EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 4, No. 1, 2011, 67-75
ISSN 1307-5543 – www.ejpam.com

The Numerical Solution Of Partial Differential-Algebraic Equations (PDAEs) By Multivariate Pade Approximation

Muhammed Yiğider [2], Ercan Celik[1,*]

[1] *Department of Mathematics, Faculty of Science, Ataturk University, Erzurum, Turkey*
[2] *Department of Mathematics, Faculty of Art and Science, Erzincan University, Erzincan, Turkey*

Abstract. In this paper, Numerical solution of Partial Diferential-Algebraic Equa- tions(PDAEs) is considered by Multivariate Padè Approximations. We applied these method to one example. First Partial Diferential-Algebraic Equation(PDAE) has been converted to power series by two-dimensional diferential transformation,Then the numerical solution of equation was put into Multivariate Padè series form. Thus we obtained numerical solution of Partial Diferential-Algebraic Equa- tion(PDAE).

2000 Mathematics Subject Classifications: 35

Key Words and Phrases: Partial Differential-Algebraic Equation (PDAs), two-dimensional differential transformation, Multivariate Padè Approximation

1. Introduction

In this study, we consider Linear Partial Differential-Algebraic Equations(PDAEs) of the form

$$Au_t(t,x)+Bu_{xx}(t,x)+Cu(t,x)=f(t,x) \tag{1}$$

Where $t\in(0,t_e)$ and $x\in(-l,l)\subset R$, $A,B,C\in R^{nxn}$ are constant matrices,
$u,f : [0,t_e]\,\mathrm{x}\,[-l,l]\to R^n$. We are interested in cases where at least one of the matrices A and $B$is singular. The two special cases $A=0$ or $B=0$ lead to ordinary differential equations or DAEs which are not considered here. Therefore in this paper we assume that none of the matrices A or B is the zero matrix [6, 7].
Many important mathematical models can be expressed in terms of Partial Differential Algebraic Equations(PDAEs). Such models arise in many areas of mathematics, engineering, the physical sciences and population growth. In resent years, much research has been focused on the numerical solution of Partial Differential-Algebraic Equations(PDAEs). Some numerical methods have been developed, using Runge-Kutta methods [8].

*Corresponding author.

Email addresses: ercelik@atauni.edu.tr (E. Celik), myigider@erzincan.edu.tr (M. Yigider)

The purpose of this paper is to consider the numerical solution of Partial Differential-Algebraic Equations(PDAEs) by using Multivariate Padé Approximations.

2. Two-Dimensional Differential Transformation

The basic definition of the two-dimensional differential transform is defined as follows [9, 2, 3, 4, 1]:

$$W(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h} w(x,y)}{\partial x^k \partial y^h}\right]_{0,0} \tag{2}$$

where $w(x,y)$ is the original function and $W(k,h)$ is the transformed function. The transformation is called T- function and lower case and upper case letters represent the original and transformed functions respectively. The differential inverse transform of $W(k,h)$ is defined as

$$w(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} W(k,h)x^k y^h \tag{3}$$

and from Eqs.(2) and (3) can be concluded

$$w(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{k!h!}\left[\frac{\partial^{k+h} w(x,y)}{\partial x^k \partial y^h}\right]_{0,0} x^k y^h. \tag{4}$$

3. Multivariate Padé Approximations

Consider the bivariate function $f(x,y)$ with Taylor series development

$$f(x,y) = \sum_{i,j=0}^{\infty} c_{ij} x^i y^j \tag{5}$$

around the origin. We know that a solution of univariate Padé approximation problem for

$$f(x) = \sum_{i=0}^{\infty} c_i x^i \tag{6}$$

is given by

$$p(x) = \begin{vmatrix} \sum_{i=0}^{m} c_i x^i & x\sum_{i=0}^{m-1} c_i x^i & \cdots & x^n \sum_{i=0}^{m-n} c_i x^i \\ c_{m+1} & c_m & \cdots & c_{m+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+n} & c_{m+n-1} & \cdots & c_m \end{vmatrix} \tag{7}$$

and

$$q(x) = \begin{vmatrix} 1 & x & \cdots & x^n \\ c_{m+1} & c_m & \cdots & c_{m+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+n} & c_{m+n-1} & \cdots & c_m \end{vmatrix} \tag{8}$$

Let us now multiply jth row in $p(x)$ and $q(x)$ by x^{j+m-1} $(j=2,\dots,n+1)$ and afterwards divide jth column in $p(x)$ and $q(x)$ by $x^{j-1}(j=2,\dots,n+1)$. This results in a multiplication of numerator and denominator by x^{mn}. Having done so, we get

$$\frac{p(x)}{q(x)}=\frac{\begin{vmatrix} \sum_{i=0}^{m} c_i x^i & \sum_{i=0}^{m-1} c_i x^i & \cdots & \sum_{i=0}^{m-n} c_i x^i \\ c_{m+1}x^{m+1} & c_m x^m & \cdots & c_{m+1-n}x^{m+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+n}x^{m+n} & c_{m+n-1}x^{m+n-1} & \cdots & c_m x^m \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ c_{m+1}x^{m+1} & c_m x^m & \cdots & c_{m+1-n}x^{m+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+n}x^{m+n} & c_{m+n-1}x^{m+n-1} & \cdots & c_m x^m \end{vmatrix}} \tag{9}$$

$$(D=\det D_{m,n}\neq 0).$$

This quotient of determinants can also immediately be written down for a bivariate function $f(x,y)$. The sum $\sum_{i=0}^{k} c_i x^i$ shall be replaced kth partial sum of the Taylor series development of $f(x,y)$ and the expression $c_k x^k$ by an expression that contains all the terms of degree k in $f(x,y)$. Here a bivariate term $c_{ij}x^i y^j$is said to be of degree $i+j$.

If we define

$$p(x,y)=\begin{vmatrix} \sum_{i+j=0}^{m} c_{ij}x^i y^j & \sum_{i+j=0}^{m-1} c_{ij}x^i y^j & \cdots & \sum_{i+j=0}^{m-n} c_{ij}x^i y^j \\ \sum_{i+j=m+1} c_{ij}x^i y^j & \sum_{i+j=m} c_{ij}x^i y^j & \cdots & \sum_{i+j=m+1-n} c_{ij}x^i y^j \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i+j=m+n} c_{ij}x^i y^j & \sum_{i+j=m+n-1}^{m} c_{ij}x^i y^j & \cdots & \sum_{i+j=m}^{m} c_{ij}x^i y^j \end{vmatrix} \tag{10}$$

and

$$q(x,y)=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \sum_{i+j=m+1} c_{ij}x^i y^j & \sum_{i+j=m} c_{ij}x^i y^j & \cdots & \sum_{i+j=m+1-n} c_{ij}x^i y^j \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i+j=m+n} c_{ij}x^i y^j & \sum_{i+j=m+n-1}^{m} c_{ij}x^i y^j & \cdots & \sum_{i+j=m}^{m} c_{ij}x^i y^j \end{vmatrix} \tag{11}$$

Then it is easy to see that $p(x,y)$ and $q(x,y)$ are of the form

$$\begin{aligned} p(x,y) &= \sum_{i+j=mn}^{mn+m} a_{ij}x^i y^j \\ q(x,y) &= \sum_{i+j=mn}^{mn+n} b_{ij}x^i y^j \end{aligned} \tag{12}$$

We know that $p(x,y)$ and $q(x,y)$ are called Padé equations [5]. So the multivariate Padé approximant of order (m,n) for $f(x,y)$ is defined as,

$$r_{m,n}(x,y)=\frac{p(x,y)}{q(x,y)} \tag{13}$$

4. Numerical Example:

The test problem considers the following Partial Differential-Algebraic Equation(PDAE) [8]:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} u_t + \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} u_{xx} + \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} u = f \tag{14}$$

$x \in [-0.5, 0.5]$, $t \in [0,1]$. Where

$$\begin{aligned} f_1 &= -x(x-1)(2\sin t + \cos t) - (e^t + t^5)(x^2 - x + 2) \\ f_2 &= x(x-1)(e^t + 5t^4 - \cos t) - 2(e^t + t^5 + \cos t) \\ f_3 &= -x(x-1)(e^t + t^5). \end{aligned}$$

The exact solution is

$$u(x,t) = \begin{pmatrix} x(x-1)\sin(t) \\ x(x-1)\cos(t) \\ x(x-1)(e^t + t^5) \end{pmatrix} \tag{15}$$

Equivalently, Equation (14) can be written as

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \end{pmatrix} + \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_{1xx} \\ u_{2xx} \\ u_{3xx} \end{pmatrix} + \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \tag{16}$$

$$\begin{aligned} u_{2t} - u_{3xx} - u_1 - u_2 - u_3 &= f_1 \\ u_{3t} - u_{2xx} - u_{3xx} - u_2 &= f_2 \\ -u_3 &= f_3 \end{aligned} \tag{17}$$

By using the basic definition of the two-dimensional differential transform and taking the transform of Equation (17) can obtain that

$$\begin{aligned} &(k+1)U_2(k+1,h) - (h+1)(h+2)U_3(k,h+2) - U_1(k,h) - U_2(k,h) - U_3(k,h) = F_1(k,h) \\ &(k+1)U_3(k+1,h) - (h+1)(h+2)U_2(k,h+2) - (h+1)(h+2)U_3(k,h+2) - U_2(k,h) = F_2(k,h) \\ &-U_3(k,h) = F_3(k,h) \end{aligned}$$

Consequently, by substituting the values of U_i. We have obtained

$$u_1(x,t) = -xt + x^2t + \frac{1}{6}xt^3 - \frac{1}{6}x^2t^3 - \frac{1}{120}xt^5 + \frac{1}{120}x^2t^5 + \frac{1}{5040}xt^7$$

$$u_2(x,t) = -x + x^2 + \frac{1}{2}xt^2 - \frac{1}{2}x^2t^2 - \frac{1}{24}xt^4 + \frac{1}{24}x^2t^4 + \frac{1}{720}xt^6 - \frac{1}{720}x^2t^6$$

$$u_3(x,t) = -x - xt + x^2 - \frac{1}{2}xt^2 + x^2t - \frac{1}{6}xt^3 + \frac{1}{2}x^2t^2 - \frac{1}{24}xt^4 + \frac{1}{6}x^2t^3 - \frac{121}{120}xt^5$$

$$+\frac{1}{24}x^2t^4 - \frac{1}{720}xt^6 + \frac{121}{120}x^2t^5 - \frac{1}{5040}xt^7 + \frac{1}{720}x^2t^6$$

The Power series $u_1(x,t), u_2(x,t)$ and $u_3(x,t)$ can be transformed into multivariate Padé approximation

$$m = 3, n = 2$$

$$\begin{aligned} p_1(x,t) &= \begin{vmatrix} -xt + x^2t & -xt & 0 \\ \frac{1}{6}xt^3 & x^2t & -xt \\ \frac{1}{6}x^2t^3 & \frac{1}{6}xt^3 & x^2t \end{vmatrix} \\ &= -0.1666666667x^3t^5 + 0.1666666667x^4t^5 - x^5t^7 + x^6t^7 \end{aligned}$$

$$\begin{aligned} q_1(x,t) &= \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{6}xt^3 & x^2t & -xt \\ \frac{1}{6}x^2t^3 & \frac{1}{6}xt^3 & x^2t \end{vmatrix} \\ &= x^4t^2 + 0.1666666667x^2t^4 + 0.02777777778x^2t^6 + 0.1666666667x^4t^4 \end{aligned}$$

$$\begin{aligned} r_1(x,t) &= \frac{p_1(x,t)}{q_1(x,t)} \\ &= \frac{-0.1666666667x^3t^5 + 0.1666666667x^4t^5 - x^5t^7 + x^6t^7}{x^4t^2 + 0.1666666667x^2t^4 + 0.02777777778x^2t^6 + 0.1666666667x^4t^4} \end{aligned}$$

$$\begin{aligned} p_2(x,t) &= \begin{vmatrix} -x + x^2 + \frac{1}{2}xt^2 & -x + x^2 & -x \\ -\frac{1}{2}x^2t^2 & \frac{1}{2}xt^2 & x^2 \\ -\frac{1}{24}xt^4 & -\frac{1}{2}x^2t^2 & \frac{1}{2}xt^2 \end{vmatrix} \\ &= -0.2500000000x^3t^4 - 0.5000000000x^5t^2 + 0.04166666667x^4t^4 \\ &\quad +0.5000000000x^6t^2 + 0.1041666667x^3t^6 + 0.2083333333x^5t^4 \end{aligned}$$

$$\begin{aligned} q_2(x,t) &= \begin{vmatrix} 1 & 1 & 1 \\ -\frac{1}{2}x^2t^2 & \frac{1}{2}xt^2 & x^2 \\ -\frac{1}{24}xt^4 & -\frac{1}{2}x^2t^2 & \frac{1}{2}xt^2 \end{vmatrix} \\ &= 0.2500000000x^4t^4 + 0.5000000000x^4t^2 + 0.2500000000x^4t^4 \\ &\quad +0.20833333333x^3t^4 + 0.0208333333x^2t^6 \end{aligned}$$

$$r_2(x,t) = \frac{p_2(x,t)}{q_2(x,t)}$$

$$= \frac{\begin{pmatrix} -0.2500000000x^3t^4 - 0.5000000000x^5t^2 + 0.04166666667x^4t^4 \\ +0.5000000000x^6t^2 + 0.1041666667x^3t^6 + 0.2083333333x^5t^4 \end{pmatrix}}{\begin{pmatrix} -0.2500000000x^3t^4 - 0.5000000000x^5t^2 + 0.04166666667x^4t^4 \\ +0.5000000000x^6t^2 + 0.1041666667x^3t^6 + 0.2083333333x^5t^4 \end{pmatrix}}$$

$$p_3(x,t) = \begin{vmatrix} -x - xt + x^2 - \frac{1}{2}xt^2 + x^2t & -x - xt + x^2 & -x \\ -\frac{1}{6}xt^3 + \frac{1}{2}x^2t^2 & -\frac{1}{2}xt^2 + x^2t & -xt + x^2 \\ -\frac{1}{24}xt^4 + \frac{1}{6}x^2t^3 & -\frac{1}{6}xt^3 + \frac{1}{2}x^2t^2 & -\frac{1}{2}xt^2 + x^2t \end{vmatrix}$$

$$= 0.0833333333x^2t^4 - 0.3333333333x^3t^3 + 0.50000000000x^4t^2 + 0.0694444444x^2t^6$$
$$-0.0416666667x^3t^5 - 0.0416666667x^2t^5 + 0.2083333333x^3t^4 + 0.083333333x^4t^4$$
$$-0.3333333333x^4t^3$$

$$r_3(x,t) = \frac{p_3(x,t)}{q_3(x,t)}$$

$$= \frac{\begin{pmatrix} 0.2083333333x^4t^4 - 0.0833333333x^3t^4 - 0.5000000000x^5t^2 + 0.5000000000x^6t^2 \\ -0.0694444444x^3t^6 - 0.1250000000x^5t^4 + 0.3333333333x^4t^3 - 0.0416666667x^3t^5 \\ -0.5000000000x^5t^3 + 0.0416666667x^4t^5 + 0.1666666667x^6t^3 \end{pmatrix}}{\begin{pmatrix} 0.0833333333x^2t^4 - 0.3333333333x^3t^3 + 0.50000000000x^4t^2 + 0.0694444444x^2t^6 \\ -0.0416666667x^3t^5 - 0.0416666667x^2t^5 + 0.2083333333x^3t^4 + 0.083333333x^4t^4 \\ -0.3333333333x^4t^3 \end{pmatrix}}$$

Table 1: Comparison of the Numerical Solution of $u_1(x,t)$ with Exact Solutions $(t = 0.01)$

x	$u_1(x,t)$	$r_1(x,t)$	$\lvert u_1(x,t) - r_1(x,t)\rvert$
-0.5	0.007499875000	0.007499874999	1.10-12
-0.4	0.005599906667	0.005599006668	1.10-12
-0.3	0.003899935000	0.003899935000	0
-0.2	0.002399960000	0.002399960000	0
-0.1	0.001099981667	0.001099981668	1.10-12
0.1	-0.008999850001	-0.008999850006	5.10-13
0.2	-0.001599973333	-0.001599973333	0
0.3	-0.002099965000	-0.002099965000	0
0.4	-0.002399960000	-0.002399960000	0
0.5	-0.002499958334	-0.002499958333	1.10-12

Table 2: Comparison of the numerical solution of $u_2(x,t)$ with exact solutions ($t = 0.01$)

x	$u_2(x,t)$	$r_2(x,t)$	$\left\|u_2(x,t) - r_2(x,t)\right\|$
-0.5	0.07499625003	0.07499625011	8.10-10
-0.4	0.5599720002	0.5599720006	4.10-10
-0.3	0. 3899805002	0. 3899805002	2.10-10
-0.2	0.2399880001	0.2399880002	1.10-10
-0.1	0.1099945000	0.1099945001	1.10-10
0.1	-0.08999550004	-0.08999550002	2.10-11
0.2	-0. 1599920001	-0. 1599929999	2.10-10
0.3	-0.2099895001	-0.2099895999	2.10-10
0.4	-0.2399880001	-0.2399889999	2.10-10
0.5	-0.2499875001	-0.2499874995	6.10-10

Table 3: Comparison of the numerical solution of $u_3(x,t)$ with exact solutions ($t = 0.01$)

x	$u_3(x,t)$	$r_3(x,t)$	$\left\|u_3(x,t) - r_3(x,t)\right\|$
-0.5	0.7575376252	0.7575376251	1.10-10
-0.4	0.5656280935	0.5656280938	3.10-10
-0.3	0. 3939195651	0. 3939195651	0
-0.2	0.2424120401	0.2424120401	0
-0.1	0.1111055184	0.1111055184	0
0.1	-0.09090451503	-0.09090451504	1.10-11
0.2	-0. 1616080267	-0. 1616080267	0
0.3	-0.2121105351	-0.2121105350	1.10-10
0.4	-0.2424120401	-0.2424120401	0
0.5	-0.2525125418	-0.2525125415	3.10-10

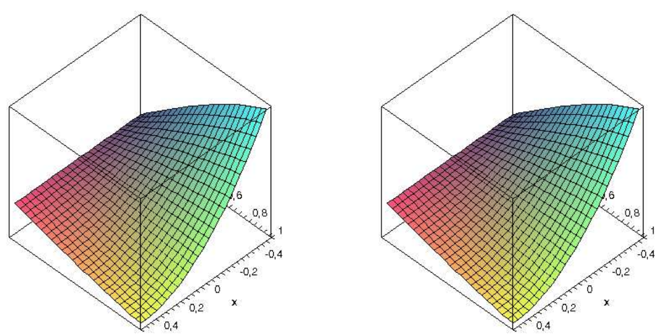

Figure 1: Values of $u_1(x,t)$ and its $r_{3,2}(x,t)$

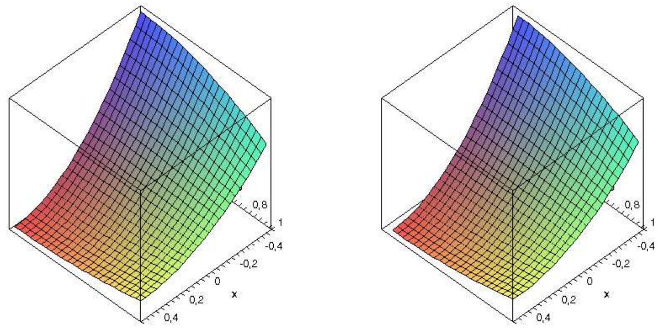

Figure 2: Values of $u_2(x,t)$ and its $r_{3,2}(x,t)$

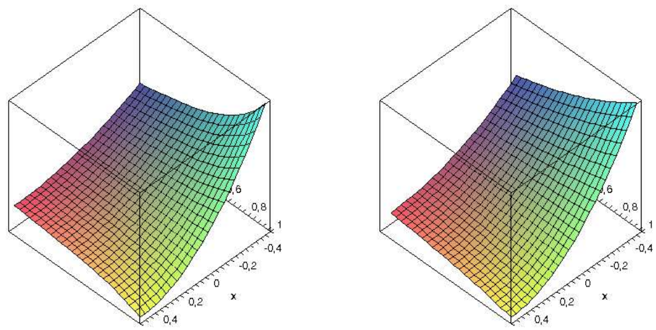

Figure 3: Values of $u_3(x,t)$ and its $r_{3,2}(x,t)$

5. Conclusions

The method has proposed for solving partial differential-algebraic equations(PDAEs). The results of example showed that exactly the same solutions have been obtained with Multivarite Padé approximation. On the other hand the results are quite reliable. Therefore, this method can be applied to many complicated Partial Differential-Algebraic Equations(PDAEs).

References

[1] G Adomian. Convergent series solution of nonlinear equations, Journal of Computational and applied Mathematatits, 11, 225-230,1984.

[2] F Ayaz. On the Two-Dimensional Differential Transform Method, Applied Mathematics and Computation, 143:361-374,2003.

[3] F Ayaz. Solutions of the System of Differential Equations by Differential Transform Method, Applied Mathematics and Computation, 147:547-567,2004.

[4] N Bildik, A Konuralp. Two-Dimensional Differential Transform Method,Adomian's Decompostion Method and Variational Iteration Method for Partial Differential Equations, International Journal of Computer Mathematics ,Vol.83, 12:973-987,2006.

[5] A Cuyt, L Wuytack. Nonlinear Methods in Numerical Analysis, Amsterdam,1987.

[6] W Lucht, K Strehmel, C E Liebenow. Linear Partial Differential-Algebraic Equations, Part I, Reports of the Institute of Numerical Mathematics, Report No. 17,1997.

[7] W Lucht, K Strehmel, C E Liebenow. Linear Partial Differential-Algebraic Equations, Part II, Reports of the Institute of Numerical Mathematics, Report No. 18, 1997,

[8] K Strehmel, K Debrabant. Convergence of Runge-Kutta Methods Applied to Linear Partial Differential-Algebraic Equations, Applied Numerical Mathematics, 53:213-229, 2005,

[9] J K Zhou. Differential Transform and its Applications for Electrical Circuits (Wuhan:Huarjung University Press), 1986