d) series can be combined to produce an $I(k)$ series, where k can range from 0 to $d-1$. In the case where $d=1$ two series are said to be cointegrated if they are non-stationary in levels, stationary in first differences and there exists a linear combination of the levels which is stationary. Although this approach has proved to be extremely fruitful in applications, it has also frustrated researchers because of its limitations.

One limitation is that it is assumed that economic time series exhibiting a trending behavior can be well approximated by processes that are integrated, usually of order one. A second limitation is that cointegrated series co-move according to a stationary process. Harris et al. [12] remark that higher frequency data appear to be more volatile than would be expected from $I(1)$ processes. They also note that series, which should co-move, often deviate

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substantially for short periods of time. They addresses those issues by introducing the concept of stochastic integration and stochastic cointegration. Briefly a process is stochastically integrated if it consists of a non-mean reverting* $I(1)$ stochastic trend plus a heteroskedastic shock term. The first difference of such a process is mean reverting but not $I(0)$. Two processes are stochastically cointegrated if they are stochastically integrated and there exists a linear combination of the two which is mean reverting but not necessarily $I(0)$. In particular heteroskedastic error correction terms are allowed.

A third limitation is that most tests for unit roots and cointegration assume a linear ARIMA or VAR framework. For example in the Dickey and Fuller [5] test for a unit root assumes a linear ARIMA model. If the series is generated by a non-linear model, the Dickey-Fuller test can lead us to conclude erroneously that the series has a unit root. A linear relationship is also assumed as the basis for the Engle-Granger and the Johansen's tests for cointegration [7, 13]. Those tests might fail to detect an adjustment to equilibrium if such an adjustment is non-linear. Enders and Ludlow [6] develop a test for reversion that does not a priori impose a particular dynamic structure of the adjustment coefficients. They use a first order Fourier approximation which allows for non-linear decay.

Because of these limitations there is a need to study a larger class of processes besides $I(d)$ processes and to develop tools to study those processes. In particular the cointegration concept needs to be generalized to non-stationary series which are not necessarily integrated and whose co-movements are not necessarily according to $I(0)$ processes.

Other generalisations of cointegration have been proposed previously. Gregoir [9, 10] uses the framework introduced by Gregoir and Laroque [11] to define integral operators to build up nonstationary time series. This type of specification may occur for time series models with more than one unit root at frequency zero and some seasonal unit roots.

In this paper we consider a class of processes which are non-stationary and includes the class of stationary processes as a subset. This is the class of Kampé de Fériet (KF) processes. Kampé de Fériet processes provide a natural extension to the class of stationary processes. The (KF) class includes modulated stationary processes, slowly changing processes and periodic stationary processes. They have been studied in engineering and signal processing. Their wavelet decomposition has also been investigated ([2, 28]. In Section 2 we consider the class (KF) studied by Kampé de Fériet and Frankel [17] and independently by Parzen [22] and Rozanov [27] under the name of asymptotically stationary processes. We show that, for example, the non linear processes considered by Enders and Ludlow [6] are of class (KF). In Section 3 we study the properties of a subclass of the (KF) class: strongly harmonizable processes. In Section 4 we investigate generalizations of the concept of cointegration to (KF) processes and consider the special case of strongly harmonizable processes. We present two concepts of cointegration. The first one is the case where there exists a stationary linear combination of (KF) processes. This would happen, for example, if two processes were generated by some (KF) non-linear systems but co-moved according to a stationary linear adjustment process. We also consider the case where we have series whose first differences are of class (KF) and for which there exists a linear combination of class (KF). Thus covering the case

*[12] defines a process as mean reverting to zero if $E(X_{t+s} \mid X_t, ..., X_1, ...) \xrightarrow{P} 0$ as $s \to \infty$.

where we might have $I(1)$ processes co-moving according to a nonlinear adjustment process or a heteroskedastic adjustment process. Section 5 concludes.

2. Class (KF)

We assume, without loss of generality, that the processes studied have zero means. To see that there is no generality loss, let X_t, t real or integer, be a real random process, we can replace X_t by $Y_t = \eta X_t$, where η is a random variable, independent of X_t for any t, such that:

$$E(\eta^2) = 1 \text{ and} E(\eta) = 0$$

This implies that $E(Y_{t+s}Y_t) = E(X_{t+s}X_t)$, for any t and s. $E(X_{t+s}X_t)$ can be considered as the covariance of a process with zero mean: Y_t.

In what follows we assume that t is an integer but the definitions and results presented generalize to the case where t is real.

2.1. Definition

A process X_t t integer, is of class (KF) if for each h integer the following limit exists:

$$r(h) = \lim_{T\to\infty} r_T(h) = \lim_{T\to\infty} \frac{1}{T}\sum_{s=0}^{T} B(s, s+h) \tag{1}$$

where $r_T(h) = \frac{1}{T}\sum_{s=0}^{T} B(s, s+h)$ and $B(s, t) = E(X_s X_t)$ is the covariance of X_t.

2.2. Classification of Class (KF) Processes

2.2.1. Stationary Processes

If X_t is real and stationary, so that $B(s, t) = B(t-s)$, then $r(h) = B(h) = B(|h|)$. This shows that every stationary process is in (KF).

2.2.2. Non-linear sequences

Theorem 1. *Let*

$$X_t = \sum_{i=1}^{k} \alpha_i(t) X_{t-i} + \varepsilon_t, \ t \geq 1 \textit{ integer} \tag{2}$$

where the $\alpha_i(t)$'s are non stochastic functions of time and ϵ_t is a white noise process with variance σ^2. If

$$\varphi(t, m) = \sum_{k_1+\ldots+k_i=m} \prod_{j=1}^{i} \alpha_{k_j}(t - \sum_{r=0}^{j-1} k_r), \ 0 \leq m \leq t, \ k_0 = 0 \tag{3}$$

the sum ranging over all partitions of m into integers k_i, then X_t belongs to the class (KF) if the $\alpha_i(\cdot)$ satisfy

$$\sum_{m=0}^{t} \left|\varphi(t,m)\right|^2 \leq M < \infty,\ t \geq 1 \tag{4}$$

Proof. The solution to the difference equation in (2) is given by:

$$X_t = \sum_{i=0}^{t-1} \varphi(t,i)\varepsilon_{t-i} + \sum_{i=t}^{t+k-1} \varphi(t,i)c_{t-i},\ t \geq 1 \tag{5}$$

where $X_i = c_i,\ i = -k+1,\ldots,0$ are the initial values [see 14, 25].

Without loss of generality the starting values can be assumed to be zero.

$$B(s,s+h) = E(X_s X_{s+h}) = \sigma^2 \sum_{m=0}^{s-1} \varphi(s,m)\varphi(s+h,m+h),\ h \geq 0,\ s \geq 1.$$

(4) implies that $|B(s,s+h)| \leq \sigma^2 M$ for all h by Cauchy inequality.

This implies that $\lim_{T\to\infty} \frac{1}{T}\sum_{s=1}^{T} B(s,s+h) = r(h)$ exists.

This model is a more general non-linear model than the one considered in [6]. Consequently this implies that the non linear processes considered in [6] are of class (KF).

2.3. Vector Class (KF) Processes

Definition 1. *An n-dimensional vector process $X_t = (X_{1t},\ldots,X_{nt})'$ is an n-dimensional class (KF) process if and only if for every $n \times 1$ vector of real numbers, w, the process $w'X_t$ is of class (KF).*

2.4. Harmonizable Processes: Definitions and Properties

In this section a few definitions and properties of harmonizable processes are summarized. For more details the reader is referred to [1, 19, 20, 21, 29].

2.5. Strongly Harmonizable Processes

A second order process X_t , t integer, is strongly harmonizable if and only if it has the quadratic mean representation:

$$X_t = \int_{-\pi}^{\pi} e^{itu} Z(du) \tag{6}$$

where $Z(\cdot)$ is a stochastic measure whose covariance is of bounded variation. Assuming that X_t is a strongly harmonizable process with zero mean Loéve [19] showed that a second order process X_t is strongly harmonizable if and only if its covariance $B(s,t) = E(X_s X_t$ has the integral representation:

$$B(s,t) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i(su-tv)} F(du,dv) \tag{7}$$

for all s and t integers, where F is a covariance of bounded variation ($\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|F(du,dv)| < \infty$), so that the integral in (7) exists. Blanc-Lapierre and Fortet [1, volume 2] showed that a strongly harmonizable process has a unique quadratic mean representation such as (6).

It is shown in [15] that oscillatory sequences are strongly harmonizable and that slowly changing processes in continuous and discrete time are also strongly harmonizable.

Gladyshev [8] proved that periodic stationary sequences are strongly harmonizable. He also showed that periodic stationary processes with continuous time are not necessarily strongly harmonizable.

2.6. Weakly Harmonizable Processes

A second order process X_t is weakly harmonizable if and only if its covariance function has the integral representation:

$$B(s,t) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i(su-tv)}F(du,dv) \tag{8}$$

for all s and t integers, where F is a positive definite and σ-additive bimeasure i.e.

$$\sup\left\{\sum\sum a_i\bar{a}_j F(A_i,A_j)/A_i \in \mathrm{B} \text{ disjoint, } \left|a_i\right| \leq 1\right\} < \infty \tag{9}$$

where B is the Borel σ-algebra of $[-\pi,\pi]$ and $F(\cdot,C)$, $F(A,\cdot)$ are complex measures on B.

In that case, integrals relative to F cannot generally be Lebesgue-Stieltjes integrals, but one can define a Morse-Transue integral [26].

Rozanov [27] proved that strongly harmonizable processes belong to the class (KF) and Rao [26] showed that many weakly harmonizable processes also belong to the class (KF). However not all weakly harmonizable processes are of class (KF) and inversely not all class (KF) processes belong to the weakly harmonizable class. Note that the Brownian motion is not harmonizable but is of class (KF) - see Figure 1.

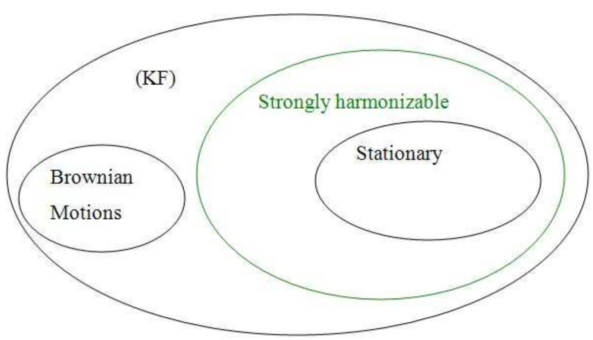

Figure 1: Demonstrating Nested Structure of Classes.

2.7. Spectral Properties of Harmonizable Processes

2.7.1. Asymptotic Stationarity

Rozanov [27] proved that every strongly harmonizable process is of class (KF) (or asymptotically stationary) and more precisely that the following theorem holds.

Theorem 2. *Let X_t be a strongly harmonizable process with spectral measure F, and let $\Delta = \{(u,v) \mid u = v\}$ be the diagonal axis of $[-\pi,\pi] \times [-\pi,\pi]$. Then for all integer h we have:*

$$\lim_{T\to\infty} \frac{1}{T} \sum_{s=0}^{T} B(s,s+h) = \iint_{\Delta} e^{ihv} F(du,dv) \tag{10}$$

2.7.2. Energy Properties of Harmonizable Processes

Let X_t be a strongly harmonizable process with spectral measure F, then F can be decomposed into F_1, F_2, F_3 where:

$$F(du,dv) = F_1(du,dv) + F_2(du,dv) + F_3(du,dv) \tag{11}$$

F_1 is absolutely continuous with spectral density f_1. F_2 is a distribution, which has its mass concentrated on a set at most denumerable, and each point carries a mass different from zero. F_3 has its mass concentrated on a set non-denumerable, and each single point carries the mass zero. Note that in the stationary case F_1, F_2 and F_3 have their total masses located on the bisector.

Lii and Rosenblatt [18] derive consistent estimators for the spectral distribution of harmonizable processes when the spectral support of the process consists of lines.

2.8. Vector Harmonizable Processes

Definition 2. *An n-dimensional vector process $X_t = (X_{1t}, \ldots, X_{nt})'$ is an n-dimensional strongly harmonizable process if and only if for every $n \times 1$ vector of real numbers, w, the process $w'X_t$ is strongly harmonizable.*

The above definition is equivalent to requiring that the covariance function of X_t be represented as

$$B(s,t) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{i(su-tv)} F(du,dv) \tag{12}$$

where $F(du,dv)$ is an $n \times n$ matrix array of covariance functions of bounded variation.

Equation (6) holds with $Z(\cdot)$ an $n \times 1$ vector of stochastic measures with covariance of bounded variation.

The matrix spectral function $F : (A,B) \to E(Z(A)Z(B)' = (F_{ij}(A,B))$, A and B Borel sets of $[-\pi,\pi]$, has F_{ii} positive definite, $F_{ij}(A,B) = F_{ji}(B,A), i \neq j$ and in the stationary case F itself is positive hermitian. In the strongly harmonizable case each F_{ij} determines a Radon measure on $[-\pi,\pi] \times [-\pi,\pi]$.

3. Generalization of the Concept of Cointegration to Class (KF) and Strongly Harmonizable Processes

3.1. Generalization 1

In this section we consider the case where there exists a stationary linear combination of class (KF) processes. This would be the case, for example, of non-stationary processes generated by nonlinear systems which co-move according to a linear adjustment process.

Definition 3. *An n-vector process X_t of class (KF) is said to be cointegrated if there exists a linear combination of the series which is stationary. This means that there exists $\beta \neq 0$ such that $\beta' X_t = \varepsilon_t$ where ϵ_t is stationary.*

Application: Slowly Changing Processes

Let X_t, t integer, be a second order n-vector process such that $E(X_t) = 0$. Assume that X_{jt}, $j = 1, \ldots, n$, can be represented as:

$$X_{jt} = \int_{-\pi}^{\pi} A_{jt}(u) e^{itu} Z_j(du) \tag{13}$$

where $Z_j(\cdot)$ is a stochastic measure with orthogonal increments:

$$E\left|Z_j(du)\right|^2 = \mu_j(du),\ \mu_j \text{ a finite positive measure,}$$
$$E(Z_i(du)\bar{Z}_j(dv)) = 0,\ u \neq v,$$
$$E\left[Z_i(du)\bar{Z}_j(du)\right] = \mu_{ij}(du)$$

and

$$A_{jt}(u) = \int_{-\pi}^{\pi} e^{itx} H_j(u, dx) \tag{14}$$

Finally, it is assumed that the generalised Fourier transform of $A_{jt}(u)$ has an absolute maximum at $x = 0$ independently of u, that $H_j(\cdot, A)$ is a Borel function and that $H_j(u, \cdot)$ is a signed measure on the Borel sets of $[-\pi, \pi]$. Then X_{jt} is said to be an oscillatory process. Thus an oscillatory process X_{jt} is defined as the output of a system with stationary input process

$$v_{jt} = \int_{-\pi}^{\pi} e^{itu} Z_j(du) \tag{15}$$

and impulse response $A_{jt}(u)$. This includes the case where the amplitudes of different frequency bands do not change at the same rate. If we want $A_{jt}(u)$ to be slowly changing with time the Fourier-Stieltjes transform of $A_{jt}(u)$ has to be highly concentrated around zero, and the measure of the concentration should be small.

If, moreover,

$$B_j(u) = \int_{-\pi}^{\pi} |x| \ \left|H_j(u, dx)\right| \leq \varepsilon,\ \forall u \in [-\pi, \pi] \tag{16}$$

$A_{jt}(u)$ is said to be ϵ-slowly changing.

A slowly changing process X_{jt} is non-stationary, and we can think of its spectrum as continuously changing. Its spectrum, however, is changing slowly over time. Priestley [23] shows that it is possible to define a spectral measure for such a process, which he calls the evolutionary spectrum.

Let $F_j(du) = E\left|Z_j(du)\right|^2$, the evolutionary power spectrum is:

$$F_{jt}(du) = \left|A_{jt}(u)\right|^2 F_j(du) \tag{17}$$

Note that $E(X_{jt}^2) = \int_{-\pi}^{\pi} F_{jt}(du)$, which implies that $F_{jt}(du)$ describes a frequency decomposition of the "total energy" of the process. When $F_j(u)$ is differentiable, $f_j(u) = F_j'(u)$ is the spectral density function of v_{jt} and we define the evolutionary spectral density function as: $f_{jt}(u) = \left|A_{jt}(u)\right|^2 f_j(u)$. $B_j(u)$ is a measure of the concentration of $H_j(u, dx)$ about zero and thus is also a measure of the rate at which $A_{jt}(u)$ is changing.

It is shown in [15] that oscillatory sequences are strongly harmonizable. It is also shown that the distribution of masses $F(du, dv)$ for an ϵ-slowly changing process has to be concentrated on a band along the bisector whose width is determined by ϵ.

Long Run Relationships Between Slowly Changing Processes: Priestley and Tong [24] consider the cross-spectrum between slowly changing processes. In the case where $n = 2$ they define the evolutionary power cross spectrum at time t by:

$$F_{12,t}(du = A_{1t}(u)\bar{A}_{2t}(u)E\left[Z_1(du)\bar{Z}_2(du)\right] = A_{1t}(u)\bar{A}_{2t}(u)\mu_{12}(du) \tag{18}$$

$F_{12,t}(du)$ can be given a physical interpretation similar to that of the cross-spectrum of a bivariate stationary process: it represents the average value of the product of the amplitudes of the corresponding frequency component in the two processes X_{1t} and X_{2t}. Since those processes are nonstationary the cross-spectrum is time dependent.

If the measure $\mu_{12}(du)$ is absolutely continuous with respect to the Lebesgue measure we have:

$$F_{12,t}(du) = f_{12,t}(u)du \tag{19}$$

where $f_{12,t}(u)$ is the evolutionary cross-spectral density function. If $\mu_1(du)$ and $\mu_2(du)$ are absolutely continuous the coherency between X_{1t} and X_{2t} can be defined as:

$$W_{12}(u) = \frac{\left|f_{12,t}(u)\right|}{\left\{f_{1,t}(u)f_{2,t}(u)\right\}^{1/2}} = \frac{\left|E\left[Z_1(du)\bar{Z}_2(du)\right]\right|}{\left\{E\left|Z_1(du)\right|^2 E\left|Z_2(du)\right|^2\right\}^{1/2}} \tag{20}$$

$W_{12}(u)$ is independent of time and can be interpreted as the modulus of the correlation coefficient between $Z_1(du)$ and $Z_2(du)$. $W_{12}(u)$ can also be interpreted as a measure of the linear relationship between the corresponding components of X_{1t} and X_{2t} at frequency u. This result generalises to more than two series using the multiple coherence. If we are interested in long run relationships between processes we need to estimate the multiple coherence in a frequency band around $u = 0$. Different techniques to estimate the evolutionary spectra are available in the engineering and statistical literature. Priestley and Tong [24] generalise Priestley [23] to the multivariate case whereas Dalhaus [4] uses a different estimation technique.

3.2. Generalization 2

We also consider the case where we have n series whose first differences are of class (KF) and for which there exists a linear combination which is of class (KF). Thus covering the case where we might have two $I(1)$ processes co-moving according to a nonlinear or a heteroskedastic adjustment process.

Definition 4 (Class (KF)-Integration). *An n-vector process X_t is class (KF)-integrated of order d, denoted by $KFI(d)$, if there exists an n-vector process w_t, which is of class (KF), such that*

$$(1-L)^d X_t = w_t$$

Definition 5 (Class (KF)-Cointegration). *An n-vector process X_t class (KF)-integrated of order 1 is said to be class (KF)-cointegrated if there exists a linear combination of the series which is $KFI(0)$. This means that there exists $\beta \neq 0$ such that $\beta' X_t = \varepsilon_t$ where ϵ_t is of class (KF).*

3.3. The Special Case of Harmonizable Cointegration

The usual concept of cointegration among integrated variables refers to cointegration at frequency zero. The concept has been generalized to cointegration at different frequencies allowing for the cointegrating vectors to be different at different frequencies [3, 16]. In this section we use the more general definition of cointegration at a specific frequency, not necessarily zero.

Definition 6. *Let X_t be a strongly harmonizable n-dimensional vector process with matrix spectral function $F(A,B)$. We will say that X_t is cointegrated at frequency ω with cointegrating vector β_ω if $\varepsilon_{t,\omega} = \beta'_\omega X_t$ is such that its matrix spectral function*

$$F\epsilon,\omega(A,B) = \beta'_\omega F(A,B)\beta_\omega = 0_n \qquad (21)$$

for all Borel sets $A = [\omega - \delta, \omega + \delta]$ and B such that $A \bigcap B = \emptyset$.

Let $\varepsilon_{t,\omega} = \beta'_\omega X_t = \int_{-\pi}^{\pi} e^{itu} Z_{\varepsilon,\omega}(du)$ then this definition implies that in the spectral decomposition of $\varepsilon_{t,\omega}$ the frequency ω is "independent" from the other frequencies. This definition also implies that $F(A,B)$ is singular and β_ω lies in its null space for all Borel sets $A = [\omega - \delta, \omega + \delta]$ and B such that $A \bigcap B = \emptyset$. If there are k distinct cointegrating vectors then $F(A,B)$ has k zero eigenvalues.

If the vectors β_ω are equal for all frequencies then $\varepsilon_{t,\omega}$ is a stationary process. A sufficient condition for X_t to be cointegrated at frequency ω is that there exists $\epsilon_{t,\omega}$ for which the ω frequency be independent from the other frequencies.

Definition 7 (Harmonizable-Integration). *An n-vector process X_t is harmonizable-integrated of order d, denoted by $HI(d)$, if there exists an n-vector process w_t , which is strongly harmonizable, such that*

$$(1-L)^d X_t = w_t$$

Definition 8 (Harmonizable-Cointegration). *An n-vector process X_t harmonizable-integrated of order* 1 *is said to be harmonizable-cointegrated if there exists a linear combination of the series which is $HI(0)$. This means that there exists $\beta \neq 0$ such that $\beta' X_t = \varepsilon_t$ where ϵ_t is strongly harmonizable.*

Note that if $(1 - L)X_t = w_t$ has an absolutely continuous distribution of masses with spectral density matrix $f_{ww}(u, v)$ this implies that $\beta' f_{ww}(u, v)\beta = 0$ at all frequencies (u, v) belonging to $[-\pi, \pi] \times [-\pi, \pi]$. Thus $f_{ww}(u, v)$ is singular and β lies in its null space.

4. Conclusion

In this paper we have shown that the class (KF) of processes is possibly the most general class of processes we might hope to study. We have generalized the concept of cointegration to class (KF) processes. We have considered the situation where we have non-stationary processes generated by nonlinear systems which co-move according to a linear adjustment process. We have also considered the case where we might have two $I(1)$ processes co-moving according to a nonlinear or heteroskedastic adjustment process.

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