EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 1, No. 1, 2008, (197-201)
ISSN 1307-5543 – www.ejpam.com

Honorary Invited Paper

Selberg-type squared matrices gamma and beta integrals

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Abstract. Although Selberg-type, single positive definite symmetric matrices, gamma and beta integrals are evaluated by several authors; see e.g., Askey and Richards (1989), Gupta and Kabe (2005), Mathai (1997), and elsewhere in the vast multivariate statistical analysis literature; the Selberg-type squared matrice gamma and beta interals appear to have been neglected. So also Selberg-type of integrals of positive signature symmetric matrices, skew symmetric matrices are neglected in the literature. This paper records Selberg-type squared matrices beta and gamma integrals.

Key words: Selberg-type integrals; squared matrices integrals; hypercomplex normal distributions;multivariate gamma and beta densities.

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1. Introduction

We first list a few Selberg-type beta and gamma readily available integrals. The following three gamma integrals are listed by Mathai (1997, p. 231)

$$\int \exp\{-\frac{1}{2}tr\ \Lambda^2\}\prod_{i<j}(\lambda_i-\lambda_j)^{2h}d\Lambda=\pi^{-\frac{1}{2}p}(\Gamma(1+h))^{-p}\prod_{i=1}^{p}\Gamma(ih+1) \tag{1}$$

where $\Lambda = diag(\lambda_1,\ldots,\lambda_p),\ 0<\lambda_i<\infty,$

$$\int \exp\{-tr;\Lambda\}|\Lambda|^{g-1}\prod_{i<j}(\lambda_i-\lambda_j)^{2h}d\Lambda=\left[\prod_{i=1}^{p}\Gamma(g+(i-1)h)\Gamma(ih+1)\right][\Gamma(1+h)]^{-p}. \tag{2}$$

and

$$\int \exp\{-tr\ \Lambda^2\}\prod_{i<j}(\lambda_i^2-\lambda_j^2)^{2h}d\Lambda=[\Gamma(1+h)]^{-p}\prod_{i=1}^{p}\gamma\left(\frac{1}{2}+(i-1)h\right)\Gamma(ih+1), \tag{3}$$

which follows from Mathai (1997, p. 114, 2.1.4).

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Askey and Richards (1989), Gupta and Kabe (2005) record

$$\int |\Lambda|^{g-1}|I-\Lambda|^{t-1}\prod_{i<j}(\lambda_i-\lambda_j)^{2h}d\Lambda = [\Gamma(1+h)]^{-p}\prod_{i=1}^{p}\frac{\Gamma(g+(i-1)h)\Gamma(t+(i-1)h)}{\Gamma(g+t+(i-h))} \quad (4)$$

and

$$\int |\Lambda|^{g-1}(1-tr\Lambda)^{t-1}\prod_{i<j}(\lambda_i-\lambda_j)^{2h}d\Lambda \quad = \frac{[\Gamma(1+h)]^{-p}\Gamma(t)\pi^{\frac{1}{2}hp(p-1)}}{\Gamma(t+pg+p(p-1)h)\prod_{i=1}^{p}\Gamma(g+(i-1)h)\Gamma(ih+1)}. \quad (5)$$

Setting

$$\Gamma_p(a) = \Pi^{tp(p-1)}\prod_{i=1}^{p}\Gamma(a-2t(p-i)), \quad (6)$$

we write

$$\int |I+\Lambda|^{-2t(n+q)}|\Lambda|^{2t(n-p+1)-1}\prod_{i<j}(\lambda_i-\lambda_j)^{4t}d\Lambda = B_p(2nt,2qt)(\Gamma(1+2t))^{-p}\prod_{i=1}^{p}\Gamma(2ti+1), \quad (7)$$

where $B_p(a,b) = \frac{\Gamma_0(a)\Gamma_p(b)}{\Gamma_p(a+b)}$.

The present paper evaluates (1), and

$$\int |I+\Lambda^2|^{-\alpha}\prod_{i<j}(\lambda_i-\lambda_j)^{4t}d\Lambda = 2^{2tp}\pi^{tp(p+1)}\Gamma_p(\alpha-\frac{t}{2}(p+1))[\Gamma(1+2t)]^{-p}\prod_{i=1}^{p}\Gamma(2ti+1). \quad (8)$$

The integral (8) is listed by Mathai (1997, p. 112, 2.1.4).

Mathai (1997, p. 231, 4.1.2) lists a positive signature symmetric matrix Selberg-type gamma integral. The positive signature symmetric matrice Selberg-type beta integrals follow on the same lines. Mathai (1997) also lists several skew symmetric matrices integrals, but not Selberg-type skew symmetric matrices integrals.

We state some useful results in the next section, and evaluate (1) and (8) in section 3. The multivariate statisticsl analysis has now become extremely vast, and search for Selberg-type skew symmetric matrix integrals is formidable.

2. Some Useful Results

Kabe (1984) develops the hypercomplex (HC) multivariate analysis distribution theory, and we record here some relevant results.

Let $X = (X_1', \ldots, X_{4t}')'$, $t = \frac{1}{4}, \frac{1}{2}, 1, 2$ be $4t\,(p\times n)$ real random matrices having the $4nt$ variate normal density

$$g(X) = (4\pi t)^{-2nt}|\Sigma_0|^{-\frac{1}{2}}\exp\left\{\frac{-tr\,\Sigma_0^{-1}XX'}{4t}\right\}, \quad (9)$$

where $\Sigma_0,\ p\times p$, is

$$[\Sigma_1-\Sigma_2-\Sigma_3-\Sigma_4-\Sigma_5-\Sigma_6-\Sigma_7-\Sigma_8\Sigma_2\Sigma_1-\Sigma_4\Sigma_3-\Sigma_6\Sigma_5\Sigma_8-\Sigma_7\Sigma_3\Sigma_4\Sigma_1$$
$$-\Sigma_2-\Sigma_7\Sigma_8\Sigma_5\Sigma_4\Sigma_4-\Sigma_3\Sigma_2\Sigma_1-\Sigma_8\Sigma_7-\Sigma_4\Sigma_3\Sigma_5\Sigma_6\Sigma_7\Sigma_8\Sigma_1-\Sigma_2-\Sigma_3-\Sigma_4\Sigma_6-\Sigma_5-$$
$$\Sigma_8-\Sigma_7\Sigma_2\Sigma_1\Sigma_4\Sigma_3\Sigma_7-\Sigma_8-\Sigma_5\Sigma_6\Sigma_3-\Sigma_4\Sigma_1\Sigma_2-\Sigma_8\Sigma_7\Sigma_6-\Sigma_5\Sigma_4\Sigma_3-\Sigma_2\Sigma_1], \quad (10)$$

where now $t=2$, and we use Hamilton's octonions, Halberstam and Ingram (1967, p. 654, equation (1)). Similar HC theory follows by using Cayley's or Young's octonions or bioctonions.

In the context octonion case Σ_1 is a real positive definite symmetric matrix, and $\Sigma_2,\ldots,\Sigma_8$ are real $p\times p$ skew symmetric matrices. Now we set

$$Y = X_1 + iX_2 + jX_3 + kX_4 + \ell X_5 + mX_6 + nX_7 + rX_8, \quad (11)$$

where i,j,k,ℓ,m,n,r octonions satisfy the multiplication rule

$$i^2 = j^2 = k^2 = \ell^2 = m^2 = n^2 = r^2 = 1 = ijk = i\ell n = irn = j\ell n = jmn = kir = knm, \quad (12)$$

and observe that

$$\bar{Y} = X_1 - iX_2 - jX_3 - kX_4 - \ell X_5 - mX_6 - nX_7 - rX_8, \quad (13)$$

is the HC conjugate of Y, and

$$dY = dX_1 dX_2 \ldots dX_8. \quad (14)$$

After some formidable algebra, Kabe (1984) writes the pn variate normal density of Y to be

$$g(Y) = \pi^{-2pnt}|\Sigma|^{-2nt}\exp\{-tr\,\Sigma^{-1}Y\bar{Y}'\}, \quad (15)$$

where

$$\Sigma = \Sigma_1 + i\Sigma_2 + j\Sigma_3 + k\Sigma_4 + \ell\Sigma_5 + m\Sigma_6 + n\Sigma_7 + r\Sigma_8, \quad (16)$$

is the HC Hermitian $p\times p$ matrix. All of its roots are real and positive. There are also positive signature HC symmetric matrices, HC skew symmetric matrices, and they can be used to generalize or evaluate integrals of type Mathai (1997, p. 231, 4.1.2).

On setting $Y\bar{Y}' = G$, the HC Hermitian $p\times p$ Wishart matrix, Kabe (1984, p. 67, equation (14)) records this Wishart density

$$g(G) = (\Gamma_p(2nt))^{-1}|\Sigma|^{-2nt}|G|^{2t(n-p-1)-1}\exp\{-tr\,\Sigma^{-1}G\}, \quad (17)$$

a gamma-type HC Hermitian $p\times p$ matrix density. The first kind beta density is

$$g(G) = [B_p(2nt,2qt)]^{-1}|I-G|^{2t(n-p-1)-1}|G|^{2t(q-p-1)-1}, \quad (18)$$

and the second kind beta density is

$$g(G) = [B_p(2nt,2qt)]^{-1}|I+G|^{-2t(n+q)}|G|^{2t(n-p-1)-1}. \quad (19)$$

3. Selberg Squared Matrices Integrals

It follows from Mathai, Provost, and Hayakawa (1995, p. 200, 4.3.2) that

$$\int \exp\{-tr\, G^2\} dG = 2^{2tp}\pi^{tp(p+1)}. \tag{20}$$

Kabe (1984, p. 68) shows that $G = O\Lambda O'$, where O is HC unitary, the Jacogian of transformation from G to Λ is given by

$$J(G;\Lambda) = \prod_{i<j}(\lambda_i - \lambda_j)^{4t}. \tag{21}$$

From (20) we conclude that

$$\int \exp\{-tr\, \Lambda^2\} \prod_{i<j}(\lambda_i - \lambda_j)^{4t} d\Lambda = 2^{2tp}\pi^{tp(p+1)}(\Gamma(1+2t))^{-p}\prod_{i=1}^{p}\Gamma(2ti+1). \tag{22}$$

Now we note that

$$\int |I + G^2|^{-\alpha} dG = \int\int \exp\{-tr(I+G^2)Z\}|Z|^{\alpha-\frac{1}{2}(p+1)} dZ dG. \tag{23}$$

Setting $G = Z^{-\frac{1}{4}}B\Sigma^{-\frac{1}{4}}$, and integrating with respect to Z first and then with respect to B the integral (23) is written as

$$\int \exp\{-tr\, Z - tr\, B^2\} dB |Z|^{\alpha - \frac{1}{4}(p+1) - \frac{1}{2}(p+1)} dZ = 2^{2tp}\pi^{tp(p+1)}\Gamma_p\left(\alpha - \frac{t}{2}(p+1)\right), \tag{24}$$

where $t = \frac{1}{4}$. When G is HC Hermitian we set in (24) $t = t$.

It follows from (24) that

$$\int |I + G^2|^{-\alpha} dG = 2^{2tp}\pi^{tp(p+1)}\Gamma_p\left(\alpha - \frac{t}{2}(p+1)\right), \tag{25}$$

and from (25) that

$$\int |I+\Lambda^2|^{-\alpha}\prod_{i<j}(\lambda_i-\lambda_j)^{4t} d\Lambda = 2^{2tp}\pi^{tp(p+1)}\Gamma_p\left(\alpha - \frac{t}{2}(p+1)\right)[\Gamma(1+2t)]^{-p}\prod_{i=1}^{p}\Gamma(2ti+1).$$

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