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Argument Estimates of Certain Meromorphically p-Valent Functions Defined by a Linear Operator

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Abstract. Making use of the linear operator $D^m_{\lambda,p}$, we obtain some argument properties of meromorphically $p-$valent functions. Also, we derive the integral preserving properties in a sector.

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1. Introduction

For any integer $n > -p$, let $\Sigma_{p,n}$ denote the class of all meromorphic functions of the form:

$$f(z) = z^{-p} + \sum_{k=n}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, \dots\}), \tag{1}$$

which are analytic and $p-$valent in the punctured unit disk

$$U^* = \{z : z \in \mathbb{C}, 0 < |z| < 1\} = U \backslash \{0\}.$$

Let f, g be analytic functions in U. Then we say that f is subordinate to g, written $f \prec g$ if there exists an analytic function $w(z)$ in U such that $|w(z)| < 1$ $(z \in U)$ and $f(z) = g(w(z))$. For this subordination, the symbol $f(z) \prec g(z)$ is used. In the case $g(z)$ is univalent in U, the subordination $f(z) \prec g(z)$ is equivalent to $g(0) = f(0)$ and $f(U) \subset g(U)$. For functions $f(z) \in \Sigma_{p,n}$ given by (1) and $g(z) \in \Sigma_{p,n}$ given by

$$g(z) = z^{-p} + \sum_{k=n}^{\infty} b_k z^k, \tag{2}$$

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we define the Hadamard product (or convolution) of f and g as

$$(f*g)(z)=z^{-p}+\sum_{k=n}^{\infty}a_kb_kz^k=(g*f)(z), \tag{3}$$

Following the recent works of Aouf and Hossen [4], Liu and Srivastava [7] and Srivastava and Patel [11], for a function $f(z)\in\Sigma_{p,n}$ given by (1), we now define a linear operator $D^m_{\lambda,p}$ $(\lambda\geq 0, p\in\mathbb{N}, m\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ by

$$\begin{aligned}
D^0_{\lambda,p}f(z)=&f(z)\\
D^1_{\lambda,p}f(z)=&D_{\lambda,p}f(z)=(1-\lambda)f(z)\frac{\lambda}{z^p}(z^{p+1}f(z))'\\
=&z^{-p}+\sum_{k=n}^{\infty}[1+\lambda(k+p)]a_kz^k,\\
D^2_{\lambda,p}f(z)=&D_{\lambda,p}(D_{\lambda,p}f(z))=z^{-p}+\sum_{k=n}^{\infty}[1+\lambda(k+p)]^2a_kz^k
\end{aligned}$$

and (in general)

$$D^m_{\lambda,p}f(z)=D_{\lambda,p}(D^{m-1}_{\lambda,p}f(z))=z^{-p}+\sum_{k=n}^{\infty}[1+\lambda(k+p)]^m a_kz^k,\quad \lambda\geq 0. \tag{4}$$

Also, we can write $D^m_{\lambda,p}f(z)$ as follows

$$\begin{aligned}
D^m_{\lambda,p}f(z)=&\left(z^{-p}+\sum_{k=n}^{\infty}[1+\lambda(k+p)]^m z^k\right)(z)\\
=&(f*\phi^m_{\lambda,p})(z),
\end{aligned} \tag{5}$$

where

$$\phi^m_{\lambda,p}(z)=z^{-p}+\sum_{k=n}^{\infty}[1+\lambda(k+p)]^m z^k.$$

It is easily verified from (4) that

$$\lambda z(D^m_{\lambda,p}f(z))'=D^{m+1}_{\lambda,p}f(z)-(1+\lambda p)D^m_{\lambda,p}f(z),\quad \lambda>0. \tag{6}$$

The operator $D^m_{\lambda,p}$ was introduced by Aouf [3].

For a function $f(z)\in\Sigma_{p,n}$ and $\upsilon>0$, the integral operator $F_{\upsilon,p}(f)(z):\Sigma_{p,n}\to\Sigma_{p,n}$ is defined by

$$F_{\upsilon,p}(f)(z)=\frac{\upsilon}{z^{\upsilon+p}}\int_0^z t^{\upsilon+p-1}f(t)dt$$

$$=z^{-p}+\sum_{k=n}^{\infty}\left(\frac{\upsilon}{\upsilon+p+k}\right)a_kz^k$$
$$=\left(z^{-p}+\sum_{k=n}^{\infty}\left(\frac{\upsilon}{\upsilon+p+k}\right)z^k\right)*f(z)\quad \upsilon>0;\, z\in U^*. \tag{7}$$

It follows from (7) that

$$z(D^m_{\lambda,p}F_{\upsilon,p}(f)(z))'=\upsilon D^m_{\lambda,p}f(z)-(\upsilon+\lambda p)D^m_{\lambda,p}F_{\upsilon,p}(f)(z). \tag{8}$$

The operator $F_{\upsilon,p}(f)(z)$ was investigated by many authors (see for example [1, 12, 13]). Let $\Sigma^*_{p,n}[\lambda,m,A,B]$ be the class of functions $f(z)\in\Sigma_{p,n}$ defined by

$$\Sigma^*_{p,n}[\lambda,m,A,B]=\left\{f(z)\in\Sigma_{p,n}:-\frac{z(D^m_{\lambda,p}f(z))'}{D^m_{\lambda,p}f(z)}\prec p\frac{1+Az}{1+Bz},\right.$$
$$\left. -1\le B<A\le 1;\quad \lambda>0; p\in\mathbb{N}; n>-p; m\in\mathbb{N}_0; z\in U^*\right\}. \tag{9}$$

We note that

(i) For $m=0$, we have $\Sigma^*_{p,n}[\lambda,0;1,-1]=\Sigma^*_{p,n}$, the well-known class of meromorphically $p-$valent starlike functions;

(ii) For $m=0, A=1-\frac{2\alpha}{p}$, $0\le\alpha<p$ and $B=-1$, we have $\Sigma^*_{p,n}[\lambda,0;1,-1]=\Sigma^*_{p,n}[\alpha]$, the well-known class of meromorphically $p-$valent starlike functions of order α (see [2]);

(iii) For $\lambda=1$ and $n=0$, the class $\Sigma^*_{p,n}[1,m;A,B]$ reduces to the class

$$\Sigma^*_{p,n}[m,A,B]=\left\{f(z)\in\Sigma_{p,n}:-\frac{z(D^m_pf(z))'}{D^m_pf(z)}\prec p\frac{1+Az}{1+Bz},\right.$$
$$\left. -1\le B<A\le 1; p\in\mathbb{N}; n>-p; m\in\mathbb{N}_0; z\in U^*\right\}.$$

where the operator D^m_p was introduced by Aouf and Hossen [4].

From (9) and by using the result of Silverman and Silvia [10], we observe that a function $f(z)$ is in the class $\Sigma^*_{p,n}[\lambda,m,A,B]$ $(-1<B<A\le 1; \lambda>0; p\in\mathbb{N}; m\in\mathbb{N}_0)$ if and only if

$$\left|\frac{z(D^m_{\lambda,p}f(z))'}{D^m_{\lambda,p}f(z)}+\frac{p(1-AB)}{1-B^2}\right|<\frac{p(A-B)}{1-B^2}\quad z\in U^* \tag{10}$$

The object of the present paper is to give some argument properties of meromorphically functions belonging to $\Sigma_{p,n}$ and the integral preserving properties in connection with the operator $D^m_{\lambda,p}$ defined by (4).

2. Main Results

Unless otherwise mentioned, we shall assume in the reminder of this paper that $\lambda > 0$, $n > -p$, $p \in \mathbb{N}$ and $m \in \mathbb{N}_0$.

In order to prove our main results, we need the following lemmas.

Lemma 1. *[5] Let $h(z)$ be convex (univalent) in U with $h(0) = 1$ and $\Re\{\beta h(z) + \gamma\} > 0$ $(\beta, \gamma \in \mathbb{C})$. If $q(z)$ is analytic in U with $q(0) = 1$, then*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec h(z),$$

implies $q(z) \prec h(z)$.

Lemma 2. *[8] Let $h(z)$ be convex (univalent) in U and $\psi(z)$ be analytic in in U with $\Re\{\psi(z)\} \geq 0$. If $q(z)$ is analytic in U and $q(0) = h(0)$, then*

$$q(z) + \psi(z)zq'(z) \prec h(z),$$

implies $q(z) \prec h(z)$.

Lemma 3. *[9] Let $q(z)$ be analytic in U, with $q(0) = 1$ and $q(z) \neq 0$, $(z \in U)$. Suppose that there exists a point $z_0 \in U$, such that*

$$|arg q(z)| < \frac{\pi}{2}\alpha \text{ for } |z| < |z_0| \tag{11}$$

and

$$|arg q(z_0)| < \frac{\pi}{2}\alpha \quad 0 < \alpha \leq 1. \tag{12}$$

Then, we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik\alpha, \tag{13}$$

where

$$k \geq \frac{1}{2}(\alpha + \frac{1}{\alpha}) \text{ when } arg q(z_0) = \frac{\pi}{2}\alpha, \tag{14}$$

$$k \geq -\ \frac{1}{2}(\alpha + \frac{1}{\alpha}) \text{ when } arg q(z_0) = -\ \frac{\pi}{2}\alpha, \tag{15}$$

and

$$q(z_0)^{\frac{1}{\alpha}} = \pm i\alpha, \quad \alpha > 0. \tag{16}$$

At first, with the help of Lemma 1, we obtain the following result:

Theorem 1. *Let h be convex univalent in U with $h(0) = 1$ and $\Re\{h\}$ be bounded in U. If $f(z) \in \Sigma_{p,n}$ satisfies the condition:*

$$-\frac{z(D_{\lambda,p}^{m+1}f(z))'}{pD_{\lambda,p}^{m+1}f(z)} \prec h(z)$$

then

$$-\frac{z(D_{\lambda,p}^{m}f(z))'}{pD_{\lambda,p}^{m}f(z)} \prec h(z)$$

for $\max_{z\in U}\Re h(z) < \left(\frac{1+\lambda p}{\lambda p}\right)$ *(provided $D_{\lambda,p}^{m}f(z) \neq 0$, $z \in U^*$).*

Proof. Let

$$q(z) = -\frac{z(D_{\lambda,p}^{m}f(z))'}{pD_{\lambda,p}^{m}f(z)}\ .$$

By using (6), we have

$$q(z) - \left(\frac{1+\lambda p}{\lambda p}\right) = -\frac{D_{\lambda,p}^{m+1}f(z)}{\lambda_p D_{\lambda,p}^{m}f(z)}. \tag{17}$$

Using logarithmic differentiation in both sides of (17) with respect to z and multiplying by z, we get

$$\frac{zq'(z)}{-pq(z) + \frac{1+\lambda p}{\lambda}} + q(z) = --\frac{D_{\lambda,p}^{m+1}f(z)}{pD_{\lambda,p}^{m}f(z)} \prec h(z)$$

From Lemma 1, it follows that $q(z) \prec h(z)$ for $\Re\left\{-h(z) + \frac{1+\lambda p}{\lambda p}\right\} > 0$, $z \in U^*$, which means

$$-\frac{z(D_{\lambda,p}^{m}f(z))'}{pD_{\lambda,p}^{m}f(z)} \prec h(z)$$

for $\max_{z\in U}\Re h(z) < \frac{1+\lambda p}{\lambda p}$. □

Using Lemmas 1 and 2 and Theorem 1, we now derive:

Theorem 2. *Let $f(z) \in \Sigma_{p,n}$, $\frac{1}{\lambda} \geq \frac{p(A-B)}{1+B}$, where $-1 < B < A \leq 1$. If*

$$\left|\arg\left(-\frac{z(D_{\lambda,p}^{m+1}f(z))'}{pD_{\lambda,p}^{m+1}g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta, \quad 0 \leq \gamma < p;\ 0 < \delta < 1$$

for some $g(z) \in \Sigma_{p,n}^*[\lambda, m+1; A, B]$ then

$$\left|\arg\left(-\frac{z(D_{\lambda,p}^{m}f(z))'}{pD_{\lambda,p}^{m}g(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha, \quad 0 < \alpha \leq 1$$

is the solution of the equation

$$\delta = \alpha + \frac{2}{\pi} tan^{-1} \left(\frac{\alpha \sin \frac{\pi}{2}[1 - t(A,B)]}{\frac{(1-B)+\lambda p(A-B)}{\lambda(1-B)} + \alpha \cos \frac{\pi}{2}[1 - t(A,B)]} \right), \tag{18}$$

when

$$t(A,B) = \frac{2}{\pi} \sin^{-1} \left(\frac{\lambda p(A-B)}{(1+\lambda p)(1-B^2) - \lambda p(1-AB)} \right). \tag{19}$$

Proof. Let

$$q(z) = \frac{1}{p-\gamma} \left(-\frac{z(D^m_{\lambda,p} f(z))'}{pD^m_{\lambda,p} g(z)} - \gamma \right).$$

Using the identity (6), we have

$$\begin{aligned}(p-\gamma)zq'(z)D^m_{\lambda,p}g(z) + (p-\gamma)q(z)z(D^m_{\lambda,p}f(z))' + \gamma z(D^m_{\lambda,p}g(z))' \\ = \frac{1+\lambda p}{\lambda} z(D^m_{\lambda,p} f(z))' - \frac{1}{\lambda} z(D^{m+1}_{\lambda,p} f(z))'.\end{aligned} \tag{20}$$

Simplifying (20), we obtain

$$q(z) + \frac{zq'(z)}{-r(z) + \frac{1+\lambda p}{\lambda}} = -\frac{1}{p-\gamma} \left(\frac{z(D^{m+1}_{\lambda,p} f(z))'}{D^{m+1}_{\lambda,p} g(z)} + \gamma \right), \tag{21}$$

where

$$r(z) = -\frac{z(D^m_{\lambda,p} g(z))'}{D^m_{\lambda,p} g(z)}.$$

Since $g(z) \in \Sigma^*_{p,n}[\lambda, m, A, B]$, from Theorem 1, we have

$$r(z) \prec p\frac{1+Az}{1+Bz},$$

using (10), we have

$$-r(z) + \frac{1+\lambda p}{\lambda} = \rho e^{i\frac{\pi}{2}\phi}$$

where

$$\frac{(1+B) - \lambda p(A-B)}{\lambda(1+B)} < \rho < \frac{(1-B) + \lambda p(A-B)}{\lambda(1+B)}$$

and $-t(A,B) < \phi < t(A,B)$, where $t(A,B)$ is given by (19).

Let h be a function which maps U onto the angular domain $\{w : |\arg w| < \frac{\pi}{2}\delta\}$ with $h(0) = 1$. Applying Lemma 2 for this h with $\psi(z) = \frac{1}{-r(z)+\frac{1+\lambda p}{\lambda}}$ we see that $\Re\{q(z)\} > 0$ in U and hence $q(z) \neq 0$ in U. If there exists a point $z_0 \in U$ such that the conditions (11) and

(12) are satisfied, then by using Lemma 3, we have (13) under the restrictions (14), (15) and (16).

At first, suppose that $q(z_0)^{\frac{1}{\alpha}} = ia(a > 0)$. Then we obtain

$$\begin{aligned}
&\arg\left[-\frac{1}{p-\gamma}\left(\frac{z(D_{\lambda,p}^{m+1}f(z_0))'}{D_{\lambda,p}^{m+1}g(z_0)}+\gamma\right)\right]\\
=&\arg\left(q(z_0)+\frac{z_0q'(z_0)}{-r(z_0)+\frac{1+\lambda p}{\lambda}}\right)\\
=&\frac{\pi}{2}\alpha+\arg\left(1+ik\alpha\left(\rho e^{i\frac{\pi}{2}\phi}\right)^{-1}\right)\\
=&\frac{\pi}{2}\alpha+tan^{-1}\left(\frac{\alpha k\sin\frac{\pi}{2}[1-\phi]}{\rho+\alpha k\cos\frac{\pi}{2}[1-\phi]}\right),\\
\geq&\frac{\pi}{2}\alpha+tan^{-1}\left(\frac{\alpha\sin\frac{\pi}{2}[1-t(A,B)]}{\frac{(1-B)+\lambda p(A-B)}{\lambda(1-B)}+\alpha\cos\frac{\pi}{2}[1-t(A,B)]}\right)\\
&-\frac{\pi}{2}\delta,
\end{aligned}$$

where δ and $t(A,B)$ are given by (18) and (19), respectively. This is a contradiction to the assumption of our theorem.

Next, suppose that $q(z_0)^{\frac{1}{\alpha}} = -ia(a > 0)$. Applying the same method as the above, we have

$$\begin{aligned}
&\arg\left[-\frac{1}{p-\gamma}\left(\frac{z_0(D_{\lambda,p}^{m+1}f(z_0))'}{D_{\lambda,p}^{m+1}g(z_0)}+\gamma\right)\right]\\
\leq&-\frac{\pi}{2}\alpha-tan^{-1}\left(\frac{\alpha\sin\frac{\pi}{2}[1-t(A,B)]}{\frac{(1-B)+\lambda p(A-B)}{\lambda(1-B)}+\alpha\cos\frac{\pi}{2}[1-t(A,B)]}\right)\\
=&-\frac{\pi}{2}\delta,
\end{aligned}$$

where δ and $t(A;B)$ are given by (18) and (19), respectively, which contradicts the assumption. Therefore we complete the proof of our theorem. □

Taking $A=1$, $B=0$ and $\delta=1$ in Theorem 2, we have the following corollary.

Corollary 1. *Let $f(z)\in\Sigma_{p,n}$. If*

$$-\Re\left\{\frac{z(D_{\lambda,p}^{m+1}f(z))'}{D_{\lambda,p}^{m+1}g(z)}\right\}>\gamma \quad 0\leq\gamma<p$$

for some $g(z) \in \Sigma_{p,n}^*$ *satisfying the condition*

$$\left| \frac{z(D_{\lambda,p}^{m+1} g(z))'}{D_{\lambda,p}^{m+1} g(z)} + p \right| < p$$

then

$$-\Re \left\{ \frac{z(D_{\lambda,p}^{m} f(z))'}{D_{\lambda,p}^{m} g(z)} \right\} > \gamma \quad 0 \le \gamma < p.$$

Taking $A = 1$, $B = 0$ and $g(z) = \frac{1}{z^p}$ in Theorem 2, we have the following corollary.

Corollary 2. *Let* $f(z) \in \Sigma_{p,n}$*, If*

$$\left| \arg \left(- z^{p+1} (D_{\lambda,p}^{m+1} f(z))' - \gamma \right) \right| < \frac{\pi}{2} \delta, \quad 0 \le \gamma < p; 0 < \delta \le 1$$

then

$$\left| \arg \left(- z^{p+1} (D_{\lambda,p}^{m} f(z))' - \gamma \right) \right| < \frac{\pi}{2} \alpha, \quad 0 < \alpha \le 1.$$

Taking $m = 0$ and $\delta = 1$ in Corollary 2, we have the following corollary.

Corollary 3. *Let* $f(z) \in \Sigma_{p,n}$*, If*

$$-\Re \left\{ z^{p+1} [\lambda z f''(z) + (1 + \lambda + \lambda p) f'(z)] \right\} > \gamma, \quad 0 \le \gamma < p,$$

then

$$-\Re \left\{ z^{p+1} f'(z) \right\} > \gamma.$$

Remark 1. *Taking* $\lambda = p = 1$ *in Corollary 3, we obtain the result obtained by Lashin [6, Corollary 2.5 with* $p = 1$*]*

By the same technique as in the proof of Theorem 2, we obtain

Theorem 3. *Let* $f(z) \in \Sigma_{p,n}$*. Choose* λ *such that* $\frac{1}{\lambda} \ge \frac{p(A-B)}{1+B}$*, where* $-1 < B < A \le 1$*. If*

$$\left| \arg \left\{ \frac{z(D_{\lambda,p}^{m+1} f(z))'}{D_{\lambda,p}^{m+1} g(z)} + \gamma \right\} \right| < \frac{\pi}{2} \delta, \quad \gamma > p; 0 < \delta < 1$$

for some $g(z) \in \Sigma_{p,n}^* [\lambda, m+1; A, B]$*, then*

$$\left| \arg \left\{ \frac{z(D_{\lambda,p}^{m} f(z))'}{D_{\lambda,p}^{m} g(z)} + \gamma \right\} \right| < \frac{\pi}{2} \alpha, \quad 0 < \alpha \le 1$$

is the solution of the equation (18).

Theorem 4. *Let h be convex univalent in U with $h(0) = 1$ and $\Re h$ be bounded in U. Let $F_{\upsilon,p}(f)(z)$ be the integral operator defined by (7). If $f(z) \in \Sigma_{p,n}$ satisfies the condition*

$$-\frac{z(D^m_{\lambda,p}f(z))'}{pD^m_{\lambda,p}f(z)} \prec h(z)$$

then

$$-\frac{z(D^m_{\lambda,p}F_{\upsilon,p}(f)(z))'}{pD^m_{\lambda,p}F_{\upsilon,p}(f)(z)} \prec h(z)$$

for $\max_{z\in U}\Re h(z) < \frac{\upsilon+p}{p}$ (provided $D^m_{\lambda,p}F_{\upsilon,p}(f)(z) \neq 0$ in U^*).

Proof. Let

$$q(z) = -\frac{z(D^m_{\lambda,p}F_{\upsilon,p}(f)(z))'}{pD^m_{\lambda,p}F_{\upsilon,p}(f)(z)} .$$

Then, by using (8), we have

$$pq(z) - (\upsilon + p) = -\upsilon\frac{D^m_{\lambda,p}f(z)}{D^m_{\lambda,p}F_{\upsilon,p}(f)(z)}. \tag{22}$$

Taking logarithmic derivatives in both sides of (22) with respect to z and multiplying by z, we get

$$q(z) + \frac{zq'(z)}{-pq(z) + (\upsilon + p)} = -\frac{z(D^m_{\lambda,p}f(z))'}{pD^m_{\lambda,p}f(z)} \prec h(z).$$

Therefore, by using Lemma 1, we have

$$-\frac{z(D^m_{\lambda,p}F_{\upsilon,p}(f)(z))'}{pD^m_{\lambda,p}F_{\upsilon,p}(f)(z)} .$$

for $\max_{z\in U}\Re h(z) < \frac{\upsilon+p}{p}$ (provided $D^m_{\lambda,p}F_{\upsilon,p}(f)(z) \neq 0$ in U^*). This completes the proof of Theorem 4. □

Theorem 5. *Let $f(z) \in \Sigma_{p,n}$ and choose a positive number υ such that $\upsilon \geq p\frac{A-B}{1+B}$, where $-1 < B < A \leq 1$. If*

$$\left|\arg\left\{-\frac{z(D^m_{\lambda,p}f(z))'}{pD^m_{\lambda,p}g(z)} - \gamma\right\}\right| < \frac{\pi}{2}\delta, \quad 0 \leq \gamma < p; 0 < \delta \leq 1,$$

for some $g(z) \in \Sigma^*_{p,n}[\lambda, m; A, B]$ then

$$\left|\arg\left(-\frac{z(D^m_{\lambda,p}F_{\upsilon,p}(f)(z))'}{pD^m_{\lambda,p}G_{\upsilon,p}(f)(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha, \quad 0 < \alpha \leq 1$$

where $F_{\upsilon,p}(f)(z)$ is the integral operator given by (7),

$$G_{\upsilon,p}(f)(z) = \frac{\upsilon}{z^{\upsilon+p}} \int_0^z t^{\upsilon+p-1} g(t)dt \quad \upsilon > 0; \tag{23}$$

is the solution of the equation

$$\delta = \alpha + \frac{2}{\pi} tan^{-1} \left(\frac{\alpha \sin \frac{\pi}{2}[1 - t(A,B,\upsilon)]}{\frac{(\upsilon+p)(1-B)+p(A-B)}{1-B} + \alpha \cos \frac{\pi}{2}[1 - t(A,B,\upsilon)]} \right), \tag{24}$$

when

$$t(A,B,\nu) = \frac{2}{\pi} \sin^{-1} \left(\frac{p(A-B)}{(\upsilon+p)(1-B^2) - p(1-AB)} \right). \tag{25}$$

Proof. Let

$$q(z) = -\frac{1}{p-\gamma} \left(\frac{z(D^m_{\lambda,p} F_{\upsilon,p}(f)(z))'}{p D^m_{\lambda,p} G_{\upsilon,p}(g)(z)} + \gamma \right).$$

Since $g(z) \in \Sigma^*_{p,n}[\lambda, m, A, B]$, from Theorem 4, $G_{\upsilon,p}(g)(z) \in \Sigma^*_{p,n}[\lambda, m, A, B]$. Using the identity (8), we have

$$(p-\gamma)q(z) D^m_{\lambda,p} G_{\upsilon,p}(g)(z) - (\upsilon+p) D^m_{\lambda,p} F_{\upsilon,p}(f)(z) = -\upsilon D^m_{\lambda,p} f(z) - \gamma D^m_{\lambda,p} G_{\upsilon,p}(g)(z).$$

Then, by a simple calculation, we have

$$(p-\gamma)\{zq'(z) + q(z)[-r(z) + \upsilon + p]\} + \gamma[-r(z) + \upsilon + p] = -\upsilon \frac{\upsilon z(D^m_{\lambda,p} f(z))'}{D^m_{\lambda,p} G_{\upsilon,p}(g)(z)}$$

where

$$r(z) = \frac{z(D^m_{\lambda,p} F_{\upsilon,p}(f)(z))'}{D^m_{\lambda,p} G_{\upsilon,p}(g)(z)}.$$

Hence, we have

$$q(z) + \frac{zq'(z)}{-r(z) + \upsilon + p} = -\frac{1}{p-\gamma} \left(\frac{z(D^m_{\lambda,p} f(z))'}{D^m_{\lambda,p} g(z)} + \gamma \right). \tag{26}$$

The remaining part of the proof is similar to that of Theorem 2 and so, we omit it. □

Taking $m = 0$ in Theorem 5, we obtain the result obtained by Lashin [6, Corollary 2.3].
Taking $m = 0$, $A = 1$, $B = 0$ and $\delta = 1$ in Theorem 5, we obtain the following result.

Corollary 4. *Let* $\upsilon > 0$ *and* $f(z) \in \Sigma_{p,n}$. *If*

$$-\Re\left\{\frac{zf'(z)}{g(z)}\right\} > \gamma \quad 0 \leq \gamma < p$$

for some $g(z) \in \Sigma_{p,n}$ *satisfying the condition*

$$\left|\frac{zg'(z)}{g(z)} + p\right| < p$$

then

$$-\Re\left\{\frac{zF'_{\upsilon,p}(f)(z)}{G_{\upsilon,p}(g)(z)}\right\} > \gamma \quad 0 \leq \gamma < p.$$

where $F_{\upsilon,p}(f)(z)$ *and* $G_{\upsilon,p}(g)(z)$ *are given by* (7) *and* (23), *respectively.*

Taking $m = 0$ $B \to A$ and $g(z) = \frac{1}{z^p}$ in Theorem 5, we have the following corollary.

Corollary 5. *Let* $\upsilon > 0$ *and* $f(z) \in \Sigma_{p,n}$. *If*

$$|\arg(-z^{p+1}f'(z) - \gamma)| < \frac{\pi}{2}\delta, \quad 0 \leq \gamma < p; 0 < \delta \leq 1,$$

then

$$|\arg(-z^{p+1}F'_{\upsilon,p}(f)(z) - \gamma)| < \frac{\pi}{2}\alpha,$$

where $F_{\upsilon,p}(f)(z)$ *is given by* (7) *and* $0 < \alpha \leq 1$ *is the solution of the equation*

$$\delta = \alpha + \frac{2}{\pi}tan^{-1}\left(\frac{\alpha}{\upsilon + p}\right).$$

By using the same argument used in proving Theorem 5, we have

Theorem 6. *Let* $f(z) \in \Sigma_{p,n}$ *and choose a positive number* υ *such that* $\upsilon \geq \frac{1+A}{1+B} - p$, *where* $-1 < B < A \leq 1$. *If*

$$\left|\arg\left\{\frac{z(D^m_{\lambda,p}f(z))'}{D^m_{\lambda,p}g(z)} + \gamma\right\}\right| < \frac{\pi}{2}\delta, \quad \gamma > p;\ 0 < \delta \leq 1,$$

for some $g(z) \in \Sigma^*_{p,n}[\lambda, m; A, B]$ *then*

$$\left|\arg\left(\frac{z(D^m_{\lambda,p}F_{\upsilon,p}(f)(z))'}{D^m_{\lambda,p}G_{\upsilon,p}(g)(z)} + \gamma\right)\right| < \frac{\pi}{2}\alpha, \quad 0 < \alpha \leq 1$$

where $F_{\upsilon,p}(f)(z)$ *and* $G_{\upsilon,p}(g)(z)$ *are given* (7) *and* (23), *respectively, and* $\alpha(0 < \alpha \leq 1)$ *is the solution of the equation* (24).

Finally, we derive

Theorem 7. *Let $f(z) \in \Sigma_{p,n}$ and choose λ such that $\frac{1}{\lambda} \geq \frac{p(A-B)}{1+B}$, where $-1 < B < A \leq 1$. If*

$$\left|\arg\left\{-\frac{z(D^m_{\lambda,p}f(z))'}{D^m_{\lambda,p}g(z)} - \gamma\right\}\right| < \frac{\pi}{2}\delta, \quad 0 \leq \gamma < p;\, 0 < \delta \leq 1,$$

for some $g(z) \in \Sigma^*_{p,n}[\lambda, m; A, B]$ then

$$\left|\arg\left(-\frac{z(D^{m+1}_{\lambda,p}F_{\upsilon,p}(f)(z))'}{D^{m+1}_{\lambda,p}G_{\upsilon,p}(g)(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta,$$

where $F_{\upsilon,p}(f)(z)$ and $G_{\upsilon,p}(g)(z)$ are given (7) and (23), respectively with $\upsilon = \frac{1}{\lambda}$.

Proof. From (6) and (8), with $\upsilon = \frac{1}{\lambda}$, we have $D^m_{\lambda,p}f(z) = D^{m+1}_{\lambda,p}F_{\upsilon,p}(f)(z)$. Therefore

$$\frac{z(D^m_{\lambda,p}f(z))'}{D^m_{\lambda,p}g(z)} = \frac{z(D^{m+1}_{\lambda,p}F_{\upsilon,p}(f)(z))'}{D^{m+1}_{\lambda,p}G_{\upsilon,p}(g)(z)}$$

and the theorem follows. □

Remark 2. *Putting $\lambda = 1$ and $n = 0$ in the above results, we obtain the results corresponding to the class $\Sigma^*_p[m; A, B]$ defined in the introduction.*

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