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SPECIAL ISSUE ON COMPLEX ANALYSIS: THEORY AND APPLICATIONS DEDICATED TO PROFESSOR HARI M. SRIVASTAVA, ON THE OCCASION OF HIS 70TH BIRTHDAY

Inclusion Properties for Certain Subclasses of Analytic Functions Defined by a Multiplier Transformation

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Abstract. The purpose of the present paper is to investigate some inclusion properties of certain subclasses of analytic functions associated with a family of Multiplier transformations, which are defined by means of the Hadamard product (or convolution).

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1. Introduction

Let $\mathscr{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathscr{S}^*(\alpha)$ and $\mathscr{K}(\alpha)$ denote the subclasses of $\mathscr{A}$ consisting of starlike and convex functions of order α $(0 \leq \alpha < 1)$ and let $\mathscr{S}^*(0) = \mathscr{S}^*$ and $\mathscr{K}(0) = \mathscr{K}$. If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g in $\mathbb{U}$, written as $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w such that $f(z) = g(w(z))$ for $z \in \mathbb{U}$.

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A function $f \in \mathscr{A}$ is said to be prestarlike of order α in $\mathbb{U}$ if

$$\frac{z}{(1-z)^{2(1-\alpha)}} * f(z) \in \mathscr{S}^*(\alpha)\ (0 \le \alpha < 1),$$

where the symbol $(*)$ means the familiar Hadamard product (or convolution) of two analytic functions in $\mathbb{U}$. We denote this class by $\mathscr{R}(\alpha)$ (see, for details, [9]). We note that a function $f \in \mathscr{A}$ is in the class $\mathscr{R}(0)$ if and only if f is convex univalent in $\mathbb{U}$, and $\mathscr{R}(1/2) = \mathscr{S}^*(1/2)$.

Let $\mathscr{N}$ be the class of all analytic functions h which are univalent in $\mathbb{U}$ and for which $h(\mathbb{U})$ is convex with $h(0) = 1$ and and $\mathrm{Re}\{h(z)\} > 0$ in $\mathbb{U}$.

For any real number s, we define the multiplier transformations I_λ^s of functions $f \in \mathscr{A}$ by

$$I_\lambda^s f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^s a_k z^k \ \ (\lambda > -1).$$

Obviously, we observe that

$$I_\lambda^s(I_\lambda^t f(z)) = I_\lambda^{s+t} f(z)$$

for all real numbers s and t. For $\lambda = 1$ and any integer s, the operator I_λ^s was studied by Uralegaddi and Somanatha [13]. Also, for $s = -1$, the operator I_λ^s is the integral operator studied by Owa and Srivastava [8]. Moreover, the operator I_λ^s is closely related to the multiplier transformation studied by Jung et al. [3] (also see [2]), and the differential operator defined by Sălăgean [10].

Let

$$f_\lambda^s(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^s z^k \ \ (s \in \mathbb{R};\ \lambda > -1)$$

and let $f_{\lambda,\mu}^s$ be defined such that

$$f_\lambda^s(z) * f_{\lambda,\mu}^s(z) = \frac{z}{(1-z)^\mu} \ \ (\mu > 0;\ z \in \mathbb{U}), \tag{2}$$

where the symbol $(*)$ stands for the Hadamard product(or convolution). Then, motivated essentially by the Choi-Saigo-Srivastava operator [1] (see also [5], [6] and [7]), we now introduce the operator $I_{\lambda,\mu}^s : \mathscr{A} \to \mathscr{A}$, which are defined here by

$$I_{\lambda,\mu}^s f(z) = \left(f_{\lambda,\mu}^s * f\right)(z) \ \ (f \in \mathscr{A};\ s \in \mathbb{R};\ \lambda > -1;\ \mu > 0), \tag{3}$$

In particular, we note that $I_{0,2}^0 f(z) = zf'(z)$ and $I_{0,2}^1 f(z) = f(z)$. In view of (2) and (3), we obtain the following relations:

$$z\left(I_{\lambda,\mu}^s f(z)\right)' = \mu I_{\lambda,\mu+1}^s f(z) - (\mu-1) I_{\lambda,\mu}^s f(z) \ \ (f \in \mathscr{A};\ \lambda > -1;\ \mu > 0) \tag{4}$$

and

$$z\left(I_{\lambda,\mu}^{s+1} f(z)\right)' = (\lambda+1) I_{\lambda,\mu}^s f(z) - \lambda I_{\lambda,\mu}^{s+1} f(z) \ \ (f \in \mathscr{A};\ \lambda > -1;\ \mu > 0). \tag{5}$$

We also define the function $\phi(a,c;z)$ by

$$\phi(a,c;z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1} \tag{6}$$
$$(z \in \mathbb{U}; a \in \mathbb{R}; c \in \mathbb{R} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- := \{-1,-2,\cdots\}),$$

where $(\nu)_k$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the Gamma function) by

$$(\nu)_k := \frac{\Gamma(\nu+k)}{\Gamma(\nu)} = \begin{cases} 1 & \text{if } k=0 \text{ and } \nu \in \mathbb{C} \setminus \{0\}, \\ \nu(\nu+1)\cdots(\nu+k-1) & \text{if } k \in \mathbb{N} := \{1,2,\cdots\} \text{ and } \nu \in \mathbb{C}. \end{cases}$$

By using the operator $I^s_{\lambda,\mu}$, we introduce the following class of analytic functions for $\gamma > 0$, $\lambda > -1$, $s \in \mathbb{R}$, $\mu > 0$ and $h \in \mathcal{N}$:

$$T^s_{\lambda,\mu}(\gamma;h) := \left\{ f \in \mathcal{A} : (1-\gamma)\frac{I^s_{\lambda,\mu} f(z)}{z} + \gamma (I^s_{\lambda,\mu} f(z))' \prec h(z) \right\}.$$

In the present paper, we derive some inclusion relations, convolution properties and integral preserving properties for the class $T^s_{\lambda,\mu}(\gamma;h)$.

The following lemmas will be required in our investigation.

Lemma 1. *[4] Let g be analytic in $\mathbb{U}$ and h be analytic and convex univalent in $\mathbb{U}$ with $h(0) = g(0)$. If*

$$g(z) + \frac{1}{\gamma} z g'(z) \prec h(z) \ (\mathrm{Re}\{\gamma\} \geq 0; \gamma \neq 0), \tag{7}$$

then

$$g(z) \prec \widetilde{h}(z) = \gamma z^{-\gamma} \int_0^z t^{\gamma-1} h(t) dt \prec h(z)$$

and $\widetilde{h}$ is the best dominant of (7).

Lemma 2. *[9] Let $f \in \mathcal{S}^*(\alpha)$ and $g \in \mathcal{R}(\alpha)$. Then for any analytic function F in $\mathbb{U}$,*

$$\frac{g*(fF)}{g*f}(\mathbb{U}) \subset \overline{co}(F(\mathbb{U}))$$

where $\overline{co}(F(\mathbb{U}))$ denotes the convex hull of $F(\mathbb{U})$.

Lemma 3. *[12] Let $0 < a \leq c$. Then*

$$\mathrm{Re}\left\{ \frac{\phi(a,c;z)}{z} \right\} > \frac{1}{2} \ (z \in \mathbb{U}),$$

where ϕ is given by (1.6).

2. Inclusion Relations

Theorem 1. *If* $0 \le \gamma_1 < \gamma_2$, *then*

$$T^s_{\lambda,\mu}(\gamma_2;h) \subset T^s_{\lambda,\mu}(\gamma_1;h).$$

Proof. Let

$$g(z) = \frac{I^s_{\lambda,\mu}f(z)}{z} \;(f \in T^s_{\lambda,\mu}(\gamma_2;h) : z \in \mathbb{U}). \tag{8}$$

Then the function g is analytic in $\mathbb{U}$ with $g(0) = 1$. Differentiating both sides of (8), we have

$$(1-\gamma_2)\frac{I^s_{\lambda,\mu}f(z)}{z} + \gamma_2(I^s_{\lambda,\mu}f(z))' = g(z) + \gamma_2 z g'(z) \prec h(z). \tag{9}$$

Hence an application of Lemma 1 with $\mu = 1/\gamma_2$ yields

$$g(z) \prec h(z). \tag{10}$$

Since $0 \le \gamma_1/\gamma_2 < 1$ and h is convex univalent in $\mathbb{U}$, it follows from (8), (9) and (10) that

$$\begin{aligned}
&(1-\gamma_1)\frac{I^s_{\lambda,\mu}f(z)}{z} + \gamma_1(I^s_{\lambda,\mu}f(z))' \\
&= \frac{\gamma_1}{\gamma_2}\left[(1-\gamma_2)\frac{I^s_{\lambda,\mu}f(z)}{z} + \gamma_2(I^s_{\lambda,\mu}f(z))'\right] + \left(1-\frac{\gamma_1}{\gamma_2}\right)g(z) \\
&\prec h(z).
\end{aligned}$$

Therefore $f \in T^s_{\lambda,\mu}(\gamma_1;h)$ and so we complete the proof of Theorem 1.

Theorem 2. *If* $0 < \mu_1 \le \mu_2$, *then*

$$T^s_{\lambda,\mu_2}(\gamma;h) \subset T^s_{\lambda,\mu_1}(\gamma;h).$$

Proof. Let $f \in T^s_{\lambda,\mu_2}(\gamma;h)$. Then

$$\begin{aligned}
&(1-\gamma)\frac{I^s_{\lambda,\mu_1}f(z)}{z} + \gamma(I^s_{\lambda,\mu_1}f(z))' \qquad (11)\\
&= \frac{\phi(\mu_1,\mu_2;z)}{z} * \left[(1-\gamma)\frac{I^s_{\lambda,\mu_2}f(z)}{z} + \gamma(I^s_{\lambda,\mu_2}f(z))'\right].
\end{aligned}$$

In view of Lemma 3, we see that the function $\phi(\mu_1,\mu_2;z)/z$ has the Herglotz representation

$$\frac{\phi(\mu_1,\mu_2;z)}{z} = \int_{|x|=1}\frac{d\mu(x)}{1-xz} \;(z \in \mathbb{U}), \tag{12}$$

where $\mu(x)$ is a probability measure defined on the unit circle $|x| < 1$ and

$$\int_{|x|=1} d\mu(x) = 1.$$

Since h is convex univalent in $\mathbb{U}$, it follows from (11) and (12) that

$$(1-\gamma)\frac{I^s_{\lambda,\mu_1} f(z)}{z} + \gamma(I^s_{\lambda,\mu_1} f(z))' = \int_{|x|=1} h(xz)d\mu(x) \prec h(z),$$

which completes the proof of Theorem 9.

Theorem 3. *If $\mu > 0$, then*

$$T^s_{\lambda,\mu+1}(\gamma; h) \subset T^s_{\lambda,\mu}(\gamma; \widetilde{h}),$$

where

$$\widetilde{h}(z) = \mu z^{-\mu} \int_0^z t^{\mu-1} h(t)dt \prec h(z).$$

Proof. Let

$$g(z) = (1-\gamma)\frac{I^s_{\lambda,\mu} f(z)}{z} + \gamma(I^s_{\lambda,\mu} f(z))' \ (f \in \mathscr{A}; z \in \mathbb{U}). \tag{13}$$

Then from (4) and (13), we have

$$zg(z) = \gamma\mu I^s_{\lambda,\mu+1} f(z) + (1-\gamma\mu) I^s_{\lambda,\mu} f(z). \tag{14}$$

Differentiating both sides of (13) and using (4) again, we obtain

$$\begin{aligned} & z(zg'(z) + g(z)) \\ = \ & \gamma\mu z(I^s_{\lambda,\mu+1} f(z)) + (1-\gamma\mu)(\mu I^s_{\lambda,\mu+1} f(z) - (\mu-1) I^s_{\lambda,\mu} f(z)). \end{aligned} \tag{15}$$

By a simple calculation with (14) and (15), we get

$$g(z) + \frac{zg'(z)}{\mu} = (1-\gamma)\frac{I^s_{\lambda,\mu+1} f(z)}{z} + \gamma(I^s_{\lambda,\mu+1} f(z))'. \tag{16}$$

If $f \in T^s_{\lambda,\mu+1}(\gamma; h)$, then it follows from (16) that

$$g(z) + \frac{zg'(z)}{\mu} \prec h(z) \ (\mu > 0).$$

Hence an application of Lemma 1 yields

$$g(z) \prec \widetilde{h}(z) = \mu z^{-\mu} \int_0^z t^{\mu-1} h(t)dt \prec h(z),$$

which shows that

$$f \in T^s_{\lambda,\mu+1}(\gamma; \widetilde{h}) \subset T^s_{\lambda,\mu}(\gamma; h).$$

Theorem 4. *If $s \in \mathbb{R}$ and $\lambda > -1$, then*

$$T^s_{\lambda,\mu}(\gamma;h) \subset T^{s+1}_{\lambda,\mu}(\gamma;\widetilde{h}),$$

where

$$\widetilde{h}(z) = (\lambda+1)z^{-(\lambda+1)}\int_0^z t^\lambda h(t)dt \prec h(z).$$

Proof. By using the same techniques as in the proof of Theorem 3 and (5), we have Theorem 11 and so we omit the detailed proof involved.

Theorem 5. *Let $\gamma > 0$, $\beta > 0$ and $f \in T^s_{\lambda,\mu}(\gamma;\beta h + 1 - \beta)$. If $\beta \le \beta_0$, where*

$$\beta_0 = \frac{1}{2}\left(1 - \frac{1}{\gamma}\int_0^1 \frac{u^{\frac{1}{\gamma}-1}}{1+u}du\right)^{-1}, \tag{17}$$

then $f \in T^s_{\lambda,\mu}(0;h)$. The bound β_0 is sharp for the function

$$h(z) = \frac{1}{1-z} \ (z \in \mathbb{U}).$$

Proof. Let

$$g(z) = \frac{I^s_{\lambda,\mu}f(z)}{z} \ (f \in T^s_{\lambda,\mu}(\gamma;\beta h + 1 - \beta); \gamma > 0; \beta > 0). \tag{18}$$

Then we have

$$\begin{aligned} g(z) + \gamma z g'(z) &= (1-\gamma)\frac{I^s_{\lambda,\mu}f(z)}{z} + \gamma(I^s_{\lambda,\mu}f(z))' \\ &\prec \beta h(z) + 1 - \beta. \end{aligned}$$

Hence an application of Lemma 1 yields

$$g(z) \prec \frac{\beta}{\gamma}z^{-\frac{1}{\gamma}}\int_0^z t^{\frac{1}{\gamma}-1}h(t)dt + 1 - \beta = (h * \psi)(z), \tag{19}$$

where

$$\psi(z) = \frac{\beta}{\gamma}z^{-\frac{1}{\gamma}}\int_0^z \frac{t^{\frac{1}{\gamma}-1}}{1-t}dt + 1 - \beta. \tag{20}$$

If $0 < \beta \le \beta_0$, where β_0 is given by (17), then from (20), we have

$$\begin{aligned}\mathrm{Re}\{\psi(z)\} &= \frac{\beta}{\gamma}\int_0^1 u^{\frac{1}{\gamma}-1}\mathrm{Re}\left\{\frac{1}{1-uz}du\right\}+1-\beta\\ &> \frac{\beta}{\gamma}\int_0^1 \frac{u^{\frac{1}{\gamma}-1}}{1+u}du+1-\beta\\ &\geq \frac{1}{2}.\end{aligned}$$

By using the Herglotz representation for ψ, it follows from (18) and (19) that

$$\frac{I_{\lambda,\mu}^s f(z)}{z} \prec (h*\psi)(z) \prec h(z),$$

since h is convex univalent in $\mathbb{U}$. This shows that $f \in T_{\lambda,\mu}^s(0;h)$.

For $h(z)=1/(1-z)$ and $f \in \mathscr{A}$ defined by

$$\frac{I_{\lambda,\mu}^s f(z)}{z} = \frac{\beta}{\gamma}z^{-\frac{1}{\gamma}}\int_0^z \frac{t^{\frac{1}{\gamma}-1}}{1-t}dt+1-\beta,$$

it is easy to verify that

$$(1-\gamma)\frac{I_{\lambda,\mu}^s f(z)}{z}+\gamma(I_{\lambda,\mu}^s f(z))' = \beta h(z)+1-\beta.$$

Thus $f \in T_{\lambda,\mu}^s(\gamma;\beta h+1-\beta)$. Furthermore, for $\beta > \beta_0$, we have

$$\mathrm{Re}\left\{\frac{I_{\lambda,\mu}^s f(z)}{z}\right\} \text{ to } \frac{\beta}{\gamma}\int_0^1 \frac{u^{\frac{1}{\gamma}-1}}{1+u}du+1-\beta < \frac{1}{2}\ (z\to -1),$$

which implies that $f \notin T_{\lambda,\mu}^s(0;h)$. Hence the bound β_0 cannot be increased when $h(z)=1/(1-z)$ $(z\in\mathbb{U})$.

3. Convolution Properties

Theorem 6. *If $f \in T_{\lambda,\mu}^s(\gamma;h)$ and*

$$\mathrm{Re}\left\{\frac{g(z)}{z}\right\} > \frac{1}{2}\ (g\in\mathscr{A}; z\in\mathbb{U}),$$

then

$$f*g \in T_{\lambda,\mu}^s(\gamma;h).$$

Proof. Let $f \in T^s_{\lambda,\mu}(\gamma;h)$ and $g \in \mathscr{A}$. Then we have

$$(1-\gamma)\frac{I^s_{\lambda,\mu}(f*g)(z)}{z}+\gamma(I^s_{\lambda,\mu}(f*g)(z))'=\frac{g(z)}{z}*\psi(z),$$

where

$$\psi(z)=(1-\gamma)\frac{I^s_{\lambda,\mu}f(z)}{z}+\gamma(I^s_{\lambda,\mu}f(z))'\prec h(z).$$

The remaining part of the proof of Theorem 6 is similar to that of Theorem 2 and so we omit the details involved.

Corollary 1. *Let $f \in T^s_{\lambda,\mu}(\gamma;h)$ be given by (1). Then the function*

$$\sigma_m(z)=\int_0^1\frac{S_m(tz)}{t}dt\ (z\in\mathbb{U}),$$

where

$$S_m(z)=z+\sum_{n=1}^{m-1}a_{n+1}z^{n+1}\ m\in\mathbb{N}\setminus\{1\};z\in\mathbb{U}),$$

is also in the class $T^s_{\lambda,\mu}(\gamma;h)$.

Proof. We have

$$\sigma_m(z)=z+\sum_{n=1}^{m-1}\frac{a_{n+1}}{n+1}z^{n+1}=(f*g_m)(z)\ (m\in\mathbb{N}\setminus\{1\}),\tag{21}$$

where

$$f(z)=z+\sum_{n=1}^{\infty}a_{n+1}z^{n+1}\in T^n_{\lambda,\mu}(\gamma;h)$$

and

$$g_m(z)=z+\sum_{n=1}^{m-1}\frac{z^{n+1}}{n+1}\in\mathscr{A},$$

while, it is known [11] that

$$\mathrm{Re}\left\{\frac{g_m(z)}{z}\right\}=\mathrm{Re}\left\{1+\sum_{n=1}^{m-1}\frac{z^n}{n+1}\right\}>\frac{1}{2}\ (m\in\mathbb{N}\setminus\{1\};z\in\mathbb{U}).\tag{22}$$

In view of (21) and (22), an application of Theorem 6 leads to $\sigma_m\in T^s_{\lambda,\mu}(\gamma;h)$.

Theorem 7. *If $f\in T^s_{\lambda,\mu}(\gamma;h)$ and*

$$g(z)\in R(\alpha)\ (g\in\mathscr{A};z\in\mathbb{U}),$$

then

$$f*g\in T^s_{\lambda,\mu}(\gamma;h).$$

Proof. By using a similar method as in the proof of Theorem 21, we have

$$(1-\gamma)\frac{I^s_{\lambda,\mu}(f*g)(z)}{z}+\gamma(I^s_{\lambda,\mu}(f*g)(z))'=\frac{g(z)*(z\psi(z))}{g(z)*z}\ (z\in\mathbb{U}), \tag{23}$$

where

$$\psi(z)=(1-\gamma)\frac{I^s_{\lambda,\mu}f(z)}{z}+\gamma(I^s_{\lambda,\mu}f(z))'\prec h(z).$$

Since h is convex univalent in $\mathbb{U}$, it follows from (23) and Lemma 2 that Theorem 7 holds true.

If we take $\alpha=0$ and $\alpha=1/2$ in Theorem 7, we have the following corollary.

Corollary 2. *If $f\in T^s_{\lambda,\mu}(\gamma;h)$ and $g\in\mathscr{A}$ satisfies one of the following conditions:*

(i) $g(z)$ *is convex univalent in* $\mathbb{U}$

or

(ii) $g(z)\in S^*(\frac{1}{2})$,

then $f*g\in T^s_{\lambda,\mu}(\gamma;h)$.

4. Integral Operators

Theorem 8. *If $f\in T^s_{\lambda,\mu}(\gamma;h)$, then the function F defined by*

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt\ (\text{Re}\{c\}>-1) \tag{24}$$

is in the class $T^s_{\lambda,\mu}(\gamma;\widetilde{h})$, where

$$\widetilde{h}(z)=(c+1)z^{-(c+1)}\int_0^z t^c h(t)dt\prec h(z).$$

Proof. Let $f\in T^s_{\lambda,\mu}(\gamma;h)$. Then from (24), we obtain

$$(c+1)f(z)=zF'(z)+cF(z). \tag{25}$$

Define the function G by

$$zG(z)=(1-\gamma)I^s_{\lambda,\mu}F(z)+\gamma z(I^s_{\lambda,\mu}F(z))'\ (z\in\mathbb{U}). \tag{26}$$

Differentiating both sides of (26) with respect to z, we get

$$G(z)+zG'(z)=(1-\gamma)\frac{I^s_{\lambda,\mu}(zF'(z))}{z}+\gamma(I^s_{\lambda,\mu}(zF'(z)))'. \tag{27}$$

Furthermore, it follows from (25), (26) and (27) that

$$
\begin{aligned}
&(1-\gamma)\frac{I^s_{\lambda,\mu}f(z)}{z}+\gamma(I^s_{\lambda,\mu}f(z))' \\
&=(1-\gamma)z^{-1}I^s_{\lambda,\mu}\left(\frac{zF'(z)+cF(z)}{c+1}\right)+\gamma\left(I^s_{\lambda,\mu}\left(\frac{zF'(z)+cF(z)}{c+1}\right)\right)' \\
&=G(z)+\frac{1}{c+1}zG'(z). \qquad (28)
\end{aligned}
$$

Since $f\in T^s_{\lambda,\mu}(\gamma;h)$, from (28), we have

$$G(z)+\frac{1}{c+1}zG'(z)\prec h(z)\ (\mathrm{Re}\{c\}>-1),$$

and so an application of Lemma 1 yields

$$G(z)\prec\widetilde{h}(z)=\frac{c+1}{z^{c+1}}\int_0^z t^c h(t)dt\prec h(z).$$

Therefore we conclude that

$$F\in T^s_{\lambda,\mu}(\gamma;\widetilde{h})\subset T^s_{\lambda,\mu}(\gamma;h).$$

Theorem 9. *If $f\in\mathscr{A}$ and F be defined as in Theorem 8. If*

$$(1-\alpha)\frac{I^s_{\lambda,\mu}F(z)}{z}+\alpha\frac{I^s_{\lambda,\mu}f(z)}{z}\prec h(z)\ (\alpha>0), \qquad (29)$$

then $F\in T^s_{\lambda,\mu}(0;\widetilde{h})$, where

$$\widetilde{h}(z)=\frac{c+1}{\alpha}z^{-\frac{\alpha}{c+1}}\int_0^z t^{\frac{c+1}{\alpha}-1}h(t)\prec h(z)\ (\mathrm{Re}\{c\}>-1).$$

Proof. Let

$$G(z)=\frac{I^s_{\lambda,\mu}F(z)}{z}\ (z\in\mathbb{U}). \qquad (30)$$

Then G is analytic in $\mathbb{U}$ with $G(0)=1$ and

$$zG'(z)=(I^s_{\lambda,\mu}F(z))'-G(z). \qquad (31)$$

It follows from (25), (29), (30), and (31) that

$$
\begin{aligned}
&(1-\alpha)\frac{I^s_{\lambda,\mu}F(z)}{z}+\alpha\frac{I^s_{\lambda,\mu}f(z)}{z} \\
&=(1-\alpha)\frac{I^s_{\lambda,\mu}F(z)}{z}+\frac{\alpha}{c+1}\left[\frac{cI^s_{\lambda,\mu}F(z)}{z}+(I^s_{\lambda,\mu}F(z))'\right]
\end{aligned}
$$

$$= G(z) + \frac{\alpha}{c+1} zG'(z) \prec h(z) \ (\text{Re}\{c\} > 1; \alpha > 0).$$

Therefore, by Lemma 1, we conclude that Theorem 9 holds true as stated.

Theorem 10. *Let* $F \in T^s_{\lambda,\mu}(\gamma; h)$*. If the function* f *is defined by*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \ (c > -1), \tag{32}$$

then

$$\frac{f(\sigma z)}{\sigma} \in T^s_{\lambda,\mu}(\gamma; h),$$

where

$$\sigma = \sigma(c) = \frac{\sqrt{1+(c+1)^2} - 1}{c+1}. \tag{33}$$

The bound σ *is sharp for the function*

$$h(z) = \beta + (1-\beta)\frac{1+z}{1-z} \ (\beta \neq 1; z \in \mathbb{U}). \tag{34}$$

Proof. We note that for $F \in \mathscr{A}$,

$$F(z) = F(z) * \frac{z}{1-z} \text{ and } zF'(z) = F(z) * \frac{z}{(1-z)^2}.$$

Then from (32), we have

$$f(z) = \frac{cF(z) + zF'(z)}{c+1} = (F * g)(z) \ (c > -1; z \in \mathbb{U}), \tag{35}$$

where

$$g(z) = \frac{1}{c+1} \left(c\frac{z}{1-z} + \frac{z}{(1-z)^2} \right) \in \mathscr{A}. \tag{36}$$

Next, we show that

$$\text{Re}\left\{ \frac{g(z)}{z} \right\} > \frac{1}{2} \ (|z| < \sigma), \tag{37}$$

where $\sigma = \sigma(c)$ is given by (4.10). Letting

$$\frac{1}{1-z} = Re^{i\theta} \ (|z| = r < 1; R > 0),$$

we see that

$$\cos\theta = \frac{1 + R^2(1-r^2)}{2R} \text{ and } R \geq \frac{1}{1+r}. \tag{38}$$

Then for (36) and (38), we have

$$2\mathrm{Re}\left\{\frac{g(z)}{z}\right\} = \frac{2}{c+1}\left[cR\cos\theta + R^2(2\cos^2\theta - 1)\right]$$
$$= \frac{R^2}{c+1}\left[c(1-r^2) + R^2(1-r^2)^2 - 2\right] + 1$$
$$\geq \frac{R^2}{c+1}\left[c+1-2r-(c+1)r^2\right] + 1.$$

This evidently gives (37), which is equivalent to

$$\mathrm{Re}\left\{\frac{g(\sigma z)}{z\sigma}\right\} > \frac{1}{2}\; z \in \mathbb{U}). \tag{39}$$

Let $F \in T^s_{\lambda,\mu}(\gamma;h)$. Then, by using (35) and (39), an application of Theorem 6 yields

$$\frac{f(\sigma z)}{\sigma} = F(z) * \frac{g(\sigma z)}{\sigma} \in T^s_{\lambda,\mu}(\gamma;h).$$

For h given by (34), we consider the function $F \in \mathscr{A}$ defined by

$$(1-\gamma)\frac{I^s_{\lambda,\mu}F(z)}{z} + \gamma(I^s_{\lambda,\mu}F(z))' = \beta + (1-\beta)\frac{1+z}{1-z}\; (\beta \neq 1; z \in \mathbb{U}). \tag{40}$$

Then from (26), (28) and (40) , we find that

$$(1-\gamma)\frac{I^s_{\lambda,\mu}f(z)}{z} + \gamma(I^s_{\lambda,\mu}f(z))'$$
$$= \beta + (1-\beta)\frac{1+z}{1-z} + \frac{z}{c+1}\left(\beta + (1-\beta)\frac{1+z}{1-z}\right)'$$
$$= \beta + \frac{(1-\beta)(c+1+2z-(c+1)z^2)}{(c+1)(1-z)^2}$$
$$= \beta\; (z = -\sigma).$$

Therefore we conclude that the bound $\sigma = \sigma(c)$ cannot be increased for each c $(c > -1)$.

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